Quark Confinement

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A model of confined quarks is presented. It involves an abelian (or non-abelian) gauge field coupled to a quark field. The theory is quantized on a discrete space-time lattice. It shows quark confinement for strong coupling. Specifically, for strong coupling isolated quarks have infinite mass and quark-antiquark pairs with a large separation \( r \) have an energy proportional to \( r \). The theory has local gauge invariance as an exact symmetry in strong coupling. The strong coupling theory is far from covariant due to the lattice.

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Quark confinement is of great interest at present. It is important that the confinement mechanism be soft. That is, the potential energy of two quarks should be small when they are close together so that quarks can respond as free particles in deep inelastic electron scattering. The potential energy should become large when the two quarks are well separated, to prevent any quarks from escaping a nucleon and becoming observable quark-like particles. Kogut and Susskind have speculated that the potential energy of two quarks (to be precise, a quark and an antiquark) will be proportional to the distance $r$ between them when $r$ is large. This is the result found in one-dimensional quantum electrodynamics for the potential energy of an electron-positron pair.

A model which exhibits quark confinement and a linear potential between quarks can be described in this paper. The model is a gauge field coupled to a quark field, quantized on a discrete space-time lattice. The model has one major drawback: It is noncovariant due to the lattice. The lattice provides an ultraviolet momentum cutoff. The model can be solved explicitly only in a limit.
where all physical masses are much larger than this cutoff, so the explicit solution is far from covariant. The problem of achieving covariance will be discussed further at the end of this paper. The properties of the model are sufficiently remarkable to make it worth study despite the lack of covariance.

A detailed description of the model is given in two preprints. Only some of the principal features will be described here.

The Lagrangian of the model depends on three parameters: the bare coupling constant $g$, the bare quark mass $m_0$, and the lattice spacing $a$. The model can be solved explicitly in two limits. One limit is the weak coupling limit ($g$ small). The other limit is $g$ and $m_0 a$ both large (but not infinite). The principal features of the large $g, m_0 a$ limit are as follows:

1) An isolated quark has infinite mass.

2) A well-separated quark-antiquark pair has a finite energy proportional to their separation $r$, namely proportional to $(\ln g^2)r/a^2$.

3) The theory has local gauge invariance as an exact symmetry of the quantized theory (for weak coupling the local gauge symmetry is spontaneously broken).

4) The infinite quark mass is a consequence of exact local gauge invariance.
Local gauge invariance means the following. For example consider the quark propagator on the lattice. This is

\[ S(n) = \langle \Omega | T \psi_n \psi_0 | \Omega \rangle \]  

(1)

where \( |\Omega\rangle \) is the vacuum state and \( n \) is a four-vector with integral components [e.g. \( n = (0,1,2,1) \)] labelling a lattice site.

A local gauge transformation is a transformation

\[
\begin{align*}
\psi_n &\to e^{-i n \varepsilon} \psi_n \\
\bar{\psi}_n &\to e^{i n \varepsilon} \bar{\psi}_n
\end{align*}
\]  

(2)

(3)

Exact symmetry under this transformation means that

\[ S(n) = e^{i(n \cdot \varepsilon)} S(n) \]  

(4)

which is possible only if

\[ S(n) = \delta_{n0} \]  

(5)

This means quarks cannot propagate, i.e. they have infinite mass. To be precise the model will be quantized in a Euclidean metric (imaginary time). In the Schrödinger representation the propagator, for \( n_0 > 0 \), is

\[ S(n) = \langle \Omega | \psi_n e^{-H_n a} \bar{\psi}_0 | \Omega \rangle \]  

(6)

The vanishing of \( S(n) \) for \( n_0 > 0 \) is the symptom of a more general result:

\[ e^{-H_n a} \bar{\psi}_0 | \Omega \rangle = 0 \]  

(7)

which means \( \bar{\psi}_0 | \Omega \rangle \) is a state with infinite energy.

