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STABILITY AND EQUILIBRIUM IN QUANTUM STATISTICAL MECHANICS*

by

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This is a report on a common work with Rudolf Haag and Eva Trych-Pohlmeyer, which is technically connected with the harmonic analysis of non-commutative dynamical systems. The general aim of this work is to provide a derivation of the Gibbs Ansatz, base of the equilibrium Statistical Mechanics, from a stability requirement. By the same token a relation is established between stability and the positivity of the hamiltonian in the zero temperature case.

Rather than the Gibbs Ansatz pertaining to finite systems, we derive in fact the so-called Kubo-Martin-Schwinger (K.M.S.) condition, a substitute of the Gibbs Ansatz for infinite quantum systems. Since these concepts are not generally familiar to functional analysts (although the second now plays a central role in the theory of Von Neumann Algebras) we shall first describe them by sketching the way in which temperature equilibrium states are obtained mathematically in standard Quantum Statistical Mechanics. This is done in two steps :

1) One first considers finite systems i.e., physically, systems describing a finite portion of the substance under consideration enclosed in a cubic box of length L, with appropriate boundary conditions on the walls of the box. Mathematically this has two consequences :
   - first, the algebra of observables can be chosen to be the algebra $\mathfrak{A}$ of the compact operators on some Hilbert space $\mathcal{H}$;
   - second, the dynamical evolution of the system is described by a positive self-adjoint operator H with a pure point spectrum[1], each point having a finite multiplicity, so that $e^{-\beta H}$ is trace class for each positive β . This hamiltonian H induces a one-parameter group $t \to \alpha_t$ of automorphisms of $\mathfrak{A}$ in the following way : for $A \in \mathfrak{A}$

   (1) $$\alpha_t(A) = e^{itH} A\, e^{-itH} .$$

   The Gibbs Ansatz for describing the state ω of the finite system corresponding to the inverse temperature β then consists in assuming that

   (2) $$\omega(A) = \frac{\operatorname{Tr}\{e^{-\beta H}A\}}{\operatorname{Tr}\{e^{-\beta H}\}}$$

   This formula defines a state (normalized positive functional) of $\mathfrak{A}$, whose physical meaning is that $\omega(A)$ is the mean value of the observable A in the physical state corresponding to the inverse temperature β , a description in accordance with the manner in which Quantum Mechanics describes physical states.

2) The second step consists in performing the so-called thermodynamical limit namely letting the length L of the box tend to infinity, whilst putting into the box a

(1) general feature of the energy eigenvalue problem for a finite quantum system in a box.

number of particles proportional to its volume. The existence of one or several limit states is then guaranteed by the analytical properties of the hamiltonian, as can be proved with sufficient ingenuity and mathematical skill (the proof has been given for a number of models -the investigation of this "thermodynamical limit" is one of the principal aims of what one might call "Constructive Statistical Mechanics", which is not our concern in this work).

What is, now, the K.M.S. condition ? We obtain it in the following way : notice that, for an $A \in \mathcal{O}\!\mathcal{L}$ which is an analytic vector for the one-parameter group (1) one can extend α to an imaginary value $i\beta$ of time :

$$\alpha_{i\beta}(A) = e^{-\beta H} A e^{\beta H}$$

One has then, for all $B \in \mathcal{O}\!\mathcal{L}$, from (2),

$$\omega(B\alpha_{i\beta}(A)) = \frac{Tr\{e^{-\beta H} B e^{-\beta H} A e^{\beta H}\}}{Tr\{e^{-\beta H}\}} = \frac{Tr\{e^{-\beta H} BA\}}{Tr\{e^{-\beta H}\}}$$

whence the K.M.S condition :

