Fully Symmetric Extension of the Dual Pion Model

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Abstract: The fully symmetric dual model (Shapiro-Virasoro model) is extended by introducing anticommuting operators similar to those of the dual pion model. In this new fully symmetric model, there is no tachyon on the leading trajectory, which begins with a massless spin two particle. Such a model might prove useful for Lagrangian theories of spin two gauge fields like gravitation.

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1. Introduction

A completely symmetric dual amplitude was proposed [1] some time ago, and was shown [2,5] to be factorizable and to have enough gauge properties to cancel all the ghosts. In that symmetric dual amplitude, the leading trajectory has intercept $\alpha_0 = 2$, and its ground-state is a spin zero tachyon. In this note, we construct another fully symmetric amplitude by introducing anticommuting operators similar to those of the dual pion model [3]. We show that this new amplitude can also be reformulated in another, more convenient Fock space [4], where there is no tachyon on the leading trajectory, which still has intercept two. We then argue that the zero slope limit of this model could give new insights on the Lagrangian field theory of massless spin two particles like gravitation.

2. Dual Pion Model

For reference, let us recall some generalities about the dual pion model. Commuting and anticommuting harmonic oscillator operators are defined, such that:

\[ [a^\mu_m, a^\nu_n] = -m \ g^{\mu\nu} \delta_{m+n,0} \]  
\[ \alpha^\mu_0 = \sqrt{2} \ p^\mu \]  
\[ \{b^\mu_m, b^\nu_n\} = -g^{\mu\nu} \delta_{m+n,0} \]  
\[ [b^\mu_m, a^\nu_n] = 0 . \]

For the $a$'s, the indices take all positive and negative values. For the $b$'s, all half-integer values. Our metric is $g^{00} = -g^{ii} = 1$. 

72/19
Virasoro gauge operators are defined as:

\[ L_n = -\frac{1}{2} \alpha_o \alpha_o - \sum_{m=1}^{\infty} \alpha_{-m} \alpha_m - \sum_{m=\frac{1}{2}}^{\infty} m b_{-m} b_m \]  
(2.5)

and for \( n > 0 \)

\[ L_n = -\sum_{m=0}^{n-1} \alpha_{-m} \alpha_{m+n} - \frac{1}{2} \sum_{m=1}^{\infty} \alpha_{-m} \alpha_{n-m} - \frac{1}{2} \sum_{m=\frac{1}{2}}^{\infty} (n + 2m) b_{-m} b_{n+m} \]  
(2.6)

They obey to the algebra

\[ [L_m, L_n] = (m - n) L_{m+n} \]  
(2.7)

except when \( m + n = 0 \), in which case an additional constant is to be added.

The vertex is given by

\[ V(k, z) = k H(z) V_0(k, z) \]  
(2.8)

where

\[ H^\alpha(z) = \sum_{m=-\infty}^{\infty} b_{\frac{1}{m}} z^{-m} \]  
(2.9)

and

\[ \sqrt{2} k \sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n} z^n = -\sqrt{2} k \sum_{n=1}^{\infty} \frac{\alpha_{n}}{n} z^{-n} \]

\[ V_0(k, z) = e^{ik \cdot x} e^{\sum_{n=1}^{\infty} \frac{\alpha_{n}}{n} z^{-2k \cdot p}} \]  
(2.10)

In this model, \( |z| = 1 \). The vertex transforms by the gauge transformations following
\[ [L_n, V(k,z)] = z^n \left[ z \frac{d}{dz} + n \left( \frac{1}{2} - k^2 \right) \right] V(k,z) \quad (2.11) \]

The \( N \)-point amplitude is then
\[
A_N(k_1, k_2, \ldots, k_N) = \int d\mu_N(z) \prod_{i=1}^{N} z_i^{-\frac{n}{2}} \left| z_i - z_{i+1} \right|^{\frac{n}{2} - k^2}
\]
\[
< 0 | V(k_1, z_1) V(k_2, z_2) \ldots V(k_N, z_N) | 0 >
\]

with \( d\mu_N(z) = (z_1 - z_N) dz_3 dz_4 \ldots dz_{N-1} \prod_{i=1}^{N-1} \frac{\theta(z_i - z_{i+1})}{z_i - z_{i+1}} \) and \( z_{N+1} \equiv z_1 \).

