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TRACING BACK RESONANCES TO FAMILIES OF REGGE TRAJECTORIES.

NEW FINITE ENERGY SUM RULES

by

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Abstract : We suppose that an amplitude A can be expressed for large enough energies as a sum of contributions of Regge poles.

Calling family of trajectories the set of trajectories which differ by integers from one of them, we prove that a correspondence, such that the energy and width of a given resonance depend on only one family of trajectories, can be established between resonances of the amplitude and families of trajectories.

We also show that the contribution to the amplitude of each family of trajectories satisfies the same finite energy sum rules as does the amplitude itself. In these sum rules the resonance approximation can be made where the only resonances that will appear are those which are in correspondence with the family.

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Introduction

In a series of articles (1)(2), we developed a new outlook on duality.

We consider an amplitude which for large enough energies can be expressed as a sum of contribution of Regge terms* and we study the singularities of the analytical continuation of this series. Among those singularities there might be poles in the unphysical sheet which can be interpreted as resonances.

Our study of this amplitude is based on the fact that the sum of contribution of Regge poles which we write as $F(s,t) = \sum_n a_n(t)\, s^{\alpha_n(t)}$ is not given as a Taylor series in s but as a more general power series, since the exponents $\alpha_n(t)$ of s are not in general integers. Such a general power series in most conveniently written as a Dirichlet series in the variable $\xi = \mathrm{Log}\, s$:

$$F(s,t) = f(\xi,t) = \sum_n a_n(t)\, e^{-\lambda_n(t)\,\xi}$$

with $\lambda_n(t) = -\alpha_n(t)$.

Now the theory of Dirichlet series contains theorems corresponding to those on Taylor series. For instance a Dirichlet series has an abscissa of convergence (corresponding to a radius of convergence in the s plane) and under general conditions there is at least a singularity of f with this abscissa.

But what is more important for us are theorems, most of which are due to S. Mandelbrojt (5) , which pertain to the fact that the $\lambda_n(t)$ are not in general integers.

To express these properties it is useful to introduce two quantities :

1) The upper density $\dot{D}$ of the λ_n , which is defined as $\dot{D} = \overline{\lim}\, \frac{n}{\lambda_n}$

2) The fractional part λ_n^* of λ_n which is defined as

*It would of course be desirable, i.e. more realistic, to consider amplitudes which are only asymptotically of this form, which amounts to considering asymptotic Dirichlet series : we intend in a subsequent article to generalize our results to this case by making use of adherent Dirichlet series (7).

$\mu_n = \mu_n^* +$ integer with $0 \leq \mu_n^* < 1$. The only fractional parts which have an influence on the singularities of f are, of course, those to which corresponds an infinite number of μ_n having this fractional part, so that in the statements we will make, concerning the set $\{\mu_n^*\}$ of fractional parts μ_n^* we will neglect the fractional parts to which correspond only a finite number of μ_n having this fractional part.

We will call the trajectories which correspond to $\mu_n(t)$ having a same fractional part $\mu_i^*(t)$, a family of trajectories, or the family of trajectories defined by $\mu_i^*(t)$.

When we have a series $\sum a_n e^{-\mu_n s}$, the sum of the terms which correspond to the μ_n which have a same fractional part μ_i^* will be called the contribution to the amplitude of this family.

Let us now summarize the guiding ideas and the main results of our articles. In this introduction we express them in an intuitive manner. For details, restrictions on our statements, and rigour the reader is refered to the corresponding articles.

In a first article (1) we used a theorem which states that a Dirichlet series with a finite abscissa of convergence and finite upper density $\dot{D}$ has at least one singularity in every horizontal band of width $2\pi\dot{D}$. In the s plane that means that there is a singularity of $F(s,t)$ in every angle of opening $2\pi\dot{D}$ centered at the origin. Thus if we want to avoid singularities in the physical sheet we must have $\dot{D} \geq 1$ (which means that for n large enough the trajectories must be spaced by at most one unit for all t), and the limiting case $D = 1$ is physically acceptable only if, as in the Veneziano model, the singularities are on the real axis. The Veneziano model thus appears as a limiting case.