The model will be quantized using the Feynman path
integral framework, hence the quantity of interest is the action. The first step in defining the model is to construct the action on a discrete space-time lattice. The action will be defined so that it is exactly gauge invariant on the lattice. The crucial part of the lattice action will be the free gauge field piece, but it is convenient to consider first the quark part. Consider the free quark continuum action

\[ A = \int d^4x \, \bar{\psi}(x) (i\gamma^\mu \partial_\mu - m_0) \psi(x) \] (8)

Converting this to a lattice action in the most naive way one makes the replacements

\[ \int d^4x \rightarrow a^4 \Sigma_n \] (9)

\[ \bar{\psi}(x), \psi(x) \rightarrow \bar{\psi}_n, \psi_n \] (10)

\[ \nabla_\mu \psi(x) \rightarrow (\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}})/2a \] (11)

where \( \hat{\mu} \) is a unit lattice vector (length \( a \)) along the axis \( \mu \).

This gives

\[ A = \Sigma_n \left\{ \frac{a^2}{2} \bar{\psi}_n \gamma^\mu (\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}) - m_0 a \bar{\psi}_n \psi_n \right\} \] (12)

The product \( \bar{\psi}_n \psi_{n+\hat{\mu}} \) is not gauge invariant. To make it gauge invariant one uses the standard procedure: one inserts an exponential of the line integral of the gauge field. A lattice approximation to the line integral from \( n \) to \( n+\hat{\mu} \) is just \( g a A_{n\mu} \), where \( A_{n\mu} \) is the gauge field at lattice site \( n \).
The expression

$$\bar{\psi}_n \gamma^\mu \psi_{n+\mu}$$

is gauge invariant provided that the gauge transformation law for $A_{\mu}$ is

$$A_{\mu} \rightarrow A_{\mu} - (\hat{\phi}_{n+\mu} - \phi_n)/g_{\mu}$$

Thus a gauge-invariant action for the quark field is

$$\Lambda = \sum_n \left\{ \frac{g_{\mu}}{2} (\bar{\psi}_n \gamma^\mu \psi_{n+\mu} + \bar{\psi}_{n+\mu} \gamma^\mu \psi_n) - ig_{\mu} n_{\mu} \right\}$$

In this action the variable $g_{\mu} n_{\mu}$ appears as an angle. That is, the action is periodic in the variable $g_{\mu} n_{\mu}$ with period $2\pi$. This is no accident. The gauge transformation variable $\theta_i$ is an angle so to preserve gauge invariance $g_{\mu} n_{\mu}$ is an angle, too.

It is natural to try to write the free gauge field action to preserve periodicity in $g_{\mu} n_{\mu}$. First we need a lattice approximation to $F_{\mu\nu}(x)$, namely

$$F_{\mu\nu} = \frac{1}{a} \left\{ A_{n+\mu,\nu} - A_n A_{\nu,\mu} + A_{\mu,\nu} \right\}$$

Let

$$B_{\mu} = g_{\mu} n_{\mu}$$

$$f_{\mu\nu} = g_{\mu}^2 F_{\mu\nu}$$

$$f_{\mu\nu} = B_{n+\mu,\nu} - B_{n,\nu} B_{n+\mu,\mu} + B_{\mu\nu}$$
Then a gauge field action which is gauge invariant and periodic in the variables $B_{\mu}^n$ is (\(A_{GF}\) means gauge field action)

\[
A_{GF} = \frac{1}{2g^2} \sum_n \sum_\mu \Sigma_\nu \epsilon^{\mu\nu} \text{if } \eta_{\mu\nu}
\]  

(19)

It is evident that this action is periodic. To show that it is gauge invariant, it is useful to represent $B_{\mu}^n$ graphically by a directed line from the lattice site $n$ to the site $n+\hat{\mu}$ (Fig. 1). A gauge transformation acts as follows: at the site $n+\hat{\mu}$, where the line is incoming, one has $B_{\mu}^n \rightarrow B_{\mu}^n + \phi_n^\mu$. At the site $n$ where the line is outgoing the transformation has the opposite sign: $B_{\mu}^n \rightarrow B_{\mu}^n - \phi_n^\mu$. Then $f_{\mu\nu\lambda}$ is represented by a closed square (Fig. 2) with the four corners of the square being the sites $n$, $n+\hat{\mu}$, $n+\hat{\mu}+\hat{\nu}$, and $n+\hat{\nu}$. (Backwards lines represent $a - B$, e.g. from $n+\hat{\nu}$ to $n$ represents $-B_{\nu}^n$.) It is easily seen that the gauge transformations cancel at each corner of the square. Thus $f_{\mu\nu\lambda}$ is gauge invariant.