(3) $$\omega(B\alpha_{i\beta}(A)) = \omega(AB)$$

This condition (3) has two advantages : first it persists in the thermodynamic limit (as can be checked on various models) and thus affords a substitute of (2) valid for infinite systems (for which (2) itself makes no sense), substitute which contains, as our experience shows, the same amount of information as (2). Thus we can replace the Gibbs Ansatz (2) by the K.M.S. condition (3) of more general validity, yielding the fundament of the Quantum Statistical Mechanics of infinite systems rather than of finite ones. This is important in that the infinite systems are the ones whose features correspond to a "thermodynamical behaviour"(which does not show up in finite systems -this motivates the necessity of performing the thermodynamic limit in the traditional approach described above). Physically, a system containing 10^{23} particles is best idealized by considering an infinite number of particles.
A second advantage of the K.M.S. condition (3) is that it has become one of the central items in the theory of Von Neumann algebras, and is therefore, mathematically, a beautiful object. We close this discussion of traditional Quantum Statistical Mechanics by noting that, if we introduce the functions

(4) $$\begin{cases} F_{AB}(t) = \omega(B\alpha_t(A)) - \omega(A)\,\omega(B) \\ G_{AB}(t) = \omega(\alpha_t(A)B) - \omega(A)\,\omega(B) \end{cases} \qquad A,B \in \mathcal{O}\!\mathcal{L},\ t \in \mathbb{R}$$

(3) can be written equivalently

$$\hat{F}_{AB}(E) = e^{\beta E}\,\hat{G}_{AB}(E) \quad , \quad A, B \in \mathfrak{A} \tag{5}$$

in terms of the Fourier transforms $\hat{F}_{AB}$ and $\hat{G}_{AB}$ of the functions (4) (if we assume for the automorphism group α the natural continuity property that $t \in R \longrightarrow \varphi(\alpha_t(A))$ should be continuous for all $A \in \mathfrak{A}$ and all states φ of $\mathfrak{A}$, $\hat{F}_{AB}$ and $\hat{G}_{AB}$ are bounded measures, for which the condition (5) has been written in a somewhat sloppy way as if these measures were functions of E (the energy), which will in fact be the case owing to further assumptions). The reason why we mention the alternative (5) to the classical K.M.S. condition (3) is that (5) naturally lends itself to our proof (note, also, that (5) can be stated without the restriction that A be an analytic element of the one parameter group α).

We now turn to our objective, which is to give the K.M.S. condition (5), the status of a theorem rather than that of an Ansatz, starting from scratch. For this we consider, from the start, an infinite quantal system, which we idealize as a "C^*-system", i.e. a pair $\{\mathfrak{A}, \alpha\}$ of a C^*-algebra $\mathfrak{A}$ and a continuous one-parameter group α of automorphisms of $\mathfrak{A}$ (the continuity assumption is the natural one that all numerical functions $t \in R \longrightarrow \varphi(\alpha_t(A))$, $A \in \mathfrak{A}$ φ a state of $\mathfrak{A}$, shall be continuous -one could require, equivalently, that the map $t \in R \longrightarrow \alpha_t(A)$ be continuous for all $A \in \mathfrak{A}$). This notion of C^*-system is a mathematical abstraction of the general frame of quantum mechanics for the description of a physical system together with its time evolution (= dynamics). It is both relevant for finite systems (in which case $\mathfrak{A}$ can be chosen as the algebra of compact operators on some Hilbert space) and for infinite systems : then $\mathfrak{A}$ is an "antiliminar" C^*-algebra (= possessing a maze of inequivalent representations) whose complexity reflects that of the infinite system. The elements of $\mathfrak{A}$ represent physically (norm limits of) local observables, whereby $\alpha_t(A)$, $A \in \mathfrak{A}, t \in R$, represents the observable obtained from A by a shift t in time. Physical states are defined by the states (= normalized positive functionals) of $\mathfrak{A}$, the value $\omega(A)$ of the state φ for $A \in \mathfrak{A}$ representing the mean value of the observable A in the state φ . Our program is now to derive the K.M.S. condition (5) (or positivity of the energy, the limiting case of (5) for zero temperature) from physically natural requirements on equilibrium states. The three constitutive properties required for an equilibrium state ω are the following :

(i) ω is invariant under α ;

(ii) ω is an extremal element of the convex set of α-invariant states ;

(iii) ω is stable for local perturbations of the dynamics.

