ABSTRACT : We describe a set of 18 invariant amplitudes free from kinematical
singularities in $\nu$ and $t$ ($\nu$ is proportional to the incoming photon energy in
the laboratory frame; $t$ is the squared momentum transfer) for Compton
scattering of off-shell photons on polarized nucleons. Special cases related to
the forward and the non-forward scattering limits are discussed.

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I - INTRODUCTION

We shall be concerned with Compton scattering of off-shell photons from polarized nucleons. The corresponding amplitude is defined by the following expression

\[ C_{\mu,\nu}^{A, A'} \left( P = \frac{1}{2} (p + p'), q, q' \right) (2\pi)^4 \delta^{(4)} (p + q - p' - q') = \int d^4x d^4x' e^{i q \cdot x} e^{-i q' \cdot x'} \left< p', A' \left| T^* \left( J_{\mu}(x) \overline{J}_{\nu}(x') \right) \right| p, A \right>, \quad (1.1) \]

where \( J_{\mu}(x) \) is the electromagnetic hadronic current and \|p, A\rangle\rangle, \|p', A'\rangle\rangle are one-nucleon states with energy momenta \( p \) and \( p' \) and polarizations \( A \) and \( A' \). The \( T^* \) product is understood to be the covariant time-ordered product. A diagrammatic representation of the amplitude \( C_{\mu,\nu}^{A, A'} (P, q, q') \) is given in Fig. 1.

These general non-forward Compton amplitudes appear in many physical processes. As examples we quote the two-photon exchange interference with the Born amplitude in elastic electron-proton scattering (see Fig. 2); the so-called Compton diagrams in the Bethe-Heitler processes (see Fig. 3b and 4b). It is conceivable that sufficiently precise experiments will be performed which are sensitive to these general nucleon Compton amplitudes. In particular, with the use of polarized hydrogen targets, it is possible to obtain precious information on observables governed by off-shell nucleon Compton amplitudes. An example is the elastic scattering of electrons from polarized protons where there already exist experimental results [1]. In this case, the up-down asymmetry is sensitive to the absorptive part of the spin independent as well as the spin dependent nucleon Compton amplitudes [2]. The absorptive Compton amplitude satisfies positivity conditions, which have been discussed recently in details by De Rújula and de Rafael [3]. These positivity restrictions are most easily expressed using a set of invariant amplitudes, the so-called transversity amplitudes, which however are not free from kinematical singularities.
In view of possible applications it would be nice to have a set of invariant amplitudes for Compton scattering of off-shell photons from polarized nucleons which are free from kinematical singularities. This is already an interesting problem per se, since a full exploitation of dispersion relations in Compton scattering requires, as a preliminary "exercise", the construction of such a set of amplitudes. The purpose of this paper is precisely to give the details of the construction of a set of invariant amplitudes free from kinematical singularities in the variables $\nu$ and $t$

$$\nu = (p, q) = (P, q'); \quad t = (p - p')^2 < 0$$

Such invariant amplitudes have already been partially constructed by various authors [4,5,6]. Their results, however, are given in spinorial form which requires further algebraic elaboration to use them in applications. Moreover we have not seen an explicit construction which takes into account the gauge invariance of the Compton amplitude and avoids kinematical singularities in $\nu$ and $t$. The choice of a set of tensor covariants for the construction of the nucleon Compton amplitudes is rather delicate; indeed, due to an unsuitable choice of the set of tensor covariants, many authors [7, 8,9] have obtained, either invariant amplitudes with kinematical singularities, or an incorrect number of independent invariant amplitudes.

The organization of the paper is as follows. In section II, we define the general Compton amplitude $C_{\mu'\nu}^{\lambda'\lambda}(P, q, q')$ and the corresponding kinematics. The absorptive part $W_{\mu'\nu}^{\lambda'\lambda}(P, q, q')$ of the general Compton amplitude plays an important role in the applications; we give the definition of $W_{\mu'\nu}^{\lambda'\lambda}$ and indicate the relation between $W_{\mu'\nu}^{\lambda'\lambda}$ and $C_{\mu'\nu}^{\lambda'\lambda}$. Then we describe the method used for the construction of invariant amplitudes free from kinematical singularities. On some crucial points a brief review of the literature is given. The selection rules due to time reversal invariance, to crossing symmetry properties and to hermiticity are discussed in sections II.1 to II.3.
In section III, we study the forward scattering limit \((t = 0)\), which is related to deep inelastic electron-proton scattering. The relation between our invariant amplitudes and the usual structure functions \(^1\) \(W_i\), \(W_i'\), \(G_i\), and \(G_i'\) is given.

In section IV, we consider some particular non-forward scattering limits. The cases \(q^2 = 0\) or \(q'i^2 = 0\) which are respectively related to photoproduction of electron (or muon) pairs on nucleons and bremsstrahlung of electron (or muon) on nucleons are discussed in section IV.1. In section IV.2 we study the case \(q^2 = q'i^2\) and in section IV.3 the real Compton scattering case \((q^2 = q'i^2 = 0)\).

For the sake of clarity the results of sections II to IV are summarized in an appendix where the reader can immediately find the tensors corresponding to each case which has been studied.
II - DECOMPOSITION OF THE GENERAL COMPTON AMPLITUDE INTO INVARIANT AMPLITUDES FREE FROM KINEMATICAL SINGULARITIES.

The amplitude $\mathcal{C}_{\mu\nu}^{A'}(T,q,q')$ for Compton scattering of off-shell photons on polarized nucleons has already been defined by Eq. (1.1). The states $|p,A\rangle$ and $|p',A'\rangle$ are one-nucleon states with momenta $p$ and $p'$ and spin projection eigenvalues $\lambda, \lambda'$ corresponding to directions $n(p)$ and $n'(p')$ such that:

$$n(p).p = 0, \quad n(p)^2 = -1$$

$$n'(p').p' = 0, \quad n'(p')^2 = -1$$

and

$$n'(p') = \Lambda_{p-p'} n(p)$$

where $\Lambda_{p-p'}$ is the pure Lorentz transformation (boost) which brings $p$ into $p'$. $\Lambda_{p-p'}$ can be written explicitly in terms of the momenta and the nucleon mass $M$. The conservation of energy-momentum is contained in the definition (1.1) of $\mathcal{C}_{\mu\nu}^{A'}(T,q,q')$; we have

$$p + q = p' + q'$$

For convenience, we introduce the following 4-vectors

$$P = \frac{i}{2} \{ p + p' \}$$

$$Q = \frac{i}{2} \{ q + q' \}$$

$$\Delta = \frac{i}{2} \{ q - q' \}$$

The invariants $\gamma$ and $t$ have been defined in Eq. (1.2). It is easy to see that $q^2, q'^2, \gamma$ and $t$ are a set of independent invariants for the problem.

The absorptive part $W_{\mu\nu}^{A'}(T,q,q')$ of the nucleon Compton amplitude is defined by

$$W_{\mu\nu}^{A'}(T,q,q') =$$

$$= \frac{i}{2} \sum_p (2\pi)^4 \delta^{(4)}(p+q-p') \langle p',A'|J_\mu(0)|p,A\rangle$$

$$= \frac{i}{2} \sum_p (2\pi)^4 \delta^{(4)}(p+q-p') \langle p',A'|J_\mu(0)|p,A\rangle$$

(2.5)
The summation $\sum_{\tau} T_{\tau}$ is extended to all possible on mass-shell strongly interacting states $|\rho, \lambda\rangle$ with phase space integration understood.