Projective invariance allows us to let \( z_1 \to \infty \), \( z_2 \to 1 \), and \( z_N \to 0 \) and put the amplitude in the form
\[
A_N = < 0 | k_1 b_\frac{1}{2} V(k_2) D V(k_3) D \ldots D V(k_{N-1}) k_N b_{-\frac{1}{2}} | 0 > \quad (2.13)
\]
where the propagator is
\[
D = \int_0^1 dx \left( 1 - x \right)^{-k^2 - \frac{3}{2}} x^{k^2 + L_0 - 3/2} \quad (2.14)
\]

Everywhere we use the notation \( V(k) = V(k,1) \).

For the pion mass \( k^2 = -\frac{1}{2} \),
\[
D = (L_0 - 1)^{-1} \quad (2.15)
\]

and the Virasoro subsidiary conditions \( L_n q = 0 \quad (n = 1,2,\ldots) \) hold.

Of special interest are the operators
\[
G_m = \sum_{n=0}^{\infty} \alpha_n b^{m-n} \quad (2.16)
\]

which have the following algebraic properties:
\begin{align*}
\{G_m, G_n\} &= 2L_{m+n} \quad (2.17) \\
\{L_m, G_n\} &= (\frac{1}{2}m - n)G_{m+n} \quad (2.18) \\
\{G_m, V_o(k)\} &= -\sqrt{2} V(k) \quad (2.19) \\
\{G_m, V(k)\} &= -\frac{1}{\sqrt{2}} \left[(L_o - m - \frac{1}{2})V_o(k) - V_o(k)(L_o - \frac{1}{2})\right] \quad (2.20)
\end{align*}

The amplitude can be rewritten as

\[ A_N = <0 | G_{\frac{1}{2}} V(k_2) \frac{1}{L_o - \frac{1}{2}} V(k_3) \ldots \frac{1}{L_o - \frac{1}{2}} V(k_{N-1}) G_{\frac{1}{2}} |0> \quad (2.21) \]

The operator $G_{\frac{1}{2}}$ can be moved to the left by using the anticommutation relation (2.20):

\[ V(k_{N-1}) G_{\frac{1}{2}} = -G_{\frac{1}{2}} V(k_{N-1}) - \frac{1}{\sqrt{2}} \left[(L_o - 1)V_o(k_{N-1}) - V_o(k_{N-1})(L_o - \frac{1}{2})\right]. \]

The second term cancels the propagator and the third one is zero, since $(L_o - \frac{1}{2}) |0> = 0$. By repeating this procedure, the amplitude reduces to

\[ A_N = <0 | V(k_2) \frac{1}{L_o - \frac{1}{2}} V(k_3) \ldots \frac{1}{L_o - \frac{1}{2}} V(k_{N-1}) |0> \quad (2.22) \]

plus terms in which at least one propagator is missing. The argument of ref. 4 to ignore those additional terms was as follows : they correspond to Koba-Nielsen type integrals in which two adjacent integration variables, $z_i$ and $z_{i+1}$ say, are equal. Hence, the term of the form $(z_i - z_{i+1})^{-2k_i k_{i+1}}$ makes such an amplitude vanish. In this paper, we shall need another version of this proof, based on integrations by parts of the type:

\[ 1 = (L_o - 1) \int_0^1 dx x^{L_o - 2} \quad (2.23) \]
All this allows a reinterpretation of the model in terms of a new Fock space \( \mathcal{F}_2 \) (the primitive one being called \( \mathcal{F}_1 \) in the following), in which the ground state with \( k^2 = -\frac{1}{2} \) is the pion itself and the propagator is \( (L_0 - \frac{1}{2})^{-1} \). In this space, there is no tachyon in the \( \rho \) trajectory. States corresponding to ancestors also disappear and else the \( G_m \)'s become on-shell gauge operators.