Before starting with our argument we briefly comment upon these conditions. First, from a physical point of view, it should be clear that (i), (ii) and (iii) are natural requirements for characterizing thermodynamical equilibrium states : (i) is obvious ; (ii) corresponds to the fact that we want to describe "pure thermodynamicam phases" rather than quantal mixtures of them ; as for (iii), its physical meaning is clear : a local disturbance of the dynamics (e.g., an impurity in a crystal, a boat on an ocean) should not upset the original state, but merely cause a gentle distortion. Our second comment is that, of course, the conditions (i), (ii), (iii) above have to be statedmathematically in a precise way. For (i) this is obviously done as follows :

<u>Assumption (i)</u> : (invariance)

(6) $$\omega(\alpha_t(A)) = \omega(A) \qquad \text{for all } A \in \mathfrak{A} \text{ and } t \in R$$

Conditions (ii) and (iii), on the other hand, will be given precise formulations as we need them in our proof (in fact the technical conditions which we will need will turn out to be somewhat strong *mathematical exegeses of (ii) and (iii) as phrased* above, which we hope future progress will help to release).

Down to work ! Since the first part of our argument consists in exploiting (iii) in combination with (6), we now need to formulate (iii). For this we need a mathematical formulation of "local perturbations of the dynamics". That is done as follows : consider $h = h^* \in \mathfrak{A}$ and define the differentiable function

(7) $$t \in R \longrightarrow \mathcal{P}_t^{(h)} \in \mathfrak{A}$$

by the following differential equation and boundary condition at $t = 0$

(8) $$\begin{cases} i \dfrac{d\mathcal{P}_t^{(h)}}{dt} = \mathcal{P}_t^{(h)} \alpha_t(h) \\ \mathcal{P}_0^{(h)} = I \end{cases}$$

(these entail the existence and uniqueness of the function (7)). One easily shows that the solution $\mathcal{I}^{(h)}$ of (8) is a unitary cocycle in the following sense :

$$\text{(9)} \qquad \begin{cases} \mathcal{I}_t^{(h)*} = \mathcal{I}_t^{(h)-1} = \alpha_t\left(\mathcal{I}_{-t}^{(h)}\right) \\ \mathcal{I}_{s+t}^{(h)} = \mathcal{I}_s^{(h)}\alpha_s\left(\mathcal{I}_t^{(h)}\right) \end{cases} , \quad s,t \in R,$$

allowing the definition

$$\text{(10)} \qquad \alpha_t^{(h)}(A) = \mathcal{I}_t^{(h)}\alpha_t(A)\mathcal{I}_t^{(h)*} \quad , A \in \mathfrak{A} , t \in R,$$

of a "perturbed one-parameter group" $\alpha_t^{(h)}$, depending upon the choice of the self-adjoint $h \in \mathfrak{A}$. The fact that we have, here, a description of a "local perturbation of the dynamics" stems from the property[1]

$$\text{(11)} \qquad \left.\frac{d}{dt}\right|_{t=0} \alpha_t^{(h)}(B) = i \left.\frac{d}{dt}\right|_{t=0} \alpha_t(B) + [h, B]$$

easily derived from (8) for a "differentiable" $B \in \mathfrak{A}$ (one for which $t \to \alpha_t(B)$ is differentiable) : (11) shows that in a representation of $\mathfrak{A}$ where α is obtained from a hamiltonian H as in (1), $\alpha^{(h)}$ is likewise obtained from the hamiltonian $H + h$, a local dynamical perturbation since h represents a local observable. Equipped with this description of local perturbations[2] we are now ready to formulate precisely

Assumption (iii) : (stability)

For each self-adjoint h in $\mathfrak{A}$ there is a map $\lambda \longrightarrow \omega^{(\lambda h)}$ of a neighbourhood $\mathcal{V}_h$ of zero in R to the state space of $\mathfrak{A}$ such that :

a) $\omega^{(\lambda h)}$ is invariant for the perturbed dynamics $\alpha^{(\lambda h)}$:

$$\text{(12)} \qquad \omega^{(\lambda h)}\left(\alpha_t^{(\lambda h)}(A)\right) = \omega^{(\lambda h)}(A) \quad , t \in R, A \in \mathfrak{A},$$

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(1) $[\ \ ]$ denotes a commutator.