Using translation invariance for the electromagnetic hadronic current $J_{\gamma}(x)$ and the corresponding transformation of states $|\rho, \lambda\rangle$ and $|\rho', \lambda'\rangle$, it can be shown that

$$C_{\mu\nu}^{(n', \rho', \lambda')}(T, q, q') = \int d^4x e^{iq'x} <p, \lambda | T^{*}(J_{\gamma}(x) J_{\mu}(x)) | p, \lambda> \quad (2.6)$$

By means of the definition of the $T^{*}$-product and using again translation invariance, we have for the absorptive part of $C_{\mu\nu}^{(n', \rho', \lambda')}(T, q, q')$ the expected result

$$\text{Abs } C_{\mu\nu}^{(n', \rho', \lambda')}(T, q, q') = W_{\mu\nu}^{(n', \rho', \lambda')} \quad (2.7)$$

We shall separate $C_{\mu\nu}^{(n', \rho', \lambda')}$ and $W_{\mu\nu}^{(n', \rho', \lambda')}$ into nucleon spin independent and dependent parts. For this purpose we write $C_{\mu\nu}^{(n', \rho', \lambda')}$ as a $2 \times 2$ matrix in the spin space of the nucleons

$$\frac{i}{2M} \ C_{\mu\nu}^{(n', \rho', \lambda')} = C_{\mu\nu}^{(n', \rho', \lambda')} + i \ C_{\mu\nu}^{(n', \rho', \lambda')} \ U_{\mu\nu}^{(n', \rho', \lambda')} \quad (2.8)$$

$$\frac{i}{2\pi M} \ W_{\mu\nu}^{(n', \rho', \lambda')} = W_{\mu\nu}^{(n', \rho', \lambda')} + i \ W_{\mu\nu}^{(n', \rho', \lambda')} \ U_{\mu\nu}^{(n', \rho', \lambda')} \quad (2.9)$$

The matrices $U_{\mu\nu}$ are the usual Pauli matrices, $n'(p')$ are three 4-vectors such that $(i = 1, 2, 3)$

$$U_{i}^{(p')} \cdot p' = 0, \quad U_{i}^{(p')} \cdot U_{j}^{(p')} = -\delta_{ij} \quad (2.10)$$

The terms $C_{\mu\nu}^{(n', \rho', \lambda')}$ and $W_{\mu\nu}^{(n', \rho', \lambda')}$ are spin independent tensors; they are second rank tensors in Minkowski space. The terms $C_{\mu\nu}^{(n', \rho', \lambda')}$ and $W_{\mu\nu}^{(n', \rho', \lambda')}$ are third rank spin pseudotensors. The quantization axis $n'(p')$ (see Eqs. (2.1) and (2.2)) of the outgoing nucleon is chosen along the third direction $n_3(p')$. We emphasize that the choice of the 4-vectors $n'_i(p') (i = 1, 2, 3)$ is arbitrary, but the tensors $C_{\mu\nu}^{(n', \rho', \lambda')}$, $W_{\mu\nu}^{(n', \rho', \lambda')}$ and $W_{\mu\nu}^{(n', \rho', \lambda')}$ are intrinsic objects.
Let us assume the following decomposition for the spin independent tensors

\[ C_{\mu\nu}(P,q,q') \text{ and } W_{\mu\nu}(P,q,q') \]

\[ C_{\mu\nu}(P,q,q') = \sum_i T_{\mu\nu}^{i} \Omega_i(q^2,q'^2,\nu,\tau) \] \hspace{1cm} (2.11)

\[ W_{\mu\nu}(P,q,q') = \sum_i T_{\mu\nu}^{i} \omega_i(q^2,q'^2,\nu,\tau) \] \hspace{1cm} (2.12)

The tensors \( T_{\mu\nu}^{i} \) are tensor covariants built up with the three independent 4-vectors \( P, q, \) and \( q' \), and \( \Omega_i(q^2,q'^2,\nu,\tau) \) and \( \omega_i(q^2,q'^2,\nu,\tau) \) are invariant amplitudes. Then with the definitions (2.8), (2.9) and Eq. (2.7) we have

\[ \omega_i(q^2,q'^2,\nu,\tau) = \frac{i}{\pi} \text{Im} \Omega_i(q^2,q'^2,\nu,\tau) \] \hspace{1cm} (2.13)

The same relation holds for invariant amplitudes associated with the spin tensors.

The general structure of Compton amplitudes for scattering of off-shell photons on polarized nucleons in terms of tensor covariants and invariant amplitudes can be specified under the restrictions of Lorentz invariance, gauge invariance and discrete symmetry requirements (parity and time reversal). As mentioned in the left hand side of Eq. (1.1) we shall use a set of basic variables the three 4-vectors \( P, q, q' \) which allow to express easily the gauge invariance and the crossing properties of \( C_{\mu\nu} \). These variables will also be very useful to deal with the problem of kinematical singularities and to discuss directly the particular cases studied in sections III and IV.

In the Dirac formalism, we can write

\[ C_{\mu\nu}(P,q,q') = 2 M \sqrt{(p+q)^2} \bar{u}(p') A_{\mu\nu}(P,q,q') u(p,\lambda) \] \hspace{1cm} (2.14)

where \( A_{\mu\nu} \) is the most general second rank tensor constructed with the Dirac matrices, the three 4-vectors \( P, q, q' \) and invariant amplitudes \( a_i(q^2,q'^2,\nu,\tau) \)

\[ A_{\mu\nu}(P,q,q') = \sum_i T_{\mu\nu}^{i} a_i(q^2,q'^2,\nu,\tau) \] \hspace{1cm} (2.15)

The following list contains all the possible tensor covariants \( T_{\mu\nu}^{i} \) that we can consider at first sight.
In writing this list, we have already taken into account parity conservation which allows to eliminate all the tensor covariants containing \( \gamma_5 \). Moreover Dirac equation has been used to keep only

\[
\begin{align*}
\bar{u}(p', \Lambda') \gamma^\mu u(p, \Lambda) & = \bar{u}(p', \Lambda') \gamma^\nu q(p, \Lambda) = \bar{u}(p', \Lambda') \gamma^\nu \Lambda \gamma(p, \Lambda) \\
\bar{u}(p', \Lambda') \gamma^\nu q(p, \Lambda) & = M \bar{u}(p', \Lambda') q(p, \Lambda)
\end{align*}
\]

Let us remark that \( Q \) and \( \Lambda \), which will be used in the following, have be understood as a shortening for \( \frac{1}{2} (q + q') \) and \( \frac{1}{2} (q - q') \), since \( p \), \( q \) and \( q' \) are the basic variables. \( t_{\mu\nu}^{15} \) and \( t_{\mu\nu}^{16} \) have been chosen in this particular way (instead of \( P_\mu q_\nu \phi \) and \( P_\nu q'_\mu \phi \)) in order to preserve simple crossing properties in further eliminations.

Notice that a simple helicity counting yields to 32 independent amplitudes. Unfortunately we have 34 tensor covariants in (2.16) ; thus there must be two independent relations among the 34 tensor covariants \( t_{\mu\nu}^{1} \). Following a method described in Ref. [6], and based on the fact that the exterior product of five arbitrary 4-vectors vanishes identically, we can obtain two independent relations by computing successively.
These are
\[
\begin{align*}
2 ( p_\mu p_\mu' - p_\mu' p_\mu ) \Phi &+ 2 M \zeta_{\mu\nu} ( (p\cdot\omega) + (p\cdot\omega') ) - ( M^2 + (p\cdot p') ) (\zeta_{\mu\nu} \Phi + \Phi \zeta_{\mu\nu} ) \\
- M ( p_\mu p_\mu' ) ( \gamma^\mu \Phi - \Phi \gamma^\mu ) + M ( p_\mu p_\mu' ) ( \gamma^\mu \Phi - \Phi \gamma^\mu ) \\
- 2 \zeta_{\mu\nu} ( p_\mu (p\cdot\omega) - p_\mu' (p\cdot\omega) ) + 2 \gamma^\nu ( p_\mu (p\cdot\omega) - p_\mu' (p\cdot\omega) ) &= 0
\end{align*}
\]
(2.18)

These relations are understood to be taken between Dirac spinors \( \tilde{\mathbf{u}} ( p_\mu \gamma^\nu ) \) and \( \mathbf{u} ( p_\mu \gamma^\nu ) \). The first relation is the same as one of the relations given by Gerstein [4] and de Calan and Stora [6]. The relation (2.19) differs from that given by de Calan and Stora by numerical coefficients. We have checked the second relation given by Gerstein, which he has chosen differently to our Eq. (2.19). We disagree with the two relations given by Conway [5] in some of the numerical coefficients. Equations (2.18) and (2.19) have been checked by means of a Bouchiat-Michel type formula [10]. These two independent relations (2.18) and (2.19) allow to eliminate two tensor covariants from the list given in (2.16) without introducing any kinematical singularities. We have chosen to eliminate \( t_{\mu\nu}^{16} \) and \( t_{\mu\nu}^{20} \). Then we are left with a set of tensor covariants which, under crossing (see section II.2), are either invariant or transform one into another. Some authors have not used relations (2.18) and (2.19) (or analogous relations) and their choice of amplitudes suffers from various diseases.
Next we express gauge invariance with respect to each photon, i.e.,

\[ q^\mu A_{\mu\nu}(T, q, q') = 0 \]  
\[ A_{\mu\nu}(T, q, q') q'^\nu = 0 \]

We obtain 16 equations, and 14 of them are independent\(^8\). Thus we are left with \( 34 - 2 - 14 = 18 \) independent invariant amplitudes for Compton scattering of off-shell photons on polarized nucleons.