3. Virasoro-Shapiro Model

Now let us add two new sets of creation and annihilation operators, \( a' \) and \( b' \), with the same rules (2.1), (2.3), (2.4) among themselves and else satisfying

\[
[\alpha^{n \mu}_n, \alpha^m_\nu] = [\alpha^{n \mu}_n, b^m_\nu] = [b^{n \mu}_n, b^m_\nu] = [b^{n \mu}_n, \alpha^m_\nu] = 0. \tag{3.1}
\]

Operators \( L'_m \) and \( G'_m \) are defined in just the same way of eqs. (2.5), (2.6), and (2.16), and have analogous properties. Total Virasoro operators are obtained as the sum \( L_m + L'_m \). Next we define \( V', V'_0, \) and \( H' \) as in (2.8-10), but changing \( z \) in its complex conjugate \( \bar{z} \). We also drop out the factors \( \sqrt{z} \) in eqs. (2.2), (2.10), (2.19), and (2.20).

Eq. (2.11) becomes

\[
[L_n, V(k,z)] = z^n \left[ \frac{d}{dz} + \frac{n}{2} (1 - k^2) \right] V(k,z). \tag{3.2}
\]

The new vertex is

\[
V_T(k,z,\bar{z}) = k.H(z) k.H'(\bar{z}) V_0(k,z) V_0'(k,\bar{z}). \tag{3.3}
\]

The variable \( z \) is now allowed to take values all over the complex
plane. The new amplitude is

$$ A_N = \langle 0 \mid G_{1/2} G'_{1/2} V_T(k_2) \frac{1}{L_0 + L'_0 - 2} V_T(k_3) \frac{1}{L_0 + L'_0 - 2} \cdots V_T(k_{N-1}) G_{-1/2} G'_{-1/2} \mid 0 \rangle $$

(3.4)

where now $V_T(k) \equiv V_T(k,1,1)$.

The propagator may be factorized by writing [5]

$$ \frac{1}{L_0 + L'_0 - 2} = \int d^2z \frac{L_0 - 2 L'_0 - 2}{z - L'_0 - 2}. \quad (3.5) $$

Eq. (3.4) is well defined only when the integration region of eq. (3.5) is the inside of the unit circle ($|z| < 1$). The outside ($|z| > 1$) corresponds to another permutation of the external momenta. As in ref. 2, one should write down eq. (3.4) for each permutation, and sum all the contributions from different permutations: the result of this sum is just the integration over the whole $z$ plane in eq. (3.5).

Since primed operators commute with unprimed and both sets have the same effect on Fock primed and unprimed states, respectively, this corresponds to working in a product $\mathcal{F}_1 \otimes \mathcal{F}'_1$. The integrand really factorizes into two parts, which only differ by the interchange of $z$ and $\bar{z}$.

For better insight, we sketch the calculations for the 4-point case. We break up the amplitude

$$ A_4 = \frac{i}{2} \int dz \, d\bar{z} \langle 0 \mid G_{1/2} G'_{1/2} V_T(k_2) z^{L_0 - 2} z^{L'_0 - 2} V_T(k_3) G_{-1/2} G'_{-1/2} \mid 0 \rangle \quad (3.6) $$

into two parts, as done in ref. 2, one coming from $|z| < 1$ and the other from $|z| > 1$.  

\[ A_{1234} = \frac{i}{2} \int \frac{dz}{|z|} < \frac{1}{0} G_{\frac{1}{2}} V(k_2) z^{L_0 - 2} V(k_3) G_{\frac{1}{2}} G'_{\frac{1}{2}} V'(k_2) z^{L_0' - 2} V'(k_3) G'_{\frac{1}{2}} |0 > \\
= \int_0^1 r dr \int_0^{2\pi} d\theta | < \frac{1}{0} G_{\frac{1}{2}} V(k_2) V(k_3, r e^{i\theta}) G_{\frac{1}{2}} |0>|^2 r^{-3-s} .
\]

Direct calculations lead to

\[ A_{1234} = \int_0^1 r dr \int_0^{2\pi} d\theta |1 - re^{i\theta}|^{2k_2 k_3} | < \frac{1}{0} |1 - k_2, b \frac{1}{2}, k_3, H(1) k_3, H(re^{i\theta}) k_4 b_{\frac{1}{2}} |0>|^2 r^{-3-s} \\
= \int_0^1 r dr \int_0^{2\pi} d\theta |1 - re^{i\theta}|^{2k_2 k_3} (k_2, k_3)^2 .
\]

The contribution from \(|z| > 1\) corresponds to the permutation \(k_2 \otimes k_3\).