A second important guiding idea is the so called principle of arithmetical separation of the exponents, due to S. Mandelbrojt. It can roughly be stated as saying that the singularities which arise in the analytical continuation of the contribution of a family of trajectories, cannot be annihilated by the singularities which arise from the contributions of other

families (4). In other words the singularities in s which arise from the contribution of each family are singularities of the amplitude itself. This is, of course, different from what happens if we add two series corresponding to exponents which have a same fractional part. In this case the set of singularities of the sum of the two series does not necessarily contain the set of singularities of each series. For instance $\frac{2}{1-s^2} = \sum 2 s^{2n}$ has the singularities $s = -1$ and $s = +1$, $\frac{1}{s-1} = \sum - s^n$ has the singularity $s = 1$ and $\sum_n 2 s^{2n} - s^n$ only has the singularity $s = -1$ (the singularity $s = +1$ has disappeared).

As a first consequence of this principle we proved (2) that from the fact that the number and position in the s plane of resonances, of the amplitude $F(s,t)$, must not depend on t, results that a mother trajectory $\alpha(t)$ and its daughters $\alpha_n(t)$ must be parallel for n large enough. The reason is that the principle of arithmetical separation of the exponents imposes that the number of elements of the set $\{\alpha_n^*(t)\}$ of fractional parts, must be independent of t if the number and position of the singularities in s of $F(s,t)$ is to be independent of t. This in turn implies that the $\alpha_n(t)$ are parallel (for n large enough) since a deviation from parallelism would modify the number of fractional parts as t varies.

In the present article we again use the principle of arithmetical separation of the exponents.

Our point is that if there is a finite number of families, summing up the contributions to the amplitude of each family introduces no further singularities than those which arise from each family. Each of these singularities, in turn, from the principle of arithmetical separation of the exponents, are singularities of the amplitude so that finally <u>each resonance of the amplitude can be traced back to a family. We thus predict a correspondence between families and resonances</u>. This correspondence is not necessarily one to one, as it could happen, by chance, that two families share a same resonance, but the point is that if one obtains a resonance (or more generally a singularity) from the contribution of one family, then the presence of other families can neither annihilate nor modify the position (i.e. energy and width for a resonance) of this singularity.

Furthermore, from the fact that the contribution of each family has no more singularities than the amplitude and from the fact which also results from our proof, that it does not behave "worse" at infinity than the amplitude, results that if the amplitude satisfies a finite energy sum rule, then the contribution of each family satisfies a similar sum rule.

Using furthermore the hypothesis of Harari-Freund which attributes all the background to the pomeron we can make on one side of the sum rule satisfied by the contribution of a family, the resonance approximation, and the only resonances which will appear in this sum rule will be the ones which correspond to this family.

Let us finally note that a general consequence of our considerations is that the concept of mother and daughter trajectories parallel and spaced by one unit (i.e. belonging to a same family) find a new justification entirely different from the one for which they were initially introduced in the literature.

I - Introducing Dirichlet series

From now on, in order to facilitate comparison with other articles on duality and finite energy sum rules, we will use the variable $\nu = \frac{s-u}{2M}$ and denote by $A^{\mp}(\nu, t)$ the antisymmetric and symmetric amplitudes.

Furthermore we will omit the variable t which is irrelevant for the considerations of this article. Thus we suppose that we have amplitudes $A^{\mp}(\nu)$ which for $|\nu| > R$ can be expanded as a sum of Regge terms

(1) $$A^{\mp}(\nu) = \sum_j \beta_j \frac{1 \mp e^{-i\pi\alpha_j}}{\sin \pi\alpha_j} \nu^{\alpha_j}$$

Let us introduce the variable $\xi = \mathrm{Log}\, \nu$, $\xi = \sigma + i\tau$ (σ and τ real) and call $\alpha_m = -\mu_m$, (we have $\mu_m > \mu_{m-1}$ and $\mu_m \to \infty$),

$$\beta_m \frac{1 - e^{-i\pi\alpha_m}}{\sin \pi\alpha_m} = d_m .$$

$A^{-}(\nu) = f(\xi)$ can then be written as a Dirichlet series

(2) $$f(\xi) = \sum_m d_m e^{-\mu_m \xi}$$

The series on the right hand side of equation (2), is called a Dirichlet series in the variable ξ (*), it converges for $\sigma > \mathrm{Log}\, R$

Let us briefly state a few properties of Dirichlet series and give a few definitions which will be useful in section II.