The elimination of 14 amplitudes from Eqs. (2.20) and (2.21) must be done in such a way as to preserve the crossing properties of the remaining amplitudes. We shall only give an example of the method used. The following equations come from Eq. (2.20) (coefficient of \( q^\nu \)) and Eq. (2.21) (coefficient of \( q'^\mu \))

\[ a_1 + q^2 a_3 + (P q) a_5 + (q q') a_{10} - 2 (P q) a_{32} = 0 \]  
\[ a_1 + q^2 a_4 + (P q') a_6 + (q q') a_{10} + 2 (P q') a_{32} = 0 \]

It is possible to solve these equations without introducing kinematical singularities in \( \nu \) and \( t \). Moreover we can preserve crossing properties by a symmetrical elimination, i.e.,

\[ a_3 = -\frac{1}{q^2} \left( a_1 + (P q) a_5 + (q q') a_{10} - 2 (P q) a_{32} \right) \]  
\[ a_4 = -\frac{1}{q'^2} \left( a_1 + (P q') a_6 + (q q') a_{10} + 2 (P q') a_{32} \right) \]

The list (2.16) is a guide for this choice, since \( t^3_{\mu\nu} \) and \( t^4_{\mu\nu} \) transform into each other by crossing. The same method can be applied to the remaining 12 equations. We have checked that two equations (the coefficient of \( q'_\nu \) \( \Phi \) in Eq. (2.20) and that of \( q'_\mu \) \( \Phi \) in Eq. (2.21)) are identically satisfied by our solution. From this solution of Eqs. (2.20) and (2.21) we can write the general tensor \( A_{\mu\nu}(T, q, q') \) with 18 independent invariant amplitudes \( \alpha_i(q^\xi, q'^\zeta, \nu, t) \) free from kinematical singularities in \( \nu \) and \( t \), in the following way\(^6\)
The 4-vectors $E$, $E'$, $S$ and $S'$ are defined by

$$E = \gamma q, \quad E' = \gamma q', \quad S = \gamma \frac{q - q'}{q^2}, \quad S' = \gamma \frac{q' - q}{q'^2}$$

It is easy to verify that each tensor coefficient in the decomposition (2.26) is gauge invariant; let us remark that

$$E \cdot q = S' \cdot q = R' \cdot q = S \cdot q' = 0$$

In order to obtain the general Compton amplitude $C_{\mu\nu} (\vec{P}, q, q')$ under the form (2.8) we can use a Bouchiat-Michel type formula [10]. We shall choose it in a way best adapted to our purposes:

$$u (p, \Lambda) A_{\mu\nu} (\vec{P}, q, q') u (p, \Lambda') = \mathcal{F} \left\{ u (p, \Lambda) \otimes \tilde{u} (p', \Lambda') A_{\mu\nu} (\vec{P}, q, q') \right\}$$

Eq. (2.26) requires the evaluation of the traces

$$\mathcal{F} \left\{ u (p, \Lambda) \otimes \tilde{u} (p', \Lambda') \Sigma \right\}$$

where $\Sigma = \mathcal{M}, y_{\mu}, \sigma_{\mu\nu}, \sigma_{\mu\nu} y^{\tau} + y^{\tau} \sigma_{\mu\nu}$. The computation of these four traces can be done easily; much more tedious is the use of these traces by means of Eq. (2.29) in order to put $C_{\mu\nu} (\vec{P}, q, q')$ in the form (2.8). We find that there are five invariant amplitudes.
\[ \Omega_\xi (q^i, q'^i, \nu, t) \]

For the spin independent tensor \( \mathcal{C}_{\mu
u} (T, q, q') \)

and thirteen invariant amplitudes \( \Xi_j (q^i, q'^i, \nu, t) \)

for the spin dependent pseudotensor \( \mathcal{C}_{\mu
u\rho} (T, q, q') \)

(see Appendix, Eqs. (A.1) and (A.4)). The relation between invariant amplitudes \( \Omega_\xi \), \( \Xi_j \) and \( \alpha_t \) is given by (2.31)

\[
\begin{align*}
\Omega_1 (q^i, q'^i, \nu, t) &= \frac{i}{M} \left[ \left( 2 M^2 - \frac{t}{z} \right) a_i + 2 M \nu a_\| \right] \\
\Omega_2 (q^i, q'^i, \nu, t) &= i M \left[ \left( 2 M^2 - \frac{t}{z} \right) a_2 + 2 M \nu a_{12} + 2 M (a_{16} + a_{12}) \right] \\
\Omega_3 (q^i, q'^i, \nu, t) &= i M \left[ \left( 2 M^2 - \frac{t}{z} \right) a_3 + 2 M \nu a_{15} + 2 M a_{25} \\
&\quad + 2 \nu a_{27} - (q^2 - q'^2) a_{31} - 2 a_{33} \right] \\
\Omega_4 (q^i, q'^i, \nu, t) &= i M \left[ \left( 2 M^2 - \frac{t}{z} \right) a_4 - 2 M \nu a_6 + 2 M a_{26} \\
&\quad - 2 \nu a_{28} - (q^2 - q'^2) a_{32} - 2 a_{33} \right] \\
\Omega_5 (q^i, q'^i, \nu, t) &= i M \left[ \left( 2 M^2 - \frac{t}{z} \right) a_{10} - 2 \nu (a_{15} - a_{32}) \right]
\end{align*}
\]

and

\[
\Xi_1 (q^i, q'^i, \nu, t) = 2 i M^2 a_\|
\Xi_2 (q^i, q'^i, \nu, t) = 2 i M^4 a_{12}
\Xi_3 (q^i, q'^i, \nu, t) = 2 i M^4 a_{15}
\Xi_4 (q^i, q'^i, \nu, t) = 2 i M^2 a_{21}
\Xi_5 (q^i, q'^i, \nu, t) = 2 i M^2 a_{22}
\Xi_6 (q^i, q'^i, \nu, t) = 2 i M^2 a_{25}
\Xi_7 (q^i, q'^i, \nu, t) = 2 i M^2 a_{26}
\Xi_8 (q^i, q'^i, \nu, t) = -4 i M^3 a_{27}
\Xi_9 (q^i, q'^i, \nu, t) = -4 i M^3 a_{28}
\Xi_{10} (q^i, q'^i, \nu, t) = -4 i M^3 a_{31}
\Xi_{11} (q^i, q'^i, \nu, t) = -4 i M^3 a_{32}
\Xi_{12} (q^i, q'^i, \nu, t) = 4 i M a_{33}
\Xi_{13} (q^i, q'^i, \nu, t) = -8 i M^2 a_{34}
\]
The invariant amplitudes $\Omega_i$ and $\Xi_j$ are free from kinematical singularities in $\nu$ and $t$. They allow to write dispersion relations in the variable $y$ at fixed $t$. Powers of the nucleon mass $M$ have been introduced in Eqs. (2.31) and (2.32) in order to have invariant amplitudes $\Omega_i$ and $\Xi_j$ with same dimension.

The explicit result for the general Compton scattering of off-shell photons on polarized nucleons in terms of the invariant amplitudes $\Omega_i(q^1, q'^1, \nu, t)$ and $\Xi_j(q^2, q'^2, \nu, t)$ is given in the Appendix (see Eqs. (A.1) to (A.6)). Each tensor covariant in Eqs. (A.2) and (A.5) is gauge invariant.

1. Restrictions due to Time Reversal Invariance.

From time reversal invariance, the general Compton scattering satisfies the following condition

$$C^\mu_{\nu'}(\hat{P}, q, q') = (-1)\lambda_{\mu'} \left( C^-_{\lambda\nu'}(\hat{\tilde{p}}, \tilde{q}, \tilde{q}') \right)^*$$

(2.33)

With $p = (p_0, \vec{p})$, the $\sim$ operation on the 4-momenta is simply $\tilde{p} = (p_0, -\vec{p})$. The symbol $*$ means complex conjugate. Eq. (2.33) implies restrictions on the intrinsic tensors $C_{\mu\nu}(\hat{P}, q, q')$ and $C_{\mu'\nu'}(\hat{P}, q, q')$. We have

$$C^\mu_{\nu'}(\hat{P}, q, q') = - \left( C^{\mu'\nu}(\hat{\tilde{p}}, \tilde{q}, \tilde{q}') \right)^*$$

(2.34)

and

$$C^\mu_{\nu'}(\hat{P}, q, q') = - \left( C^{\mu'\nu}(\hat{\tilde{p}}, \tilde{q}, \tilde{q}') \right)^*$$

(2.35)

The restrictions due to Eqs. (2.34) and (2.35) impose that $\Omega_i(q^1, q'^1, \nu, t)$ and $\Xi_j(q^2, q'^2, \nu, t)$ are pure imaginary invariant amplitudes.