(2) which we owe to Derek Robinson and Huzihiro Araki (and which constitutes basically a bounded operator version of the old Dirac-Tomonaga-Dyson perturbation expansion).

b) $\lambda \longrightarrow \omega^{(\lambda h)}$ is differentiable, in the weak sense, for $\lambda = 0$, with derivative $\omega_1^{(h)}$.

(13) $$\frac{d}{d\lambda}\Big|_{\lambda=0} \omega^{(\lambda h)}(A) = \omega_1^{(h)}(A) \quad , A \in \mathfrak{A},$$

c) $\omega_1^{(h)}$ is a normal form of the representation π_ω of $\mathfrak{A}$ generated by ω [1].

From this stability assumption a very simple argument allows to proceed towards our aim of proving condition (5) : from (12) immediately follows that, for each differentiable $B \in \mathfrak{A}$,

$$0 = \omega^{(\lambda h)}\Big(\frac{d}{dt}\Big|_{t=0} \alpha_t^{(\lambda h)}(B)\Big),$$

whence, using (11),

$$0 = \omega^{(\lambda h)}\Big(i\frac{d}{dt}\Big|_{t=0} \alpha_t(B) + \lambda[h, B]\Big).$$

Replacing $\omega^{(\lambda h)}$ by $\omega^{(\lambda h)} - \omega + \omega$ and dividing by λ, one obtains, taking account of (6) differentiated with respect to t for $t = 0$,

$$0 = \omega([h, B]) + \frac{\omega^{(\lambda h)} - \omega}{\lambda}\Big(i\frac{d}{dt}\Big|_{t=0} \alpha_t(B) + \lambda[h, B]\Big),$$

whence, by (13), for $\lambda \longrightarrow 0$,

(14) $$0 = i\,\omega_1^{(h)}\Big(\frac{d}{dt}\Big|_{t=0} \alpha_t(B)\Big) + \omega([h, B]).$$

If, in this equation, we take B to be the differentiable

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(1) c) is not strictly necessary for our proof and is perhaps too strong a formulation. Cf. discussion below.

$$= \int_S^T \alpha_t(A)\,dt \quad , \quad \begin{array}{l} A \in \mathfrak{A} \\ S, T \in R \end{array} ,$$

with (as immediately checked)

$$i \frac{d}{dt}\Big|_{t=0} \alpha_t(B) = \alpha_T(A) - \alpha_S(A) ,$$

we obtain that, for all $S, T \in R$,

(15) $$\omega_1^{(h)}\left(\alpha_T(A) - \alpha_S(A)\right) = i \int_S^T \omega\left([h, \alpha_t(A)]\right) dt .$$

We have not yet used condition c) in Assumption (iii) above : c) will be used in combination with a (strengthened form of) the extremality assumption (ii) which we formulate as follows :

Assumption (ii) : (a strengthened form of the extremality of the invariant state ω).

We assume that :

a) the C^*-system $\{\mathfrak{A}, \alpha\}$ is asymptotically abelian in the sense that, for any $A, B \in \mathfrak{A}$, and state φ of $\mathfrak{A}$,

(16) $$\varphi\left([A, \alpha_t(B)]\right) \xrightarrow[t \to \infty]{} 0$$

b) the state ω is hyperclustering in the following sense : there is a dense, self-adjoint set $\mathcal{A}$ of $\mathfrak{A}$ such that, for arbitrary $A, B \in \mathcal{A}$ there is a majoration

(17) $$\left|\omega\left(A\alpha_t(B)\right) - \omega(A)\omega(B)\right| < \frac{C}{\{1+|t|\}^{1+\delta}} ,$$

where C and δ are positive constants. Furthermore analogous majorarations hold for the truncated expectation values up to order 6 [1].