In the limit $\alpha \rightarrow 0$ the exponential (19) reduces to the usual continuum action. This is seen as follows. For a small, $f_{\mu\nu\lambda}$ is also small [from Eq. (17)], hence

\[
A_{GF} = \frac{1}{2g^2} \sum_n \sum_\mu \Sigma_\nu \left\{ 1 + i g a^2 F_{\mu\nu} \right\} - \frac{1}{2} g^2 a^4 F_{\mu\nu\lambda}^2 - \ldots \} \]  

(20)

A constant term has no effect on physics; it can be ignored. The term linear in $F_{\mu\nu\lambda}$ is zero because $F_{\mu\nu}$ is odd under the interchange $\mu \rightarrow \nu$. Hence the double sum over $\mu$ and $\nu$ of $F_{\mu\nu\lambda}$
gives zero. Thus the dominant term is the $F_{\mu \nu}^2$ term. The factors of $g^2$ cancel, and the factor $\frac{1}{4}$ converts the sum over $n$ to an integral. Higher order terms ($F_{\mu \nu}^3$, etc.) have higher powers of $a$ and therefore vanish in the $a \to 0$ limit.

The above analysis of the $a \to 0$ limit is only valid classically: It was assumed in this analysis that $F_{\mu \nu}$ has a limit (i.e. $F_{\mu \nu}(\infty)$) for $a \to 0$ and this is true classically but not quantum mechanically. This will be discussed further below.

The complete action of the quark-gluon lattice theory is

$$A = \sum_{n} \left[ \frac{1}{g^2} \sum_{\mu} \bar{\psi}^\dagger_{n+\mu} i \gamma_\mu \psi_n - a^0 \sum_{\mu} \bar{\psi}^\dagger_{n+\mu} i \gamma_\mu \psi_n \right] + \frac{1}{2g^2} \sum_{\mu} \bar{\psi} e^{i f_{\mu \nu}}$$

(21)

This choice for the action is not unique. One could have used $f_{\mu \nu}$ for $e^{i f_{\mu \nu}}$, for example; this would violate only the ad hoc requirement of periodicity. However only an exponential form such as given above generalizes easily to nonabelian gauge theories. In the nonabelian case I cannot find a gauge invariant form on the lattice for $f_{\mu \nu}$, but $e^{i f_{\mu \nu}}$ is easily generalized. In the nonabelian case one has a set of variables $B^\alpha_{\mu \nu}$. For each choice of $n$ and $\mu$, the variables $B^\alpha_{\mu \nu}$ parameterize a finite element of the gauge group (not an infinitesimal element).
One replaces $e^{-iB_{\mu}}$ by a unitary transformation $U[B_{\alpha}]$ representing this element in the quark representation. In place of $e^{-iH_{\nu}}$ one substitutes the trace of a product of four unitary transformations:

$$e^{-iH_{\nu}} \rightarrow \text{Tr} U[B_{\alpha}] U[B_{\alpha}] U^{-1} [B_{\alpha}] U^{-1} [B_{\nu}]$$

This trace is gauge invariant under the nonabelian gauge group.