2. Crossing Symmetry Properties.

With our conventions the crossing symmetry properties read

$$C^\lambda_{\mu'}(\hat{P}, q, q') = C^{\lambda\mu'}(\hat{\tilde{p}}, -q', -q)$$

(2.36)
The corresponding restrictions on the invariant amplitudes are as follows (see Appendix, Eqs. (A.2), (A.5) and (A.7))

\[ \Omega_i (q^\mu, q^\nu, \nu, t) = \Omega_i^* (q^\mu, q^\nu, \nu, t) \quad i = 1, 2, 5 \]  
\[ \Omega_3 (q^\mu, q^\nu, \nu, t) = -\Omega_4 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_{i3} (q^\mu, q^\nu, \nu, t) = -\Sigma_{i3} (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_4 (q^\mu, q^\nu, \nu, t) = \Sigma_5 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_6 (q^\mu, q^\nu, \nu, t) = \Sigma_7 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_8 (q^\mu, q^\nu, \nu, t) = \Sigma_9 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_{10} (q^\mu, q^\nu, \nu, t) = \Sigma_{11} (q^\mu, q^\nu, \nu, t) \]  

3. Restrictions due to the Hermiticity of the Compton Amplitude.

From the definition of the general Compton amplitude in Eq. (1.1), and the hermiticity of the electromagnetic current, one can show that

\[ \mathcal{C}_{\mu\nu}^{aA'} (p, q, q') \]  
obeys the following hermiticity property

\[ \mathcal{C}_{\mu\nu}^{aA'} (p, q, q') = (\mathcal{C}_{\nu\mu}^{N^A} (p, q, q'))^* \]  

The corresponding restrictions on the invariant amplitudes are the following:

\[ \Omega_i (q^\mu, q^\nu, \nu, t) = \Omega_i^* (q^\mu, q^\nu, \nu, t) \quad i = 1, 2, 5 \]  
\[ \Omega_3 (q^\mu, q^\nu, \nu, t) = -\Omega_4 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_{i3} (q^\mu, q^\nu, \nu, t) = -\Sigma_{i3} (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_4 (q^\mu, q^\nu, \nu, t) = \Sigma_5 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_6 (q^\mu, q^\nu, \nu, t) = \Sigma_7 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_8 (q^\mu, q^\nu, \nu, t) = \Sigma_9 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_{10} (q^\mu, q^\nu, \nu, t) = \Sigma_{11} (q^\mu, q^\nu, \nu, t) \]  

\[ \mathcal{C}_{\mu\nu}^{aA'} (p, q, q') \]  
obeys the following hermiticity property

\[ \mathcal{C}_{\mu\nu}^{aA'} (p, q, q') = (\mathcal{C}_{\nu\mu}^{N^A} (p, q, q'))^* \]  

The corresponding restrictions on the invariant amplitudes are the following:

\[ \Omega_i (q^\mu, q^\nu, \nu, t) = \Omega_i^* (q^\mu, q^\nu, \nu, t) \quad i = 1, 2, 5 \]  
\[ \Omega_3 (q^\mu, q^\nu, \nu, t) = -\Omega_4 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_{i3} (q^\mu, q^\nu, \nu, t) = -\Sigma_{i3} (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_4 (q^\mu, q^\nu, \nu, t) = \Sigma_5 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_6 (q^\mu, q^\nu, \nu, t) = \Sigma_7 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_8 (q^\mu, q^\nu, \nu, t) = \Sigma_9 (q^\mu, q^\nu, \nu, t) \]  
\[ \Sigma_{10} (q^\mu, q^\nu, \nu, t) = \Sigma_{11} (q^\mu, q^\nu, \nu, t) \]
Using time reversal invariance and hermiticity, the invariant amplitudes $\Omega_i$, $\Xi_j$ have the following properties under the exchange $q^2 \rightarrow q'^2$

\begin{align*}
\Omega_i \left( q^2, q'^2, v, t \right) &= \Omega_i \left( q'^2, q^2, v, t \right) \quad i = 1, 2, 5 \\
\Omega_3 \left( q^2, q'^2, v, t \right) &= \Omega_4 \left( q'^2, q^2, v, t \right) 
\end{align*}

and

\begin{align*}
\Xi_i \left( q^2, q'^2, v, t \right) &= \Xi_i \left( q'^2, q^2, v, t \right) \quad i = 1, 2, 12, 13 \\
\Xi_3 \left( q^2, q'^2, v, t \right) &= -\Xi_3 \left( q'^2, q^2, v, t \right) \\
\Xi_4 \left( q^2, q'^2, v, t \right) &= -\Xi_4 \left( q'^2, q^2, v, t \right) \\
\Xi_6 \left( q^2, q'^2, v, t \right) &= -\Xi_6 \left( q'^2, q^2, v, t \right) \\
\Xi_8 \left( q^2, q'^2, v, t \right) &= -\Xi_8 \left( q'^2, q^2, v, t \right) \\
\Xi_{10} \left( q^2, q'^2, v, t \right) &= -\Xi_{10} \left( q'^2, q^2, v, t \right) 
\end{align*}

In the case of equal "photon masses" $q^2 = q'^2$, these relations play an important role to decrease the number of invariant amplitudes (see Section IV.2).
III - THE FORWARD SCATTERING LIMIT (t = 0).

We will be concerned in this section with the forward scattering limit (t = 0) of the absorptive Compton amplitude. It is well-known that this situation is closely related to deep inelastic scattering of electrons (or muons) on nucleons (see Fig. 5). Taking into account hermiticity and time reversal invariance, there are two invariant amplitudes for the spin independent tensor and two invariant amplitudes for the spin dependent tensor. This result is obtained setting

\[ q = q', \quad p = p' \quad (3.1) \]

in Eqs. (A.2) and (A.5). Then we have

\[ S = S' = 0 \]
\[ R = R' = 0 - \frac{(p-g)}{q^2} q \quad (3.2) \]
\[ \Delta = 0 \]
\[ E_{\alpha\xi\nu} = E_{\alpha\xi\nu} = E_{\alpha\xi\nu} - E_{\alpha\xi\nu} q^\mu q^\nu \]
\[ H_{\alpha\xi\nu} = H_{\alpha\xi\nu} = E_{\alpha\xi\nu} \]

and we are left with the following tensor covariants

\[ T_1^{\mu\nu} = \frac{q^\mu q^\nu}{q^2} \quad (3.3) \]
\[ T_2^{\mu\nu} = \frac{1}{M^2} R^\mu R^\nu \]

and

\[ S_8^{\mu\nu} = \frac{1}{M^2} p^\alpha q^\beta R^\mu E_{\alpha\xi\nu} \]
\[ S_9^{\mu\nu} = \frac{1}{M^2} p^\alpha q^\beta R^\nu E_{\alpha\xi\nu} \]
\[ S_{12}^{\mu\nu} = \frac{1}{M} \left( q^\chi q^\delta q^\nu E_{\alpha\xi\nu} - q^\gamma q^\mu E_{\alpha\xi\nu} + E_{\delta\xi\nu} \right) \]
\[ S_{13}^{\mu\nu} = \frac{1}{M^3} \left( -M^2 E_{\delta\xi\nu} q^\chi + p^\gamma E_{\alpha\xi\nu} p^\delta q^\nu \right) \]

The comparison between relations (3.3) and the usual definition\(^1\) of \( W_{\mu\nu} \) in terms of structure functions \( W_1(q^\gamma q^\nu) \) and \( W_2(q^\gamma q^\nu) \) yields the result\(^9\),\(^10\)
\[ \omega_1 (q^2, \nu) = - W_1 (q^2, \nu) \]
\[ \omega_2 (q^2, \nu) = W_2 (q^2, \nu) \]

(3.5)