Why is this assumption a stronger form of extremality for ω ? Because, for asymptotically abelian systems, we have the fact that extremality of an α -invariant state ω is synonimous with the fact that, for $A, B \in \mathfrak{A}$, ,

(18) $$\omega(A\alpha_t(B)) \xrightarrow[t=\infty]{} \omega(A)\omega(B) \quad \text{in mean.}$$

Physical reasons for strengthening this latter condition can be given (work is in progress on this point). We uncritically adopt the above formulation of Assumption (ii) to conclude that, for $A, B \in \mathfrak{A}$,

(19) $$\int_{-\infty}^{+\infty} \omega([h, \alpha_t(A)])\, dt = 0 ,$$

(with independent limits in the integration). This conclusion follows from performing the limits $S \longrightarrow -\infty$, $T \longrightarrow +\infty$ in (15), noting that (18) (a fortiori (19)), combined with asymptotic abelianness, entails that

$$\alpha_t(A) \xrightarrow[t=\pm\infty]{} \omega(A)\, I ,$$

σ-weakly, in the representation generated by the state ω , whence the vanishing of the l.h.s. of (15) in the limit. It is now apparent that we achieved progress towards proving the K.M.S. condition (5) since (19) can alternatively be written

(20) $$\int_{-\infty}^{+\infty} F_{Ah}(t)\, dt = \int_{-\infty}^{+\infty} G_{Ah}(t)\, dt$$

(Cf. (4)), which is nothing but the special case of (5) for $E = 0$.

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(1) We refrain from stating these conditions precisely, because they are used in a later stage of our proof which will only be sketched here. For a precise proof the reader is referred to [1] .

The rest of our work consists in heaving the "K.M.S. condition at zero energy" (20) to finite values of the energy. This is done by using the following trick (we here sketch the proof, omitting details needed for rigour) : since (20) holds for arbitrary $\ell, A \in \mathcal{O}\!\mathcal{L}$, it is liable to make $\ell = \ell_1 \alpha_u(\ell_2)$, $A = A_1 \alpha_t(A_2)$ with $\ell_1, \ell_2, A_1, A_2 \in \mathcal{S}$ and $u \in \mathbb{R}$. Using the fact that, by the hyperclustering property of ω,

$$\omega(\ell_1 \alpha_u(\ell_2)\, \alpha_t(A_1)\, \alpha_u(\alpha_t(A_2))) \xrightarrow[t=\infty]{} \omega(\ell_1 \alpha_t(A_1))\,\omega(\ell_2 \alpha_t(A_2)),$$

and taking (allowable) limits under the integral, one derives that

(21) $$\int_{-\infty}^{+\infty} F_{A_1\ell_1}(t)\, F_{A_2\ell_2}(t)\, dt = \int_{-\infty}^{+\infty} G_{A_1\ell_1}(t)\, G_{A_2\ell_2}(t)\, dt\,, \quad \ell_1, \ell_2, A_1, A_2 \in \mathcal{S};$$

and also, by an iteration of the same trick, that

(22) $$\int_{-\infty}^{+\infty} F_{A_1\ell_1}(t)\, F_{A_2\ell_2}(t)\, F_{A_3\ell_3}(t)\, dt = \int_{-\infty}^{+\infty} G_{A_1\ell_1}(t)\, G_{A_2\ell_2}(t)\, G_{A_3\ell_3}(t)\, dt\,, \quad \ell_1, \ell_2, \ell_3, A_1, A_2, A_3 \in \mathcal{S}$$

Now, from (21) written with $\alpha_s(A_2)$ instead of A_2, integration of both sides with respect to s after multiplication by e^{-iEs} yields that

(23) $$\hat{F}_{A_1\ell_1}(-E)\, \hat{F}_{A_2\ell_2}(E) = \hat{G}_{A_1\ell_1}(-E)\, \hat{G}_{A_2\ell_2}(E)$$

or, using the evident fact

(24) $$\hat{F}_{A\ell}(-E) = \hat{G}_{\ell A}(E)\,,$$

that

(25) $$\hat{G}_{\ell_1 A_1}(E)\, \hat{F}_{A_2\ell_2}(E) = \hat{F}_{\ell_1 A_1}(E)\, \hat{G}_{A_2\ell_2}(E)\,.$$