- The quantities μ_m are called the exponents of the series.
- The fractional part μ_m^* of μ_m is defined by $0 \leq \mu_m^* < 1$ and $\mu_m - \mu_m^* = n$, n being an integer. We will denote the set of values taken by μ_m by $\{\mu_m\}$ and the set of values taken by the fractional parts μ_m^* by $\{\mu_m^*\}$.

A theorem of S. Mandelbrojt, which we shall prove in section II, shows that the introduction of the concept of fractional parts of the exponents is mathematically useful and natural. It is also natural from the physical point of view as the members of a family of Regge trajectories consisting of a mother and daughter trajectories, and any trajectory which is exchange degenerate with one of them, all correspond to the same fractional part. They are caracterized by the fractional part α_i^* of the exponent α_i of the mother trajectory.

- A Dirichlet series converges in a half plane $\sigma > \sigma_c$. σ_c is called the abscissa of convergence of the series,
- It converges absolutely for $\sigma > \sigma_a \geq \sigma_c$. σ_a is called the abscissa of absolute convergence. If $\underline{\lim} (\mu_m - \mu_{m-1}) = h \neq 0$ (h is called the step of the series), then $\sigma_a = \sigma_c$.
- In any compact domain contained in $\sigma > \sigma_a$, the Dirichlet series converges uniformly.
- We denote by the same notation $f(\xi)$ the Dirichlet series and its analytic continuation. If $h \neq 0$, $f(\xi)$ has a singularity on its abscissa of convergence; this relates $\sigma_a = \overline{\lim} \frac{\mathrm{Log}|d_n|}{\mu_n}$ to the singularity of $f(\xi)$ which has the largest real part.
- We shall say that $f(\xi)$ increases no faster than $e^{C(\tau)}$, outside of its singularities, in a half plane $\sigma > \sigma_0$, if in the domain $\sigma > \sigma_0$, and outside of circles of radius ε around each singularity of $f(\xi)$, we have

$$|f(\xi)| < A_\varepsilon e^{C(\tau)}$$

(*) The Riemann surface of ν is mapped unto the ξ plane.

A_ε being a constant which depends on ε .

II - Arithmetical decomposition of the amplitude

In this section we start by proving a simplified version of a theorem of S. Mandelbrojt (4) .

Theorem I

Consider a set of exponents $\{\mu_n\}$ which can be separated into two sets $\{\nu_n\}$ and $\{\lambda_n\}$ with the following properties : the elements ν_m of $\{\nu_m\}$ all have the same fractional part ν^* , and each fractional part λ_n^* satisfies $|\lambda_n^* - \nu^*| \geqslant \delta$, with $0<\delta<1$.

Suppose that $f(\zeta) = \sum_n d_n e^{-\mu_n \zeta}$ has abscissa of absolute convergence $\sigma_a = 0$ (the condition $\sigma_a = 0$ simplifies the proof but can be replaced by $\sigma_a < \infty$), and that $f(\zeta)$ is a uniform function(x).

Suppose furthermore that $f(\zeta)$ has only isolated singularities(xx) for $\sigma > \sigma_0$, and that for $\sigma > \sigma_0$ $f(\zeta)$ does not increase, outside of its singularities, faster than $e^{C(\tau)}$, $C(\tau)$ being an increasing concave function with

$$\int_1^\infty \frac{C(\tau)}{\tau^2}\, d\tau < \infty$$

Denote by $f_1(\zeta) = \sum_j d_{n_j} e^{-\mu_{n_j}\zeta} = \sum_m c_m e^{-\nu_m \zeta}$ the partial series obtained by summing up the terms of $f(\zeta)$ whose exponents have the fractional part ν^*, i.e. the contribution of this family (c_m is the subset of d_m which correspond to the exponents ν_m).