By using \(\alpha_s = s + 2\), \(\alpha_s + \alpha_t + \alpha_\mu = 2\) and by making a transformation \(r \to r^{-1}\), we arrive at

\[ A_{1324} = \int_1^\infty r dr \int_0^{2\pi} d\theta |1 - re^{i\theta}|^{2k_3 k_2} (k_2, k_3)^2 .
\]

Therefore,

\[ A_4 = (\frac{1}{2} \alpha_\mu)^2 \int d^2 z |1 - z|^{-2k_2 k_3} |z|^{-2} = \frac{\Gamma(1 - \frac{\alpha_\mu}{2}) \Gamma(1 - \frac{\alpha_t}{2}) \Gamma(1 - \frac{\alpha_\mu}{2})}{\Gamma(\frac{\alpha_t}{2}) \Gamma(\frac{\alpha_\mu}{2}) \Gamma(\frac{\alpha_\mu}{2})} .
\]

(3.7)

To go from the Hilbert space \(F_1\), where duality is evident, to the Hilbert space \(F_2\), where the spectrum is easier to study, one can use the same techniques as mentioned in sec. 2, and in ref. 4; terms left over

72/19
from the commutation of \( G_{\frac{1}{2}} \) and \( G'_{\frac{1}{2}} \), have the form

\[
\int dz \, d\bar{z} \, \cdots \, z^{L_0-2} (L_0 - 1) \cdots f(z)
\]

where \( f \) is some function of \( z \) only. By using the formal independence of \( z \) and \( \bar{z} \), one can integrate over \( z \) and thus argue that such a term gives zero. That this is indeed the case can be checked by going back to polar coordinates, \( z = r \exp i\theta \); the integration over \( \theta \) vanishes because of the periodicity. The integration over \( r \) vanishes only when the full range \( 0 \) to \( \infty \) is taken: each individual term, \( 0 \) to \( 1 \) and \( 1 \) to \( \infty \) does not give zero.

In the new Hilbert space, the propagator is

\[
\frac{1}{L_0 + L'_0 - 1} = \int d^2z \, z^{L_0-3/2} \bar{z}^{L'_0-3/2},
\]

so that the absence of the tachyon on the leading trajectory is manifest.

The on-shell gauge operators are just \( G_m \) and \( G'_m \). There are two kinds of trajectories spaced by one unit: even trajectories with even intercepts, which in \( \mathcal{F}_2 \times \mathcal{F}'_2 \) have an odd number of \( b \) and \( b' \) excitations, and odd trajectories, with odd intercepts. The first odd state, which is the analogue of the pion in ref. 3 occurs at \( m^2 = -1 \).

The even sector of the model has no tachyon, the ground state being a set of two massless particles, one with spin 2 and another with spin zero.

4. Conclusion

We wish to conclude with some comments on the possible relevance of such an unrealistic model. It has been remarked [6] that, in the zero-
slope limit, there is a very close connection between the "gauge invariance" of the conventional dual models (with unit intercept) and the usual gauge invariance of massless Yang-Mills fields. However, the two gauges act in completely different ways: the dual theory gauge tends to implement Moebius transformations on the Koba-Nielsen variables, whereas the field theory gauge gives divergence conditions on the Feynman diagrams. By combining the two properties, one could obtain interesting methods of regularizing loop amplitudes in those field theories. Namely, it has been remarked that dual N-loop diagrams are characterized by a set of N Moebius transformations. The multipliers of these transformations are Moebius invariants, and divergences of dual loops appear when one integrates over those multipliers. Restricting the range of integration gives a finite, Moebius invariant, hence gauge-invariant theory. This procedure could presumably be translated into a field-theory language, and used to regularize Yang-Mills fields in a gauge-invariant fashion. But this is not so urgent, since other gauge-invariant regularizations of Yang-Mills fields have been found recently [7]. What we want to suggest is that the dual theory approach might prove useful for massless spin two theories, like gravitation, where divergence difficulties in loop diagrams are much worse than for Yang-Mills theories: this could be done by a close examination of the symmetric dual models discussed in this paper, and by using their zero-slope limit.

Note added in proof: after completion of this work, we received a preprint by J.H. Schwarz (Princeton University report PURC 3072-06) in which, among other things, he considers this model.
References


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