This means already, in view of the arbitrariness of choice of ℓ_1, ℓ_2, A_1, A_2 in $\mathcal{A}$, that we have a universal function $\phi(E)$ for which

(26) $$\hat{F}_{A\ell}(E) = \phi(E)\,\hat{G}_{A\ell}(E) \quad , \; A, \ell \in \mathfrak{A},$$

provided we have the guarantee that ℓ_1 and A_1 can be chosen so that $\hat{G}_{\ell_1 A_1}(E) \neq 0$ for a preassigned $E \in \mathbb{R}$. Assuming this for a while, we note that (26) is identical with (5) if $\phi(E) = e^{-\beta E}$. That the latter is the case now follows from the positivity of ϕ (due to the positivity of $\hat{F}_{AA^*}$ and $\hat{G}_{AA^*}$), and its multiplicativity :

(27) $$\phi(E' + E'') = \phi(E')\,\phi(E'') \quad ,$$

which itself follows from

(28) $$\phi(-E) = \phi(E)^{-1}$$

(immediate consequence of (24)), combined with (22) exploited in a manner analogous to the step of passing from (21) to (23). We thus proved (5) if we know that, to each $E \in \mathbb{R}$, there is a choice of $A_1, \ell_1 \in \mathfrak{A}$ with $\hat{G}_{\ell_1 A_1}(E) \neq 0$. This restriction is now settled by the

Proposition

Let $\{\pi, U\}$ be the covariant representation of the C*-system $\{\mathfrak{A}, \alpha\}$ generated by an invariant state ω of $\mathfrak{A}$ and assume that Assumption (ii) above holds. Then the spectrum $Sp(U)$ of the representation U of $\mathbb{R}$ is either one-sided (= lies on the non-negative or the non-positive reals) or coincide with the whole real line $\mathbb{R}$.

The alternative stated by this Proposition allows to conclude that if $Sp(U)$ is not one-sided, then $Sp(U) = \mathbb{R}$, whence the possibility of choosing, to each $E \in \mathbb{R}$, $A_1 = \ell_1^* \in \mathfrak{A}$ with $\hat{G}_{A_1^* A_1}(E) \neq 0$. We thus have the

Theorem :

Let $\{\mathfrak{A}, \alpha\}$ be a C*-system, with ω a state of $\mathfrak{A}$ satisfying assumptions (i), (ii) and (iii) above. Either the spectrum of $Sp(U)$ defined in the preceeding Proposition) is one-sided, or ω fulfills the K.M.S. condition (5) for some real temperature β.

We now sketch the proof of the Proposition. For $A \in \mathfrak{A}$ we denote by $Sp^{\alpha}(A)$ the support of the operator valued distribution $\hat{X}_A$, Fourier transform of the function $X_A : t \in R \rightarrow \alpha_t(A) \in \mathfrak{A}$. Since $X_{AB} = X_A X_B$, and Fourier transforms turn products into convolutions, it is intuitive that $Sp^{\alpha}(AB) \subset Sp^{\alpha}(A) + Sp^{\alpha}(B)$ for $A, B \in \mathfrak{A}$. Now with π, U and Ω the G.N.S. construction afforded by the state ω, i.e.

$$
(29) \qquad \begin{cases} \omega(A) = (\Omega, \pi(A)\Omega) & t \in R \\ \pi(\alpha_t(A)) = U_t \, \pi(A) \, U_t^* \quad , & A \in \mathfrak{A} \;) \\ U_t \Omega = \Omega \end{cases}
$$