The set of tensors in Eq. (3.4) seems to indicate that there are four invariant amplitudes for the spin dependent part whereas we expect three structure functions or two with time reversal invariance plus hermiticity. In fact, using the identity
\[ \epsilon_{\mu \nu \rho \sigma} V_\nu + \epsilon_{\nu \mu \rho \sigma} V_\nu + \epsilon_{\mu \nu \sigma \rho} V_\sigma + \epsilon_{\nu \sigma \rho \mu} V_\rho = 0 \]
where \( V \) is any 4-vector, it can be shown (with \( V = p \)) that
\[ \left\{ S^g_{\mu \nu f} - S^g_{\mu \nu g} - \frac{(p \cdot q)}{M^2} S^{12}_{\mu \nu f} - S^{13}_{\mu \nu g} \right\} \mathcal{N}_i^f (p) = 0 \]

(3.7)

Thus we have actually three independent amplitudes for the spin dependent part, which is the correct result. In order to avoid kinematical singularities in \( V \) and to preserve crossing symmetry between \( S^{13}_{\mu \nu g} \) and \( S^g_{\mu \nu f} \), we choose to eliminate \( S^{13}_{\mu \nu g} \). Then
\[ W_{\mu \nu f} (p, q) = S^g_{\mu \nu f} g_1 (q^2, \nu) + S^g_{\mu \nu g} g_2 (q^2, \nu) + S^{12}_{\mu \nu f} g_3 (q^2, \nu) \]

(3.8)

where, by definition
\[ g_1 (q^2, \nu) = \Gamma_8 (q^2, \nu) + \Gamma_{13} (q^2, \nu) \]
\[ g_2 (q^2, \nu) = \Gamma_9 (q^2, \nu) - \Gamma_{13} (q^2, \nu) \]
\[ g_3 (q^2, \nu) = \Gamma_{12} (q^2, \nu) - \frac{(p \cdot q)}{M^2} \Gamma_{13} (q^2, \nu) \]

(3.9)

The comparison between (3.8) and the usual definition \(^1\) of \( W_{\mu \nu f} \) in terms of structure functions \( G_1 (q^2, \nu) \), \( G_2 (q^2, \nu) \) and \( G_3 (q^2, \nu) \) yields the result
\[ g_1 (q^2, \nu) = \frac{M^2}{(p \cdot q)} \left( - G_1 (q^2, \nu) - i G_3 (q^2, \nu) \right) \]
\[ g_2 (q^2, \nu) = \frac{M^2}{(p \cdot q)} \left( G_1 (q^2, \nu) - i G_3 (q^2, \nu) \right) \]
\[ g_3 (q^2, \nu) = G_1 (q^2, \nu) + G_2 (q^2, \nu) \]

(3.10)
Using now time reversal invariance plus hermiticity for the invariant amplitudes $\gamma_j(q^2,\nu)$ (see Eq. (2.43)) we have

$$g_1(q^2,\nu) = -g_2(q^2,\nu)$$

i.e., in terms of structure functions $G_1, G_3(q^2,\nu) = 0$, as is well-known. Finally, we choose $g_2(q^2,\nu)$ and $g_3(q^2,\nu)$ as basic invariant amplitudes for the spin dependent part of $W^{\lambda\lambda'}(p,q)$.

$$W_{\mu
\nu}(p,q) = \left( S^g_{\mu
\nu} - S^g_{\mu
\nu} \right) g_2(q^2,\nu) + S^h_{\mu
\nu} g_3(q^2,\nu) \quad (3.11)$$

The relation between $g_3,g_2$ and $G_1,G_2$ is

$$g_2(q^2,\nu) = \frac{M}{(p,q)} G_1(q^2,\nu) \quad (3.12)$$

$$g_3(q^2,\nu) = G_1(q^2,\nu) + G_2(q^2,\nu)$$

In the Appendix we write explicitly the tensor covariants corresponding to the situation we have just described. Of course each tensor covariant in Eqs.(A.11) and (A.13) is gauge invariant.
IV - STUDY OF PARTICULAR NON-FORWARD SCATTERING LIMITS.

We want to consider in this section some particular non-forward scattering limits, for the general Compton amplitude $C_{\mu\nu}^{\lambda\lambda'}(P, q, q')$ defined in Eq. (1.1).

1. One Real Photon, One Off-Shell Photon.

Two particular cases directly related to physical processes can be considered with one real photon and one off-shell photon.

A) Real incoming photon ($q^2 = 0$), off-shell outgoing photon.

A physical process related to this situation is the photoproduction of electron (or muon) pairs on nucleons. To lowest order in the fine structure constant $\alpha = \frac{1}{137}$, the diagrams which contribute to this process are given in Fig. 4. More precisely we have in Fig. 4a one of the Bethe Heitler diagrams and in Fig. 4b the so-called Compton diagram. For the Compton amplitude which appears in this diagram, we have with $q^2 = 0$ three invariant amplitudes for the spin independent tensor $C_{\mu\nu}^{\lambda\lambda'}(P, q, q')$ and nine invariant amplitudes for the spin tensor $C_{\mu\nu\lambda}^{\lambda\lambda'}(P, q, q')$. These results, and the corresponding tensor covariants are obtained from Eqs. (A.1) and (A.4) setting the coefficient of $\frac{1}{q^2}$ equal to zero. For example, one obtains

$$\Omega_2(q, q', \nu, t) + (q' q') \Omega_4(q, q', \nu, t) = 0 \quad (4.1)$$

We solve this equation setting

$$\Omega_2(q, q', \nu, t) = \frac{M^2}{M^2} \Omega_2(q, q', \nu, t) \quad (4.2)$$

For the spin independent tensor $C_{\mu\nu}^{\lambda\lambda'}(P, q, q')$, we choose the $\Omega_3(q, q', \nu, t)$, $\Omega_5(q, q', \nu, t)$, and $\Omega_6(q, q', \nu, t)$ as basic invariant amplitudes. The corresponding tensor covariants are written in the Appendix (see Eqs. (A.14) and (A.15)), it can be verified that they are gauge invariant.
In the same way as in Eq. (4.2), we can introduce a new invariant amplitude \( \Sigma_{\mu\nu} (o, q^2, \nu, t) \) for the spin dependent tensor \( C_{\mu\nu} (P, q, q^1) \) for the spin dependent tensor \( C_{\mu\nu} (P, q, q^1) \). The nine tensor covariants for \( C_{\mu\nu} (P, q, q^1) \) are also written in the Appendix (see Eqs. (A.16) and (A.17)); each of them is gauge invariant.

B) Off-shell incoming photon, real outgoing photon \( (q'^2 = 0) \).

A physical process related to this situation is the electron bremsstrahlung in the reaction

\[
\text{electron} + \text{nucleon} \rightarrow \text{electron} + \text{nucleon} + \text{photon}.
\]

To lowest order in \( \alpha \), the diagrams which contribute to this process are given in Fig. 3. We have in Fig. 3a one of the Bethe-Heitler diagrams and in Fig. 3b the Compton diagram. For the Compton amplitude which appears in this case, we have with \( q'^2 = 0 \) three invariant amplitudes for the spin dependent tensor \( C_{\mu\nu} (P, q, q^1) \) and nine invariant amplitudes for the spin tensor \( C_{\mu\nu} (P, q, q^1) \). These results, and the corresponding tensor covariants are obtained from Eqs. (A.1) and (A.4) setting the coefficient of \( \frac{1}{q'^2} \) equal to zero. As previously, we introduce new invariant amplitudes \( \Lambda_{\text{out}} (q^o, \nu, t) \) and \( \Sigma_{\text{out}} (q^o, \nu, t) \).

The tensors \( C_{\mu\nu} (P, q, q^1) \) and \( C_{\mu\nu} (P, q, q^1) \) are explicitly written in the Appendix, Eqs. (A.18) to (A.21).

Cases A) and B) that we have just discussed are related to each other by crossing symmetry properties. In fact under crossing,

\[
\Lambda_{\text{out}} (q^o, \nu, t) = \Lambda_{\text{out}} (q^o, -\nu, t) \quad \text{and} \quad \Sigma_{\text{out}} (q^o, \nu, t) = -\Sigma_{\text{out}} (q^o, -\nu, t)
\]

and furthermore, (see Eqs. (A.15), (A.17), (A.19) and (A.21))

\[
\begin{align*}
J_{\mu\nu}^3 &\rightarrow -H_{\mu\nu}^4 \\
J_{\mu\nu}^5 &\rightarrow -H_{\mu\nu}^4 \\
J_{\mu\nu}^{\text{inc}} &\rightarrow -H_{\mu\nu}^{\text{out}}
\end{align*}
\]

(4.4)
2. Off-Shell Photons with Equal Masses \( q^2 = q'^2 \).