The quantum mechanics of the lattice gauge theory can now be defined using the Feynman path integral. The action defined above is actually in a Euclidean metric, so the resulting path integral will define vacuum expectation values for imaginary times. One can still define the full quantum mechanics in real time using the "transfer matrix" method, which is discussed in Ref. 2. The path integral involves integrating over all values of the $B_{\mu}$ plus an "integral" over the fermion variables $\psi_n$. The fermion integral is discussed in Ref. 3; denote it by $\langle \rangle$. Then the path integral for the quark propagator is

$$S(n) = \frac{\prod \int d\mu \langle \psi_n \bar{\psi}_o e^{A} \rangle}{\prod \int d\mu \langle e^{A} \rangle}$$

where $A$ is the action of Eq. (23). The periodicity of $A$ with respect to the $B_{\mu}$ makes it unnecessary to integrate over more
than 1 period (-\pi to \pi) of B_{\mu\nu}. The denominator is necessary
to remove vacuum-to-vacuum graphs. The denominator will be
denoted by Z in subsequent formulae.

In the weak coupling limit, 1/g^2 is large and therefore
the path integral will be dominated by values of f_{\mu\nu} for which
if_{\mu\nu}/g^2 is near its maximum. This means, essentially, small
f_{\mu\nu}, which allows one to make the expansion of Eq. (20), giving
back the conventional gauge theory action. There are details
that have to be straightened out in this limit but it appears
that the small g limit of the lattice theory agrees with the
usual continuum theory. (This requires that m_a also be small.)

The interesting limit is the limit g large. Calculations
are simple in this limit if m_a is also large. In this case
both the gauge field action and the \gamma_\mu terms of the quark
action can be treated as a perturbation. The lowest order
approximation is to neglect these terms altogether, giving

\[ S(n) = Z^{-1} \prod_{m} \prod_{\mu} \int_{-\pi}^{\pi} d\theta_{\mu} \langle \psi_{n} \bar{\psi}_{\bar{n}} e^{m_{0} \frac{4}{\pi} \sum_{m} \bar{\psi}_{m} \psi_{m}} \rangle \] (24)

Because the action in this expression is quadratic in the fields
\psi_{n} and \bar{\psi}_{\bar{n}}, the \langle \rangle can be calculated by Wick's theorem. The
bare propagator to be used in Wick's theorem is

\[ S_0(n) = (m_{0} a^4)^{-1} \delta_{n,0} \] (25)
The approximation (24) gives \( S(n) = S_0(n) \); therefore
\[ S(n) \approx \delta_{no} \] as predicted earlier.

In higher orders it remains true that \( S(n) \) is proportional to \( \delta_{no} \). To see this one first expands in powers of the \( \gamma_\mu \) terms in the action and picks out terms which give a nonzero fermion integral \( \langle \cdots \rangle \). It is not difficult to see that the fermion integral is nonzero only if one can combine terms of the form \( a^3 \psi^\dagger_1 \gamma^\mu \psi_1 \hat{a}^1 \) (or complex conjugate) to form a path from the origin to the lattice site \( n \); the term \( \psi^\dagger_1 \gamma^\mu \psi_1 \hat{a}^1 \) is represented by a line element from \( m \hat{\mu} \) to \( m \) in this path. The fermion integral then replaces the \( \psi \)'s and \( \overline{\psi} \)'s by products of \( (m_o a^4)^{-1} \). One is left with the integration over \( B \)'s which now has the form

\[
Z^{-1} \prod_{m, \mu} \int_{-\pi}^{\pi} dB_{o, \mu} e^{-iB_{o, \mu}} \exp \left\{ \frac{1}{2\pi^2} \sum_{m, \mu} \sum_{\nu, \nu'} \frac{if_{\mu \nu}}{\Sigma_{n B_{o, \mu}}} \right\}
\]

(26)

where \( \Sigma_{n B_{o, \mu}} \) means a sum over the \( B \)'s on the path from \( 0 \) to \( n \) defined above.