one easily checks that $\lambda \in \hat{R}$ is contained in $Sp(U)$ iff, to each neighbourhood $\mathcal{V}$ of λ, there is $A \in \mathfrak{A}$ with $Sp^{\alpha}(A) \subset \mathcal{V}$ and $\pi(A)\Omega \neq 0$ (to establish this, use that $\mu \in Sp^{\alpha}(A)$ iff $\hat{f}(\lambda) = 0$ for each $f \in L^1(R)$ with $\alpha_f(A) = 0$)[1]. Now the first step in proving our Proposition consists in establishing that $Sp(U)$ is additive, which goes as follows : given $\lambda_1, \lambda_2 \in Sp(U)$ and an arbitrary neighbourhood $\mathcal{V}$ of $\lambda_1 + \lambda_2$ there are neighbourhoods $\mathcal{V}_1, \mathcal{V}_2$ resp. of λ_1, λ_2 with $\mathcal{V}_1 + \mathcal{V}_2 \subset \mathcal{V}$; and elements A_1, A_2 of $\mathfrak{A}$ with $Sp^{\alpha}(A_i) \subset \mathcal{V}_i$ and $\pi(A_i)\Omega \neq 0$, $i = 1, 2$. Let $A = \alpha_t(A_1) A_2$, one has $Sp^{\alpha}(A) \subset Sp^{\alpha}(\alpha_t(A_1)) + Sp^{\alpha}(A_2) \subset \mathcal{V}_1 + \mathcal{V}_2 \subset \mathcal{V}$ for each $t \in R$ (observe that $Sp^{\alpha}(\alpha_t(A_1)) = Sp^{\alpha}(A_1)$) ; and t can be chosen such as to make $\pi(A)\Omega \neq 0$ since

$$\|\pi(A)\Omega\|^2 = \omega(A_2^* \alpha_t(A_1^* A_1) A_2) \xrightarrow[t \to \infty]{} \omega(A_2^* A_2)\omega(A_1^* A_1) = \|\pi(A_2)\Omega\|^2 \|\pi(A_1)\Omega\|^2 ,$$

by the assumed clustering (iii) b) of ω. We thus conclude that $\lambda_1 + \lambda_2 \in Sp(U)$: $Sp(U)$ is additive.

Rest of the proof of the Proposition : if $Sp(U)$ is not one-sided it contains $a > 0$ and $-b$, $b > 0$. If a and b are not commensurable, the set of $ma - nb$, m, n positive integers,

[1] $\alpha_f(A) = \int f(t) \, \alpha_t(A) \, dt$

will have zero set distance to 0 , whence the density of $Sp(U)$ in R, whence $Sp(U) = R$ since $Sp(U)$ is closed. If a and b are commensurable, one can use the fact that $Sp(U)$ has no isolated points (an easy consequence of Assumption (ii)) to replace b by b' , $b'>0$ not commensurable with a ; and to argue as above.

We conclude with a few remarks. First, our Theorem is satisfactory from a physical point of view, since the alternative of one-sidedness of $Sp(U)$ or K.M.S. nature of ω is what we observed in nature where the first case occurs at zero temperature (where the hamiltonian is known to be positive) and the second for a finite temperature. However what is observed is the positivity of and the occurence of K.M.S. for positive values of β . This is not explained by our work as it stands now and presents us with one of our future problems.

Second, the reader of books on Statistical Mechanics finds that what is observed is the validity of the Gibbs Ansatz (or, for that matter, K.M.S.) with the hamiltonian H replaced by $H - \mu N$, where N is the particle number operator (generator of the gauge group) and μ the chemical potential. This result is obtained by our method replacing the algebra $\mathfrak{A}$ of observables by the field algebra $\mathfrak{F}$ and looking for the α-invariant, hyperclustering states of $\mathfrak{F}$ stable for local perturbations $\alpha^{(h)}$ of α corresponding to a gauge invariant h . This theory generalizes in fact to arbitrary (non commutative) compact automorphism group commuting with the dynamical group α . Work in collaboration with Rudolf Haag on this subject is in progress.

We conclude with a sketch of an alternative technique for deriving K.M.S. from stability, within a frame less interesting for physics but more in the mood of operator theory. Apart from a possible intrinsic interest for the theory of Von Neumann algebras, this alternative approach has the merit of shedding more light on the mathematical mechanism linking modular automorphisms with stability. Consider a W^*-system i.e. a pair $\{\mathfrak{M}, \alpha\}$ of a Von Neumann algebra $\mathfrak{M}$ with a one-parameter group α of automorphisms of $\mathfrak{M}$ such that $t \in R \to \varphi(\alpha_t(A))$ is continuous for all $A \in \mathfrak{M}$ and all normal states φ of $\mathfrak{M}$. And take a normal state ω of $\mathfrak{M}$ which is α-invariant and faithful (i.e. such that $\varphi(A^*A) = 0$, $A \in \mathfrak{M}$, implies $A = 0$: this condition is a natural one in the theory of Von Neumann algebras, although not physically cogent).