(x) This of course does not imply that $A^{\mp}(\nu)$ is a uniform function of ν .

(xx) For the definition of singularities, see (Ref. 5). In the case of isolated singularities, and in particular poles, the definition becomes the usual one.

Then : ζ_n' being a singularity of f , and k_n an integer, the only possible singularities of $f_1(\zeta)$, in the half plane $\sigma > \sigma_0$, are points of affix $\zeta_n' + 2k_n \pi i$, or a limit of such points.

Proof of Theorem I :

In the ζ plane we consider a Dirichlet series :

(2) $$f(\zeta) = \sum d_n e^{-\mu_n \zeta}$$, with an abscissa of absolute convergence $\sigma_a = 0$, and we suppose that $f(\zeta)$ has only isolated singularities in the half plane $\sigma > \sigma_a$, and that $f(\zeta)$ does not increase in this half plane, outside of its singularities, faster than $e^{C(\tau)}$, $C(\tau)$ being an increasing concave function for $\tau > 0$, and $\int_1^\infty \frac{C(\tau)}{\tau^2} d\tau < \infty$.

We suppose that the set $\{\mu_n\}$ can be separated into two sets $\{\nu_n\}$ and $\{\lambda_n\}$, all elements ν_n of $\{\nu_n\}$ having the same fractional part ν^*, and that each fractional part λ_n^* satisfies $|\lambda_n^* - \nu^*| > \delta$ with $0<\delta<1$.

We then take a number $0 < \delta' < \delta$. From condition $\int_1^\infty \frac{C(\tau)}{\tau^2} d\tau < \infty$ results that we can construct explicitly (see Ref. 5) an entire function $H(\zeta)$ having the following properties.

The Fourier transform $\widehat{H}$ of H is a real, even, function which satisfies :

(3) $$\begin{cases} \widehat{H}(u) = 1 & \text{for } u = 0 \\ \quad\;\; = 0 & \text{for } |u| > \delta' \\ |H(\zeta)| \leq e^{\delta'|\tau| - C(|\zeta|)} \mathcal{L}(\sigma) & \end{cases}$$

$\mathcal{L}(\sigma)$ being an even integrable function.

We define $\widehat{H}_1(u) = \widehat{H}(u - \nu^*)$ and call H_1 its inverse Fourier transform.

We then consider the integral

(4) $$f_1(z) = \int_{d-i\infty}^{d+i\infty} f(z-\zeta)\, \varphi(\zeta)\, H_1(-i\zeta)\, d\zeta$$

where $$\varphi(\zeta) = \frac{1}{e^\zeta - 1} = \frac{e^{-\zeta}}{1-e^{-\zeta}} = \sum_n e^{-n\zeta}$$

$$z = x + iy \quad \text{(x and y real)}$$

We take z in a closed domain in the half plane $x \geqslant d' > d > \sigma_a = 0$. From (3) and from the fact that $C(\tau)$ is concave, result that integral (4) has a meaning for the values of z and d which we consider. Furthermore for these values we can replace $f(z-\zeta)$ by its Dirichlet series expansion. Replacing also $\varphi(\zeta)$ by its expansion we get

$$\int_{d-i\infty}^{d+i\infty} f(z-\zeta)\,\varphi(\zeta)\,H_1(-i\zeta)\,d\zeta = \int_{d-i\infty}^{d+i\infty} \sum_m \sum_n d_m e^{\mu_m(\zeta - z)} e^{-n\zeta} H_1(-i\zeta)\,d\zeta$$

$$= \sum_m d_m e^{-\mu_m z} \sum_n \int_{d-i\infty}^{d+i\infty} e^{-(n-\mu_m)\zeta}\, H_1(-i\zeta)\,d\zeta$$