In this case, using time reversal invariance plus hermiticity (see Eqs. (2.42) and (2.43)), we see that there are four invariant amplitudes

\[ \Omega_i \left( q^i, q^i, \nu, t \right) \]

for the spin independent tensor \( T_{\mu
u} \left( T, q, q' \right) \) and eight invariant amplitudes \( \Xi^i_j \left( q^i, q^i, \nu, t \right) \) for the spin tensor \( T_{\mu
u} \left( T, q, q' \right) \). The corresponding tensor covariants are written in the Appendix, Eqs. (A.22) to (A.25).

3. Real Compton Scattering \( q^2 = q'^2 = 0 \).

As is well-known [11], real Compton scattering is described by six invariant amplitudes, two for the spin independent part, four for the spin dependent part. These amplitudes can be obtained from the case \( q^2 = q'^2 \), setting the coefficient of \( \frac{1}{q^2} \) equal to zero in Eqs. (A.22) and (A.24). As previously, we introduce new invariant amplitudes \( \Omega_{\mu \nu} \left( \nu, t \right) \) and \( \Xi^i_j \left( \nu, t \right) \). The tensor covariants for real Compton scattering are written in the Appendix, Eqs. (A.26) to (A.29). Each of them is gauge invariant.
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We recall the decomposition of the general Compton scattering of off-shell photons on polarized nucleons into spin independent and dependent parts

\[ \frac{i}{2M} C_{\mu\nu}^{M'}(\mathcal{P}, q, q') = C_{\mu\nu}(\mathcal{P}, q, q') \delta_{M'M} + i \ C_{\mu\nu}^{M'F}(\mathcal{P}, q, q') \eta^{f}(p') \ \mathcal{M}_{M'} \]

We write for the spin independent tensor

\[ C_{\mu\nu}(\mathcal{P}, q, q') = \sum_{i=1}^{S} I_{\mu\nu}^{i}(\mathcal{P}, q, q') \Omega_{i} \left( q^2, q'^2, \nu, \ell \right) \]  
(A.1)

where

\[ I_{\mu\nu}^{1} = q_{\mu}q_{\nu} - \frac{q_{\mu}q_{\nu}}{q^{2}} - \frac{q'_{\mu}q'_{\nu}}{q'^{2}} + \frac{(q \cdot q')_{\mu} (q \cdot q')_{\nu}}{q^{2}q'^{2}} \]
\[ I_{\mu\nu}^{2} = \frac{i}{M^2} R_{\mu} R'_{\nu} \]
\[ I_{\mu\nu}^{3} = \frac{i}{M^2} R_{\mu} S_{\nu} \]
\[ I_{\mu\nu}^{4} = \frac{i}{M^2} S'_{\mu} R'_{\nu} \]
\[ I_{\mu\nu}^{5} = \frac{i}{M^2} S'_{\mu} S_{\nu} \]
(A.2)

The 4-vectors \( R, R', S \) and \( S' \) are defined by

\[ R = \mathcal{P} - \left( \frac{\mathcal{P} \cdot q}{q^2} \right) q, \quad S = q - \left( \frac{(q \cdot q')_{\mu} (q \cdot q')_{\nu}}{q^2q'^2} \right) q' \]
\[ R' = \mathcal{P} - \left( \frac{\mathcal{P} \cdot q'}{q'^2} \right) q', \quad S' = q' - \left( \frac{(q \cdot q')_{\nu}}{q^2} \right) q \]
(A.3)

and we recall that

\[ P = \frac{1}{2}(p + p') \]
\[ Q = \frac{1}{2}(q + q') \]
\[ \Delta = \frac{1}{2}(q - q') \]

We write for the spin dependent tensor

\[ C_{\mu\nu}^{f}(\mathcal{P}, q, q') = \sum_{j=1}^{13} S_{\mu\nu}^{j}(\mathcal{P}, q, q') \Xi_{j}(q^2, q'^2, \nu, \ell) \]  
(A.4)
The absorptive Compton amplitude is decomposed into spin independent
and spin dependent parts in the following way

$$\frac{1}{2\pi M} W^\Delta_{\mu
u}(\sqrt{q},q') = W^I_{\mu\nu}(\sqrt{q},q') \frac{d}{\Delta} + i W^S_{\mu\nu}(\sqrt{q},q') \kappa^I_{\mu} (p) \gamma^I_{q'}$$

We have

$$W^I_{\mu\nu}(\sqrt{q},q') = \sum_{i=1}^{5} I^i_{\mu\nu} (\sqrt{q},q') \omega_i (q,q',\nu,\tau)$$

$$W^S_{\mu\nu}(\sqrt{q},q') = \sum_{j=1}^{13} S^j_{\mu\nu} (\sqrt{q},q') \gamma_j (q,q',\nu,\tau)$$

where the tensor covariants $I^i_{\mu\nu}$ and $S^j_{\mu\nu}$ are defined in Eqs. (A.2) and

The forward scattering limit of the absorptive Compton amplitude

$$(q = q')$$

is described by two invariant amplitudes $\omega_i (q',\nu)$ and

$\omega_2 (q',\nu)$ for the spin independent part and two invariant amplitudes

$g_2(q',\nu)$ and $g_3(q',\nu)$ for the spin dependent part

$$W^\mu (p,q) = \sum_{i=1}^{5} I^i_{\mu\nu} (q = q') \omega_i (q',\nu)$$

where

$$I^1_{\mu\nu} (q = q') = g^\mu_{\nu} - \frac{p^\mu - q^\mu}{q^2} q^\nu$$

$$I^2_{\mu\nu} (q = q') = \frac{1}{M^2} (p^\mu - \frac{(p-q)^2}{q^2} q^\nu)$$

$$W^S_{\mu\nu} (p,q) = (S^g_{\mu\nu} (q = q') - S^b_{\mu\nu} (q = q')) g_2 (q',\nu)$$

$$+ S^s_{\mu\nu} (q = q') g_3 (q',\nu)$$

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where

\[ S^q_{\mu
u} (q = q') - S^q_{\mu
u} (q = q') = \]

\[ = \frac{1}{M^2} \left[ P^\alpha q^\beta \left( \left( p_\mu - \frac{q_\mu q_\nu}{q^2} q_\nu \right) \right) \xi_{\alpha\beta\mu\nu} - \left( p_\mu - \frac{q_\mu q_\nu}{q^2} q_\nu \right) \right] \xi_{\alpha\beta\mu\nu} \]  
(A.13)

\[ L^q_{\mu
u} (q = q') = \frac{1}{M} \left[ P^\alpha \left( \xi_{\alpha\beta\mu\nu} - \frac{q^\alpha}{q^2} \left( q_\mu \xi_{\alpha\beta\mu\nu} - q_\nu \xi_{\alpha\beta\mu\nu} \right) \right) \right] \]

Eqs. (A.12) and (A.13) have been obtained by taking into account hermiticity plus time reversal invariance of the Compton amplitude.

The non-forward scattering limit \( q^2 = 0 \) of the general Compton amplitude is described by three invariant amplitudes for the spin independent part and nine invariant amplitudes for the spin dependent part

\[ C_{\mu\nu} (T, q; q') \bigg|_{q^2 = 0} = \sum_{i=1,5,6} J^i_{\mu\nu} (T, q; q') \Omega_i (O, q'; \nu, t) \]  
(A.14)

where

\[ J^3_{\mu\nu} = \frac{1}{M^2} \left( - g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) (T; \hat{q}) + \hat{P}_\mu S_\nu \]

\[ J^5_{\mu\nu} = \frac{1}{M^2} \left( - g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) (q; q') + q_\mu S_\nu \]  
(A.15)

\[ J^7_{\mu\nu} = \frac{1}{M^4} \left( \hat{P}_\mu (q; q') - q_\mu (T; \hat{q}) \right) R_\nu \]

\[ C_{\mu\nu} (T, q; q') \bigg|_{q^2 = 0} = \sum_{j=1}^3 T^3_{\mu\nu} (T, q; q') \Omega_j (O, q'; \nu, t) \]  
(A.16)

where

\[ T^2_{\mu\nu} = \frac{1}{M^5} \left[ \xi_{\alpha\beta\mu\nu} \xi^\alpha \xi^\beta R_\nu \xi^\alpha (\hat{P}_\mu - \frac{q_\nu q_\nu}{q^2} \hat{P}_\nu) + \hat{P}_\mu S_\nu - q_\mu R_\nu \right] \]