Keeping the same stability requirements as above in Assumption (iii) and replacing Assumption (ii) by the requirement of "ergodicity of α" (= no α-invariant elements in $\mathfrak{M}$ but the multiples of unity), we propose to establish the K.M.S. condition (3) for ω. Since, now, ω is assumed faithful, we know from the Tomita-Takesaki theory that ω generates a modular automorphism group σ for which it is K.M.S. at temperature 1. Our game will therefore consist in proving the identity of α and σ up to a scale factor β. The strategy is the following: we first note that α and σ commute, due to the fact that ω is α-invariant. Now the faithful a. b. α- and σ-invariant ω generates a faithful representation of $\mathfrak{M}$ in which α and σ are respectively implemented by unitary representation U and V, whilst the two-parameter group $(t,s)\in\mathbb{R}^2 \to \alpha_t\sigma_s$ is implemented by UV. Further, by a mechanism analogous to that which gave rise to the Proposition above $Sp(U)$, $Sp(V)$ and $Sp(UV)$ will all be groups, the latter a subgroup of $\mathbb{R}^2$ whose projections on the x- and y-axes respectively coincide with $Sp(U)$ and $Sp(V)$. Because of the assumed ergodicity of α, $Sp(U)$ covers the whole reals. $Sp(V)$, on the other hand, can either be $\{0\}$, or $\{n\lambda ; n\in\mathbb{Z}\}$ or $\mathbb{R}$. The first of these three cases trivially gives rise to stability (it corresponds to the temperature ∞ case in physics). The two others leave us with the three following possibilities for

1) an array of horizontal lines $y = n\lambda$, $n\in\mathbb{Z}$

2) the whole $\mathbb{R}^2$-plane

3) a straight line of slope β through the origin.

We want to eliminate the two first cases and keep case 3) which leads to the desired proportionality of H and K, the infinitesimal generators of resp UV ($U_t = e^{iHt}$, $V_s = e^{iKs}$). Here is a sketch of the way in which this can be done : if we rewrite (15) in the G.N.S. construction from ω introducing the modular operator $\Delta = e^{K}$ and the modular conjugation J of σ and using the fact that $h = h^*$ implies $JhJ = \Delta^{\frac{1}{2}}h$ we obtain the condition (Ω denotes the cyclic vector obtained from ω) :

$$\text{(30)} \qquad \omega_1^{(h)}\big(\alpha_T(A) - \alpha_S(A)\big)$$
$$= \Big(i\int_S^T e^{-iHt}(I-\Delta^{\frac{1}{2}})h\Omega \,\Big|\, (I+\Delta^{\frac{1}{2}})A\Omega\Big)$$
$$, \; A = \pi(A), \; h = \pi(h)\in\pi(\mathfrak{M})$$

Now the limit $S \to -\infty$ (or $S \to -\infty$, $T \to +\infty$) will cause $\int_S^T e^{-iHt}$ to become, say, something like a principal value of $\frac{1}{H}$, whils $I - \Delta^{-t} = I - e^{-tK}$ behaves like $\sim K$ in the spectral regions where K is small. Thus, roughly, the limit $S \to -\infty$ in ( ) will make sense iff K/H is meaningful, a circumstance realized in case 3), but not in cases 1) and 2) for which there are regions of the UV -spectrum where $H = 0$ whilst K is finite. For the rigourization of this bold argument, it seems that we need a spectral concentration theorem believed to be true, but not yet formally proven, by our friends in the theory of Von Neumann algebras. So please allow a rugged, but pious physicist to end his talk with a prayer for the progress of Harmonic Analysis of Non-Commutative Systems !

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