we can replace this integral by an integral on the axis $\sigma = 0$,

$$= i \sum_m d_m e^{-\mu_m z} \sum_n \int_{-\infty}^{+\infty} e^{(\mu_m - n)i\tau}\, H_1(\tau)\,d\tau$$

$$= i \sum_m d_m e^{-\mu_m z} \sum_n \widehat{H}_1(\mu_m - n)$$

but with the choice we have made for H_1 we have

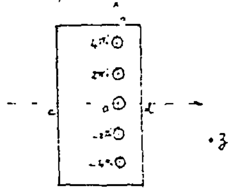
$$\widehat{H}_1(\mu_m - n) = 0 \quad \text{if} \quad \mu_m^* \neq \mathcal{D}^*$$
$$= 1 \quad \text{if} \quad \mu_m^* = \mathcal{D}^*$$

so that we finally have :

$$f_1(z) = \int_{d-i\infty}^{d+i\infty} f(z-\zeta)\,\varphi(\zeta)\,H_1(-i\zeta)\,d\zeta = i \sum_m c_m e^{-\mathcal{D}_m z}$$

We thus see that the integration on the right hand side of (4) constitutes an operator which, acting on f, picks out of the Dirichlet series representing f, the sum of these terms whose exponents have a same fractional part $\mathcal{D}^*$.

Let us now examine the singularities of $f_1(z)$ in the domain $\sigma > \sigma_0$. To do this we consider the domain D limited by the straight lines $\sigma = d$, $\sigma = c$, $(\sigma_0 < c < 0)$, small circles centered on the singularities $\zeta = 2k\pi i$ of $\varphi(\zeta)$, and two horizontal lines $\tau = \pm a$ with $a \neq 2k\pi$.



z being in the same domain as previously ($x \geqslant d' > d$).

It results from (3) and the concavity of $C(\tau)$ that the integrals on the vertical parts of the contour have a limit as $a \to \infty$, that the integrals on the horizontal part of the contour tend to zero as $a \to \infty$ and that the sum of the integrals on the small circles converge uniformly with respect to z, so that we can write

$$f_1(z) = \int_{c-i\infty}^{c+i\infty} f(z-\zeta)\,\varphi(\zeta)\,H_1(-i\zeta)\,d\zeta + \sum_k C_k$$

with

$$C_k = \int f(z-\zeta)\,\varphi(\zeta)\,H_1(-i\zeta)\,d\zeta$$

this integral being taken on the small circle centered on $2k\pi i$.

If we now take z inside the domain D, the integral

$$\int_{c-i\infty}^{c+i\infty} f(z-\zeta)\,\varphi(\zeta)\,H_1(-i\zeta)\,d\zeta$$

will remain regular since $x - c > 0$ and thus $f(z-\zeta)$ is regular (remember that the abscissa of absolute convergence of the Dirichlet series is $\sigma_a = 0$ and that f has no singularity on the right of this abscissa of convergence). As for an integral C_k, it is regular as long as $z - \zeta$ is not a singularity of f, ζ being on the small circle centered on $2k\pi i$. We thus arrive at the result that z'_n being a singular of f and k_n an integer, the only possible singularities of f_1 are of the form $z'_n + 2k_n\pi i$ or a limit of such points. This proves Theorem I.

In reference (4) this theorem is proved under more general conditions, without in particular the restriction that the singularities of f are isolated, we will call this version theorem I'.

If we now apply Theorem I' to amplitudes $A^{\mp}(\nu)$, translating theorem I' into the ν plane and using the fact that the singularities of $A^{\mp}(\nu)$ are either resonances in the unphysical sheet, or the cuts on the real axis, we get Theorem II.

Theorem :I

Consider an amplitude which can be expanded for $|\nu| > R$ as

$$A^{\mp}(\nu) = \sum_j \beta_j \frac{1 \mp e^{-i\pi\alpha_j}}{\sin\pi\alpha_j} \nu^{\alpha_j} \tag{5}$$

and consider the contribution $A_i^{\mp}(\nu)$ to $A^{\mp}(\nu)$ of the family of trajectories which contains the trajectory α_i (i fixed) :

$$A_i^{\mp}(\nu) = \sum_n \beta_{i,n} \frac{1 \mp e^{-i\pi(\alpha_i - n)}}{\sin\pi(\alpha_i - n)} \nu^{\alpha_i - n} \tag{6}$$

Then the projection on the ν plane of any singularity of A_i, is the projection on the ν plane of a singularity of A.