To clarify this formula a bit consider the case that the site \( n \) is a nearest neighbor site to the origin, say \( n = (0,1,0,0) = \hat{1} \). Then one needs only a single term, namely \( -a^3 \psi^\dagger_1 \gamma^1 \psi_1 \hat{a}^{101} \) from the expansion. That is, one writes
\[ \langle \psi_1 \bar{\psi}_0 \exp \left\{ \frac{i}{4m} \left( \bar{\psi}_n \gamma_\mu \psi_{n+1} e^{-i B_{n+1} \mu} - \bar{\psi}_{n+1} \gamma_\mu \psi_n e^{-i B_n \mu} \right) \right\} \rangle = \langle \psi_1 \bar{\psi}_0 \left\{ 1 - a^3 \bar{\psi}_1 \gamma_1 \psi_0 e^{-i B_{01}} - \ldots \right\} \rangle. \quad (27) \]

where only one term from the expansion of the exponential is written out. This bracket when calculated by Wick's theorem gives

\[ (m_o a^4)^{-2} a^3 \langle \bar{\psi}_1 \gamma_1 \psi_0 e^{-i B_{01}} \rangle. \quad (28) \]

Here the sum \( \sum_{n=0}^{\infty} B_{n}\mu \) reduces to the single term \( B_{01} \). When the exponential in (27) is expanded to higher orders there are additional terms contributing to the expression (28); these additional terms are of higher order in \( (m_o a)^{-1} \) and contain more complicated sums of \( B \)'s in the exponential.

The integral (26) is zero to all orders in \( g^{-2} \), except when \( n = 0 \). It is zero in lowest order because

\[ \int_{-v}^{v} d\tau e^{-i B} = 0 \]

and there is one such integral for each link in the path from 0 to \( n \). Expanding in powers of \( g^{-2} \), the next term has as an integrand

\[ \sum_{n=1}^{\infty} \int_{-v}^{v} d\tau e^{-i B} \]

The exponential can be represented by a more complicated path combining a square (for \( f_{\tau \tau_0} \)) with the original path from 0 to \( n \). There is no way a closed square of \( B \)'s can cancel an open path, so in order \( g^{-2} \) there remain some integrals of the form \( \int_{-v}^{v} d\tau e^{-i B} \). This is true in all higher orders as well:
hence \( S(n) \) is proportional to \( \delta_{n0} \).

The underlying reason for \( S(n) \) being proportional to \( \delta_{n0} \) is local gauge invariance. The action given in Eq. (21) is invariant to local gauge transformations. In continuum theories one has to add a gauge-fixing term, which explicitly breaks local gauge invariance. This was not necessary for the lattice theory. The reason for this is the finite range of the \( B_{n\mu} \) integrations. In the continuum theory, where the path integral over \( A_\mu(x) \) has infinite limits, exact local gauge invariance leads to divergent infinite integrations.\(^5\) No such divergences appear in the lattice theory. It was shown in the introduction that local gauge invariance implies that \( S(n) \) is proportional to \( \delta_{n0} \).

In weak coupling the quark propagator is certainly not proportional to \( \delta_{n0} \). What happens in weak coupling is that local gauge invariance is spontaneously broken. This means, as usual with spontaneous breaking, that an infinitesimal gauge breaking term must be added to the action in order to calculate \( S(n) \) correctly. The details of this spontaneous breaking are complex and have not been carefully worked out as yet.\(^6\)

Now consider a well-separated quark-antiquark pair. The quantity of interest is the minimum energy of the pair at a fixed distance \( r \). One has to form a gauge invariant state
containing this pair -- otherwise the state will have infinite energy. A simple gauge-invariant state is

$$\psi_n e^{i \sum_{o \in \mathbb{M}} \mu \psi_o}$$

with the lattice site \( n \) being at time 0 \((n_0 = 0)\) and a spatial distance \( r = |n| a \) from the origin. This state, using the shortest path from 0 to \( n \), turns out to have the minimum energy for two quarks separated by \( r \), in the limit of large \( g \) and large \( m_\alpha \). To determine its energy one can study the time dependence of the Schrödinger representation expression

$$\langle \Omega | \psi_n e^{i \sum_{o \in \mathbb{M}} \mu \psi_o} e^{-Ht} e^{i \sum_{o \in \mathbb{M}} \mu \psi_o} | \Omega \rangle$$

With \( t = n a \) this becomes in the Heisenberg representation

$$\langle \Omega | T \psi_{n_0,0} e^{-i \sum_{o \in \mathbb{M}} \mu \psi_{n_0,0}} \psi_{o,0,0} e^{i \sum_{o \in \mathbb{M}} \mu \psi_{o,0,0}} | \Omega \rangle$$