\[ T^3_{\mu\nu} = \frac{1}{M^5} \left[ \xi_{\alpha\beta\mu\nu} \xi^\alpha \xi^\beta \left( - g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) (T; \hat{q}) + \hat{P}_\mu S_\nu - q_\mu R_\nu \right] \]

\[ T^4_{\mu\nu} = \frac{1}{M^5} \left[ - \xi_{\alpha\beta\mu\nu} \xi^\alpha \xi^\beta \left( - g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \hat{P}_\mu \xi_{\alpha\beta\mu\nu} \xi^\alpha \xi^\beta \left( \hat{P}_\nu q_\nu \right) - \hat{P}_\nu \xi_{\alpha\beta\mu\nu} \xi^\alpha \xi^\beta \left( \hat{P}_\nu q_\nu \right) \right] \]
The non-forward scattering limit $q'^2 = 0$ of the general Compton amplitude is described by three invariant amplitudes for the spin independent part and nine invariant amplitudes for the spin dependent part

$$\mathcal{C}_{\mu \nu} (P, q, q') \bigg|_{q'^2 = 0} = \sum_{i=5, 6, \text{out}} \Gamma_{\mu \nu}^i (P, q, q') \Omega_i (q^2, \sigma, \nu, t) \quad (A.18)$$

where

$$\Gamma_{\mu \nu}^4 = \frac{1}{M^4} \left( -g_{\mu \nu} + \frac{P_{\mu} Q_{\nu}}{q^2} \right) (P, 0) + S^\nu_{\mu} (P, \nu)$$

$$\Gamma_{\mu \nu}^5 = \frac{1}{M^4} \left( -g_{\mu \nu} + \frac{P_{\mu} Q_{\nu}}{q^2} \right) (q, q^1) + S^\nu_{\mu} (q, \nu) \quad (A.19)$$

$$\Gamma_{\mu \nu}^{\text{out}} = \frac{1}{M^4} R_{\mu} (P, q^1 - q, (P, 0))$$

$$\mathcal{C}_{\mu \nu} (P, q, q') \bigg|_{q'^2 = 0} = \sum_j U_{\mu \nu}^j (P, q, q') \Xi_j (q^2, \sigma, \nu, t) \quad (A.20)$$

where

$$j = 2, 3, 4, 5, 6, 7, 8, 11, \text{ out}$$
The non-forward scattering limit $g = g'$ of the general Compton amplitude is described by four invariant amplitudes for the spin independent part and eight invariant amplitudes for the spin dependent part.

\[
\begin{align*}
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^2} \left[ E_{x_{\mu} f} P_x^B \left( R_{\mu} P_{\nu} - P_x^B \right) \left( \mathcal{P}_x E_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) - \mathcal{E}_{x_{\mu} f} (\mathcal{P}_x (\mathcal{P}_x) - \Omega_x (M^2 - \frac{t}{4})) - \mathcal{P}_f \mathcal{E}_{x_{\nu}} (\mathcal{P}_x^B + \Delta^x) \Omega_x (\mathcal{P}_x) \right] \\
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^3} \left[ E_{x_{\mu} f} P_x^B \left( P_x^B \right) \left( \mathcal{P}_x \mathcal{E}_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) - \mathcal{E}_{x_{\mu} f} (\mathcal{P}_x (\mathcal{P}_x) - \Omega_x (M^2 - \frac{t}{4})) - \mathcal{P}_f \mathcal{E}_{x_{\nu}} (\mathcal{P}_x^B + \Delta^x) \Omega_x (\mathcal{P}_x) \right] \\
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^3} \left[ 2 E_{x_{\mu} f} P_x^B \left( \mathcal{P}_x \mathcal{E}_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) - \mathcal{P}_f \mathcal{E}_{x_{\nu}} (\mathcal{P}_x^B + \Delta^x) \Omega_x (\mathcal{P}_x) \right] \\
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^3} \left[ 2 E_{x_{\mu} f} P_x^B \left( \mathcal{P}_x \mathcal{E}_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) - \mathcal{P}_f \mathcal{E}_{x_{\nu}} (\mathcal{P}_x^B + \Delta^x) \Omega_x (\mathcal{P}_x) \right] \\
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^3} \left[ E_{x_{\mu} f} P_x^B \left( \mathcal{P}_x \mathcal{E}_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) + \mathcal{P}_x \Omega_x S_{\mu} \mathcal{E}_{x_{\nu}} \right] \\
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^3} \left[ E_{x_{\mu} f} P_x^B \left( \mathcal{P}_x \mathcal{E}_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) - \mathcal{E}_{x_{\mu} f} (\mathcal{P}_x^B (\mathcal{P}_x) - \Omega_x (M^2 - \frac{t}{4})) \right] \\
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^3} \left[ E_{x_{\mu} f} P_x^B \left( \mathcal{P}_x \mathcal{E}_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) + \mathcal{P}_x \Omega_x S_{\mu} \mathcal{E}_{x_{\nu}} \right] \\
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^3} \left[ E_{x_{\mu} f} P_x^B \left( \mathcal{P}_x \mathcal{E}_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) + \mathcal{P}_x \Omega_x S_{\mu} \mathcal{E}_{x_{\nu}} \right] \\
\mathcal{U}_{\mu}^{\nu} &= \frac{1}{M^3} \left[ E_{x_{\mu} f} P_x^B \left( \mathcal{P}_x \mathcal{E}_{x_{\nu}} - \frac{q^x}{q^2} \mathcal{P}_x \right) \left( \mathcal{P}_x \right) + \mathcal{P}_x \Omega_x S_{\mu} \mathcal{E}_{x_{\nu}} \right] \\
\end{align*}
\]

The non-forward scattering limit $q^2 = q'^2$ of the general Compton amplitude is described by four invariant amplitudes for the spin independent part and eight invariant amplitudes for the spin dependent part.

\[
\mathcal{C}_{\mu \nu} (\mathcal{P}, q, q') \bigg|_{q^2 = q'^2} = \sum_i L_{\mu \nu}^i (\mathcal{P}, q, q') \Omega_i (q^2, q'^2, \nu, t) \tag{A.22}
\]

where

\[
L_{\mu \nu}^i (\mathcal{P}, q, q') \bigg|_{q^2 = q'^2} = \sum_l \mathcal{C}_{\mu \nu} (\mathcal{P}, q, q') \bigg|_{q^2 = q'^2} \Omega_i (q^2, q'^2, \nu, t)
\]

i = 1, 2, 3, 5
\[
L^1_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \frac{q_\mu q'_\nu}{q^2} + \frac{(q \cdot q')}{q^2} q_\mu q'_\nu
\]
\[
L^2_{\mu\nu} = \frac{1}{M^2} (P_\nu - \frac{(P \cdot q)}{q^2} q_\nu)
\]
\[
L^3_{\mu\nu} = \frac{1}{M^2} \left[ R_\mu \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) + S_\mu \left( P_\nu - \frac{(P \cdot q)}{q^2} q_\nu \right) \right]
\]
\[
L^4_{\mu\nu} = \frac{1}{M^2} S_\mu \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right)
\]
\[\epsilon_{\mu\nu j} (P, q, q') \bigg|_{q^2 = q'^2} = \sum_j V^j_{\mu\nu j} (P, q, q') \Xi_j (q^2, q^2, \nu, \ell) \quad j = 1, 2, 4, 6, 8, 10, 12, 13 \quad (A.24)\]

where (A.25)
\[
v^1_{\mu\nu j} = \frac{1}{M^3} \xi_{\alpha\beta\gamma} \hat{P}_\sigma \Delta_\sigma Q^\alpha L^4_{\mu\nu}
\]
\[
v^2_{\mu\nu j} = \frac{1}{M^3} \xi_{\alpha\beta\gamma} \hat{P}_\sigma \Delta_\sigma Q^\alpha L^4_{\mu\nu}
\]
\[
v^3_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ R_\mu \left( \xi_{\alpha\beta\gamma} \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right) + \left( P_\nu - \frac{(P \cdot q)}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} \right]
\]
\[
v^4_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^5_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^6_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^7_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^8_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^9_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^{10}_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^{11}_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^{12}_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]
\[
v^{13}_{\mu\nu j} = \frac{1}{M^3} \hat{P}_\sigma \Delta_\sigma \left[ \left( q_\nu - \frac{(q \cdot q')}{q^2} q_\nu \right) \xi_{\alpha\beta\gamma} Q^\alpha \frac{q_\nu}{q^2} \right]
\]