III - Correspondence between resonances and families of trajectories

Theorem II shows that the projection ν_p on the ν plane, of a singularity of A_i coincides with the projection of a singularity of A, but it could happen that these two singularities are on two different sheets of the Riemann surface of ν. However this possibility is extremely unlikely as it would imply that the sum of contributions of all the families other than the one which contains α_i, has two singularities having ν_p as projection : one of these singularities should be on the same sheet of the Riemann surface as the singularity of A_i, so as to be able to compensate it, and the other one should be on another sheet in order to restore in A a singularity with the projection ν_p, so as to satisfy Theorem II.

We will neglect this highly improbable case and state that if A_i is the contribution of the family of trajectories defined by α_i, then $A_i(\nu)$ has no other singularities than those of $A(\nu)$.

Thus every resonance of the amplitude A_i is a resonance of A, and if we suppose, as is natural, that there is only a finite number of families, then the set of singularities of $A = \sum_i A_i$ is simply the reunion of the sets of singularities of each A_i.

We are thus led to a correspondence between resonances of A and families of trajectories according to which each resonance of A can be traced back to a family A_i. This leads us to a new classification of resonances.

Furthermore from the fact that the contribution A_i of a family of trajectories has no more singularities than does the amplitude A (*) results that if $A(\nu)$ satisfies a finite energy s.. rule

$$\int_0^N \nu^{2n}\, \mathrm{Im}\, A^-(\nu)\, d\nu = \sum_j \beta_j \frac{1 - e^{i\pi\alpha_j}}{\sin\pi\alpha_j} \frac{N^{\alpha_j + 2n + 1}}{\alpha_j + 2n + 1} \tag{7}$$

(*) It is also easy to see that the proof of Theorem I shows that for $|\nu| \to \infty$, A_i does not behave "worse" at infinity than does A.

then $A_i(\nu)$ satisfies a similar finite energy sum rule :

$$\int_0^N \nu^{2n}\, \mathcal{I}m\, A_i^-(\nu)\,d\nu = \sum_m \beta_{i,m} \frac{1 - e^{-i\pi(\alpha_i - m)}}{\sin\pi(\alpha_i - m)} \frac{N^{\alpha_i - m + 2n}}{\alpha_i - m + 2n} \tag{8}$$

Now in this last finite energy sum rule, making use of the Harari-Freund assumption which attributes all the background to the pomeron, we can make the resonance approximation where the only resonances which will be present are those which we have put in correspondence with the family of trajectories defined by α_i .

We thus obtain from a set of finite energy sum rules satisfied by the amplitude A , as many sets of finite energy sum rules as there are families of trajectories. We thus have what we can call multicomponent finite energy sum rules.

Let us finally note that there is some hope that our considerations can be extended through the use of adherent Dirichlet series (7) , to amplitudes which are only asymptotically of Regge form.

Appendix : Equivalence of FESR and of a development of the amplitude in power series.

The purpose of this appendix is to establish the equivalence between a complete set of finite energy sum rules for an amplitude, and its development as a sum of Regge terms for ν large enough.

We first go through the usual proof that a development of the amplitude A^- as a sum of Regge terms (5) for $|\nu| > R$, and the analytical properties of A^- imply the set (11) of FESR . We then prove that conversely, the set (11) of FESR imply that A can be written as a sum (5) of Regge terms for $|\nu| > R$.