When this is computed through the Feynman path integral and the fermion integral is computed, one is left with a gauge field expectation value to compute, namely

$$Z^{-1} \int d^4 \mu \int d^4 \nu \exp \left\{ i \sum_{o \in \mathbb{M}} \mu \psi_o \right\} \exp \left\{ \frac{1}{2g^2} \sum \mu \nu \psi_o \right\}$$

where \( \mathbb{M} \) is a sum over the closed path illustrated in Fig. 3. The two horizontal (equal-time) legs are the paths originally present in the vacuum expectation value; the vertical legs are generating by expanding in powers of the \( \gamma_\mu \) terms of the
complete action. The shortest vertical path is used to get
the minimum power of \((m_0a)^{-1}\) from the fermion integral. (This
power has been omitted in the expression (33).)

The gauge field integral is easily determined for large \(g\).
In order to eliminate integrals of the form \(\int dB e^{i\int B} \), it is
necessary to expand to sufficiently high order in \(g^{-2}\) so that
the entire area enclosed by the path of Fig. 3 can be filled
by squares representing \(\tilde{f}_{\mu \nu}\)'s (Fig. 4). There is a factor
\(g^{-2}\) for each \(\tilde{f}_{\mu \nu}\). Hence the expectation value (33) behaves
as

\[
(g^{-2})^2 |n|n_0^{a}
-\text{En}_0^a
\]

This is proportional to \(e^{E}\), where \(E\) is the energy of the
quark-antiquark state. Thus

\[
E = \frac{1}{a} |n| \ln g^2 = r(z, r^2)/a^2 \tag{34}
\]

Hence the speculation of Kogut and Susskind is confirmed
for this model when \(g\) and \(m_0a\) are both large.

Consider finally the limit \(a \to 0\). One would like the
energy \(E(r)\) to be finite in this limit. This is evidently not
true if \(g^2\) is held fixed (and large) as \(a \to 0\); this limit
gives \(E(r) \to \infty\), from Eq. (34).
This is a different result than the earlier (classical) result that a limit exists for \( a \to 0 \). The classical argument fails for the quantum mechanics at large \( g \) because \( B_{\mu i} \) ranges independently from \(-\pi\) to \( \pi\) for each \( n \) and \( \mu \). Only for a negligible subset of this integration range is \( B_{\mu i} \) a slowly varying function of the lattice label \( n \), and hence only for this negligible subset do continuum functions \( A_\mu(x) \) and \( F_{\mu \nu}(x) \) exist.

The only hope to obtain a continuum limit is to let \( g \) decrease as \( a \) decreases; but this means leaving the range of \( g \) for which the large \( g \) approximation is good. Nevertheless one can discuss qualitatively the behavior of \( E(r) \) for intermediate \( g \). First, note that in weak coupling \( E(r) \) is a constant for large \( r \) (twice the quark mass); in other words the coefficient of \( r \) is zero. For sufficiently large coupling \( g \),

\[
E(r) = rc(g^2)/a^2
\]  

(35)

where \( c(g^2) \approx \ln g^2 \) for sufficiently large \( g^2 \) but there are additional terms of order \( 1/g^2, 1/g^4 \), etc. from the full \( 1/g^2 \) expansion.

There must be a critical coupling \( g_c \) where the transition from exact gauge invariance to spontaneously broken gauge invariance takes place. If the vacuum changes continuously
at $g_c$, then one would expect $c(g_c^2) = 0$ so that the energy law changes smoothly to the energy law for weak coupling. In this case one can easily define a limit $g \to g_c$ from above simultaneously with $a \to 0$ such that $E(r)$ has a continuum limit. Unfortunately it is also possible that there is a discontinuous jump at $g_c$ from a symmetric vacuum to a spontaneously broken vacuum, and in this case $c(g_c^2)$ can be nonzero. This is called a "first order" transition. In this situation no continuum limit is possible.