The non-forward scattering limit \( q^2 = q'^2 = 0 \) (real Compton scattering) of the general Compton amplitude is described by two invariant amplitudes for the spin independent part and four invariant amplitudes for the spin dependent part

\[C_{\mu\nu} (P, q, q') \bigg|_{q^2 = q'^2 = 0} = \sum_{\ell = 5, R} M_{\mu\nu}^{\ell} \{ P, q, q' \} \omega_{\ell} (\nu, \ell) \quad (A.26)\]

where
\[M_{\mu\nu}^{5} = \frac{1}{M^2} \left[ - Q_{\mu\nu} (q \cdot q') + q'_{\mu} q_{\nu} \right] \quad (A.27)\]
\[M_{\mu\nu}^{R} = \frac{1}{M^4} \left[ \hat{Q}_{\mu\nu} (P \cdot 0)^2 + \hat{Q}_{\mu\nu} (P \cdot q') - (P_\mu q_\nu + P_\nu q'_\mu) (P \cdot 0) \right] \]
\[ X^j_{\mu \nu} \mid _{q^2 = q'^2 = 0} = \sum \theta^j \chi^j_{\mu \nu} (\theta, q, q') \Xi^j_{\mu \nu} (\nu, \ell) \] 

(A.28)

where

\[ X^2_{\mu \nu} = \frac{1}{2 M^3} \left[ 2 \varepsilon_{\kappa \lambda \rho \sigma} \varepsilon^\kappa \varepsilon^\lambda \varepsilon^\rho \varepsilon^\sigma \varepsilon_{\mu \nu} + \varepsilon_{\mu \nu \sigma} \varepsilon^\sigma \varepsilon_{\sigma} (\varepsilon_{\tau \rho} - \frac{1}{4} \delta_{\tau \rho}) \right] \]

(A.29)

\[ X^6_{\mu \nu} = \frac{1}{M^3} \left[ -2 \varepsilon_{\kappa \lambda \rho \sigma} \varepsilon^\kappa \varepsilon^\lambda \varepsilon^\rho \varepsilon^\sigma \varepsilon_{\mu \nu} + \varepsilon_{\mu \nu \sigma} \varepsilon^\sigma \varepsilon_{\sigma} (\varepsilon_{\tau \rho} - \frac{1}{4} \delta_{\tau \rho}) \right] \]

(A.29)

\[ X^8_{\mu \nu} = \frac{1}{M^3} \left[ \varepsilon_{\kappa \lambda \rho \sigma} \varepsilon^\kappa \varepsilon^\lambda \varepsilon^\rho \varepsilon^\sigma \varepsilon_{\mu \nu} - \varepsilon_{\mu \nu \sigma} \varepsilon^\sigma \varepsilon_{\sigma} \varepsilon_{\tau \rho} (\varepsilon_{\tau \rho} - \frac{1}{4} \delta_{\tau \rho}) \right] \]

(A.29)

\[ X^8_{\mu \nu} = \frac{1}{2 M^3} \left[ 2 \varepsilon_{\kappa \lambda \rho \sigma} \varepsilon^\kappa \varepsilon^\lambda \varepsilon^\rho \varepsilon^\sigma \varepsilon_{\mu \nu} - (q \cdot q') \varepsilon_{\mu \nu \sigma} \varepsilon^\sigma \varepsilon_{\sigma} \varepsilon_{\tau \rho} (\varepsilon_{\tau \rho} - \frac{1}{4} \delta_{\tau \rho}) + \varepsilon_{\mu \nu \sigma} \varepsilon^\sigma \varepsilon_{\sigma} \varepsilon_{\tau \rho} (\varepsilon_{\tau \rho} - \frac{1}{4} \delta_{\tau \rho}) \right] \]
1) See e.g. ref. [3].

2) The non-zero components of the metric tensor are: \( g_{00} = 1 \), \( g_{11} = g_{22} = g_{33} = -1 \). We take \( \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \), \( \epsilon_{0123} = 1 \).

3) \( \Lambda_{p \rightarrow p'} = 1 - \frac{(p+p') \otimes (p+p')}{M^2 + (p-p')^2} + 2 \frac{p' \otimes p}{M^2} \), that is, in components \( (\Lambda_{p \rightarrow p'})_{ij} = \delta_{ij} - \frac{(p+p')_i (p+p')_j}{M^2 + (p-p')^2} + 2 \frac{p'_{i} p'_j}{M^2} \).

4) Eq. (2.6) is often used as a definition of \( C_{\mu \nu} (\tau, q, q') \). However we shall prefer Eq. (1.1) which exhibits clearly the crossing symmetry properties of the Compton amplitude.

5) As usual, \( \gamma' \) is defined by \( \gamma' = \gamma_{\mu} \gamma'^{\mu} \). We have also \( \gamma'_{\mu \nu} = \frac{1}{2} \left[ \gamma'_{\mu}, \gamma'_{\nu} \right] \).

6) Throughout this paper, any Dirac matrix is understood to be taken between Dirac spinors \( \bar{u}(\gamma', \lambda') \) and \( u(\gamma, \lambda) \).

7) For example, Gourdin [7] eliminates \( t^{33}_{\mu \nu} \) and \( t^{34}_{\mu \nu} \) and thus obtains invariant amplitudes with kinematical singularities in \( \nu \), as can be seen from Eqs. (2.18) and (2.19). Meyer [8] eliminates \( (\gamma_{\mu} (q_1 + q_2') - \gamma_{\nu} (q_1' + q_2)) \gamma' \) and \( \gamma_{\mu} \gamma'_{\nu} - \gamma_{\nu} \gamma'_{\mu} \); but the first does not appear at all in Eqs. (2.18) and (2.19) and so is actually independent while the second one appears in Eq. (2.18) with coefficient \( q^2 - q_1^2 \) which is zero when \( q^2 = q_1^2 \). Amati, Jengo and Remiddi [9] use all 34 tensor covariants and therefore have a redundant set.
Eight of the 16 equations come from Eq. (2.20) setting equal to zero the coefficients of the following quantities

\[ P_{\nu}, q_{\nu}, q'_{\nu}, P', \Phi, q, q', q'_{\nu}, f_{\nu}, r_{\nu}, \Phi - \Phi_{\nu} \]

The other eight equations come from Eq. (2.21) in the same way. Only seven equations in each of those sets are independent.

\[ \omega_i \left( q^i, q^{i'}, \nu, t \right) \quad \text{and} \quad \sum_j \left( q^i, q^{i'}, \nu, t \right) \] are real invariant amplitudes defined as (see Eq. (2.13))

\[
\omega_i \left( q^i, q^{i'}, \nu, t \right) = \frac{1}{2i} \text{Im} \Omega_i \left( q^i, q^{i'}, \nu, t \right) \quad i = 1, 2, \ldots, 5 \\
\sum_j \left( q^i, q^{i'}, \nu, t \right) = \frac{1}{2i} \text{Im} \sum_j \left( q^i, q^{i'}, \nu, t \right) \quad j = 1, 2, \ldots, 13
\]

Throughout this section, we write \( \omega_i \left( q^i, \nu \right) \) , \( \sum_j \left( q^i, \nu \right) \) as a short-hand notation for \( \omega_i \left( q^i, q^i, \nu, o \right) \) , \( \sum_j \left( q^i, q^i, \nu, o \right) \).

I would like to thank Prof. E. Stora for a discussion on this point.

We write \( \Omega \left( \nu, t \right) \) , \( \Xi \left( \nu, t \right) \) as a short-hand notation for \( \Omega \left( o, o, \nu, t \right) \) , \( \Xi \left( o, o, \nu, t \right) \).

12) We write \( \Omega \left( \nu, t \right) \) , \( \Xi \left( \nu, t \right) \) as a short-hand notation for \( \Omega \left( o, o, \nu, t \right) \) , \( \Xi \left( o, o, \nu, t \right) \).
FIGURE CAPTIONS

Figure 1 : A diagrammatic representation for Compton scattering of off-shell photons on polarized nucleons.

Figure 2 : Two-photon exchange contributions to elastic electron-proton scattering.

Figure 3 : a) One of the electron bremsstrahlung diagrams in the process: electron + nucleon → electron + nucleon + photon.
            b) Compton diagram in the same process.

Figure 4 : Diagrams for photoproduction of electron pairs on nucleons. 
           a) One of the Bethe-Heitler diagrams.
           b) Compton diagram.

Figure 5 : Inelastic electron-proton scattering to lowest order in the electromagnetic coupling.