We start by considering scattering amplitudes $A^{\mp}(\nu)$ which for $|\nu| > R$ can be expanded as sum of Regge terms

$$A^{\mp}(\nu) = \sum_j \beta_j \frac{1 \mp e^{-i\pi\alpha_j}}{\sin\pi\alpha_j} \nu^{\alpha_j} \tag{5}$$

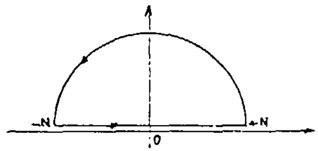
where ν is the usual variable

$$\nu = \frac{s - u}{2M}$$

and where $A^{\mp}(\nu)$ are the antisymmetric and symmetric amplitude which satisfy

$$A^{\mp}(-\nu) = \mp A^{\mp}(\nu)$$

we will only study A^- , the extension to A^+ is straightforward. The integration of A^- over a contour consisting of a semi-circle of radius $N > R$ in the upper half plane of ν and the diameter $-N, +N$ of this semi-circle, just above the real axis, gives 0 .



For the integration on the semi-circle we replace A^- by the right hand side of (5) and use

$$\nu^{\alpha_j+1} = e^{(\alpha_j+1)\,\mathrm{Log}\,\nu} = e^{(\alpha_j+1)(\mathrm{Log}|\nu| + \arg\nu)}$$

so that

$$\int_{\frown} \nu^{\alpha_j}\, d\nu = \frac{N^{\alpha_j+1}}{\alpha_j+1}\left(-e^{i\pi\alpha_j}-1\right) \tag{9}$$

The integration of A^- on the diameter gives, using the antisymmetry property of A^-

$$\int_{-N}^{+N} A^-(\nu)\, d\nu = 2i\int_0^N \mathrm{Im}\, A^-(\nu)\, d\nu$$

we thus obtain from (5) the well-known FESR

$$\int_0^N \mathrm{Im}\, A^-(\nu)\, d\nu = \sum_j \beta_j \frac{N^{\alpha_j+1}}{\alpha_j+1} \tag{10}$$

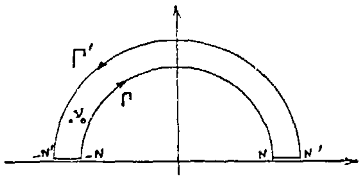
similarly $\nu^{2n} A^-(\nu)$, which has the same symmetry properties as $A^-(\nu)$, satisfies

$$\int_0^N \nu^{2n}\, \mathrm{Im}\, A^-(\nu)\, d\nu = \sum_j \beta_j \frac{N^{\alpha_j+1+2n}}{\alpha_j+1+2n} + \left(\frac{1}{2}\frac{\pi}{(2|n|-1)!} A^{-(2|n|-1)}(0),\ \text{for } n<0\right) \tag{11}$$

for any positive of negative integer n. (For $n<0$ we replace the diameter $-N, +N$ by the line $\underset{-N\qquad 0\qquad +N}{\overline{\qquad\frown\qquad}}$).

We now prove the converse property : if the set of equations (11) holds, for at least two values of $N>R$, then (5) holds for $|\nu|>R$.

To prove this we consider a domain in the upper half plane limited by semi-circles Γ and Γ' of radius $N>R$ and $N'>N$, and the two segments $(-N', -N)$ and (N, N') parallel to, and just above, the real axis, and we consider ν_0 inside this domain.



The Cauchy formula

$$A^-(\nu_0) = \frac{1}{2\pi i}\int_{\frown} \frac{A^-(\nu)}{\nu-\nu_0}\, d\nu$$

can be rewritten as

$$\begin{cases} A^-(\nu_0) = \dfrac{1}{2\pi i}\left(\displaystyle\int_{\frown}\frac{A^-(\nu)}{\nu-\nu_0}\, d\nu - \int_{\frown}\frac{A^-(\nu)}{\nu+\nu_0}\, d\nu\right) \\[2ex] A^-(\nu_0) = \dfrac{1}{4\pi i \nu_0}\displaystyle\int_{\frown}\frac{A^-(\nu)}{\nu^2-\nu_0^2}\, d\nu \end{cases} \tag{12}$$

since $\int_{\frown} \frac{A^-(\nu)}{\nu+\nu_0}\, d\nu = 0$, $-\nu_0$ being outside of the domain of integration.