The evidence from a rough mean field calculation$^2,3$ is that the transition is first order. However the analysis has so far been restricted to the case $m \approx 1$, i.e. static quarks. A much broader study needs to be made allowing for small $m \approx$, ordinary $SU(3) \times SU(3)$, and a nonabelian gauge group. This has not been attempted yet.

The importance of the lattice gauge theory is that it provides a specific quantum field theory with confined quarks. It is therefore a model one can use to study what is reasonable in such a theory. Consider for example the potential energy law (34). This form for $E(r)$ has always seemed (to the author, at least) to be in contradiction with locality. In a theory with no zero mass particles (there are no zero mass particles in the strongly coupled gauge theory) it has
usually been the rule that the potential energy of two particles decreases exponentially at large distances, apart from the rest energy.

One can make a rough argument to show that a linear potential energy is a natural consequence of locality given that isolated quarks have infinite mass. In a theory with specific states with infinite energy, the Hamiltonian is not the most well-defined operator and it is better to study $e^{-Ht}$. Consider the expectation value of $e^{-Ht}$ for a quark-antiquark $(q\bar{q})$ state with separation $r$:

$$\langle q\bar{q}|e^{-Ht}|q\bar{q}\rangle$$

For sufficiently distant $q$ and $\bar{q}$, one expects $e^{-Ht}$ to behave like a product of operators, one each $q$ and $\bar{q}$; in other words one expects

$$\langle q\bar{q}|e^{-Ht}|q\bar{q}\rangle = \langle q|e^{-F_{q}}|q\rangle \langle \bar{q}|e^{-H_{\bar{q}}}|\bar{q}\rangle + O(e^{-b(t)r}) \quad (36)$$

This is a cluster decomposition formula so one's intuition is that the error in this formula falls off exponentially in $r$, as indicated. The function $b(t)$ is unknown. If the quark and antiquark had a finite mass $m$, one would then have

$$e^{-E(r)t} = e^{-2mt} + O(e^{-b(t)r}) \quad (37)$$

i.e.

$$E(r) = 2m + O(e^{-b(t)r}) \quad (38)$$
which is the normal situation. But for infinite mass quarks the right hand side of (36) vanishes. In this case one is left with

\[ E(t) = b(t) r \]  

Clearly in this case \( b(t) \) must be proportional to \( t \) and \( E(r) \) is proportional to \( r \). Thus the linear form for \( E(r) \) appears reasonable, and the model provides an example to back up this general argument.

Ongoing studies of the lattice gauge theory model include extensive calculations of the strong coupling expansion\(^7\) and a formulation of the Hamiltonian of a nonabelian gauge theory on the lattice.\(^8\)

I have been helped by many people at Cornell, Orsay, and elsewhere to bring these ideas into focus.
References


4. This can be seen from the approximation (20) which has no $i$ multiplying $F_{\mu\nu}^2$. The anticommutators $\{\gamma_\mu, \gamma_\nu\}$ must be $\delta_{\mu\nu}$ (in place of $g_{\mu\nu}$ or $-\delta_{\mu\nu}$). Also the quark action has unwanted symmetries: it is, for example, symmetric to $\psi_n \to (-1)^n \psi_n$, $\bar{\psi}_n \to (-1)^n \bar{\psi}_n$, $B_{n1} \to B_{n1} + \pi$ which implies that quarks with lattice momentum $\pi$ have the same energy as quarks of zero momentum. This unpleasant feature can be removed by adding a (gauge invariant) lattice approximation to $\int a \bar{\psi}(x)\gamma^2 \psi(x) d^4 x$ to the action; the extra factor $a$ is to make this term vanish in the continuum limit.


6. See Refs. 2 and 3 for incomplete discussions of the spontaneous breaking.


Figure Captions

Fig. 1  Graphical representation of gauge field $B_{\mu}$.

Fig. 2  Graphical representation of $f_{\mu\nu}$.

Fig. 3  Closed loop sum of $B_{\mu}$ appearing in Eq. (33).

Fig. 4  Squares enclosed by the loop of Fig. 3; the loop sum is cancelled by the sum of $f_{\mu\nu\sigma}$ over the 12 enclosed squares.