We then develop $\frac{1}{\nu^2-\nu_0^2}$ as

$$\frac{1}{\nu^2-\nu_0^2} = -\frac{1}{\nu_0^2}\left(1+\frac{\nu^2}{\nu_0^2}+\dots+\frac{\nu^{2n}}{\nu_0^{2n}}+\dots\right) \quad \text{on } \Gamma, \text{ since } \left|\frac{\nu}{\nu_0}\right|<1 \tag{13}$$

and

$$\frac{1}{\nu^2-\nu_0^2} = \frac{1}{\nu^2}\left(1+\frac{\nu_0^2}{\nu^2}+\dots+\frac{\nu_0^{2n}}{\nu^{2n}}+\dots\right) \quad \text{on } \Gamma', \text{ since } \left|\frac{\nu_0}{\nu}\right|<1 \tag{14}$$

We then have to consider in (5) terms of the form

$$(15)\quad \int_{\Gamma} \nu^{2n} A^-(\nu)\,d\nu = \int_{-N}^{N} \nu^{2n} A^-(\nu)\,d\nu = 2i \int_0^N \nu^{2n}\, \mathrm{Im}\, A^-(\nu)\,d\nu = 2i \sum_j \beta_j \frac{N^{\alpha_j+2n+1}}{\alpha_j+2n+1}$$

and similarly

$$(16)\quad \int_{\Gamma'} \nu^{-2n} A^-(\nu)\,d\nu = -2i \sum_j \beta_j \frac{N'^{\,\alpha_j-2n+1}}{\alpha_j-2n+1}$$

(here we take n integer > 0).

Furthermore the symmetry properties of $\nu^{2n}A^-(\nu)$ and the FESR written for N and N' give

$$(17)\quad \int_{-N'}^{-N} \nu^{\pm 2n} A^-(\nu)\,d\nu + \int_{N}^{N'} \nu^{\pm 2n} A^-(\nu)\,d\nu = 2i \sum_j \beta_j \left(\frac{N'^{\,\alpha_j \pm 2n+1}}{\alpha_j \pm 2n+1} - \frac{N^{\alpha_j \pm 2n+1}}{\alpha_j \pm 2n+1} \right)$$

Inserting (13) , (14) , (15) , (16) , (17) into the right hand side of (12) and using (9) and the integration of ν^{α_j} on the two horizontal segments (-N', -N) and (N,N') , we then get

$$A^-(\nu_0) = \frac{1}{4\pi i \nu_0} \int_{\frown} \sum_j \beta_j \frac{1-e^{-i\pi\alpha_j}}{\sin \pi\alpha_j} \frac{\nu^{\alpha_j}}{\nu^2-\nu_0^2}\,d\nu = \sum_j \beta_j \frac{1-e^{-i\pi\alpha_j}}{\sin \pi\alpha_j} \nu_0^{\alpha_j}$$

which proves that (11) implies (5) .

We have then proved the Theorem :

Theorem :

A necessary and sufficient condition for the set of FESR (11) to hold for an amplitude $A^-(\nu)$ for two values of $N > R$, is that development (5) holds for $|\nu| > R$. (11) then holds for all $N > R$).

- REFERENCES -

(1) M. JACOB and J. MANDELBROJT
Nuovo Cimento 63A , 279 (1969)

(2) J. MANDELBROJT and S. MANDELBROJT
Nuovo Cimento 1 A , 274 (1971)

(3) R. DOLEN, D. HORN and C. SCHMID
Phys. Rev. Letters 19 , 402 (1967), and
Phys. Rev. 166 , 1768 (1968)

(4) S. MANDELBROJT
C.R.A.S. Série A , tome 279, p. 781 (November 1974)

(5) S. MANDELBROJT
"Séries de Dirichlet, principes et méthodes"
Gauthier-Villars, page 83 (Paris 1969)

(6) H. HARARI
Phys. Rev. Letters 20 , 1395 (1968)

P. FREUND
Phys. Rev. Letters 20 , 235 (1968)

(7) S. MANDELBROJT
"Séries adhérentes, régularisation des suites, applications"
Gauthier-Villars, (1952)