THE NUCLEAR INTERACTION POTENTIAL IN A FOLDED-YUKAWA MODEL WITH DIFFUSE DENSITIES*

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ABSTRACT

The folded-Yukawa model for the nuclear interaction potential is generalized to diffuse density distributions which are generated by folding a Yukawa function into sharp generating distributions. The effect of a finite density diffuseness or of a finite interaction range is studied. The Proximity Formula corresponding to the generalized model is derived and numerical comparison is made with the exact results.

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1. INTRODUCTION

The study of the interaction between two nuclei has become of increased interest in recent years. Several models have been suggested for the calculation of the nuclear interaction potential. For example, Krappe and Nix[1] have proposed a model in which the interaction energy is calculated as the Yukawa interaction between two nuclear distributions with sharply defined surfaces and uniform interior. The Yukawa interaction is supposed to contain the combined effect of two diffuse matter distributions interacting via some short range interaction. This procedure leads to a simple analytic potential.

A different approach is represented by the Proximity Formula,[2] which expresses the force between two gently curved leptodermous surfaces as a product of a geometrical factor proportional to the mean radius of curvature of the gap between the surfaces and a universal function equal to the interaction energy per unit area between two parallel surfaces. This latter approach is very general and has the advantage of being simple to use, once the problem involving the parallel surfaces has been solved.

In the present paper an analytical model is studied which enables one to gain insight into the accuracy of some of the various approaches, including the two mentioned above. In the model studied, each of the two interacting objects has a diffuse surface which is generated by folding a Yukawa function into a generating sharp-surface distribution. The interaction energy is subsequently obtained on the basis of a two-body Yukawa interaction.

This model can be considered a generalization of the Krappe-Nix[1] model. Hence it permits a test of the idea that the interaction can be represented as a single effective Yukawa interaction acting between sharp-surface distributions. Moreover, the model is sufficiently realistic to present a good test case for the Proximity Formula.[2] So far, such tests have only been carried out for the extreme cases of zero-diffuseness distributions (the Krappe-Nix model) or a zero-range interaction between diffuse surfaces.[2]

The model is presented in Section 2 and the effects of the matter density diffuseness and of the finite interaction range are discussed. In Section 3 the Proximity Formula is derived for the model studied and numerical comparisons are made with the exact results. Section 4 contains some concluding remarks.

2. GENERALIZED FOLDED-YUKAWA MODEL

2.1 General Expressions

The interaction energy V between two matter distributions ρ_1 and ρ_2 is given by

$$V = -C\iint \rho_1(\vec{r}_1)\, Y_{a_0}(r_{12})\, \rho_2(\vec{r}_2)\, d^3\vec{r}_1 d^3\vec{r}_2 \tag{2.1}$$

where the notation

$$Y_a(r) \equiv \frac{1}{4\pi a^3}\,\frac{e^{-r/a}}{r/a} \tag{2.2}$$

has been introduced. The strength of the interaction is governed by the constant C which is positive for an attractive interaction. The matter density distribution ρ_i ($i = 1,2$) is obtained by folding a Yukawa function

of some range a_i into a generating sharp distribution $\hat{\rho}_i$,

$$\rho_i(\vec{r}_1) = \int Y_{a_i}(r_{12})\, \hat{\rho}_i(\vec{r}_2) d^3\vec{r}_2 \tag{2.3}$$

The starting point for the analytical treatment of this model is the observation that the interaction energy may be calculated as the interaction between the two sharp generating densities arising from a composite two-body interaction $-C\mathcal{Y}$. This composite interaction is given as the folding product of the three entering Yukawa interactions,

$$\mathcal{Y} = Y_{a_0} * Y_{a_1} * Y_{a_2} \tag{2.4}$$

(the symbol * denotes the folding).

It should be stressed at this point that the above observation implies that the formulated generalized folded-Yukawa model is conceptually similar to the standard folded-Yukawa model due to Krappe and Nix[1] in that it calculates the energy by folding some kernel into generating sharp densities. The generalized model thus applies to all cases covered by the Krappe-Nix model. In particular, the modified surface-energy prescription suggested by Krappe and Nix[1] can be generalized by employing the composite kernel $\mathcal{Y}$ rather than a single Yukawa function.

2.1a Evaluation of the kernel

The composite kernel $\mathcal{Y}$ is the folding product of three Yukawa functions. It is elementary to derive the following relation for the folding product of two Yukawa functions.

$$(Y_a * Y_b)(r_{13}) \equiv \int Y_a(r_{12}) Y_b(r_{23}) d^3\vec{r}_2$$

$$= \left(1 - \frac{b^2}{a^2}\right)^{-1} Y_a(r_{13}) + \left(1 - \frac{a^2}{b^2}\right)^{-1} Y_b(r_{13}) \qquad (2.5)$$

This relation is valid in the general case of different ranges a and b. For equal ranges the corresponding expression may be obtained by taking the appropriate limit, $b \to a$,

$$(Y_a * Y_a)(r) = \frac{1}{8\pi a^3} e^{-r/a} \equiv E_a(r) \qquad (2.6)$$

The expressions for products involving more than two Yukawas are easily obtained by repeated use of Eq. (2.5). Thus,

$$\mathcal{Y} = Y_{a_0} * Y_{a_1} * Y_{a_2} = \sum_{i=0}^{2} \left(1 - \frac{a_j^2}{a_i^2}\right)^{-1} \left(1 - \frac{a_k^2}{a_i^2}\right)^{-1} Y_{a_i} \qquad (2.7)$$

where $i \neq j \neq k \neq i$. Again, the formula is valid in the general case of different Yukawa ranges. The special cases of two or three equal ranges may be derived by taking the corresponding limit. For example, in the case of $a_1 = a = a_2$ we have

$$Y_{a_0} * Y_a * Y_a = \left(1 - \frac{a^2}{a_0^2}\right)^{-2} Y_{a_0} + \left(1 - \frac{a_0^2}{a^2}\right)^{-1} \left(1 - \frac{a^2}{a_0^2}\right)^{-1} Y_a$$

$$+ \left(1 - \frac{a_0^2}{a^2}\right)^{-1} E_a \qquad (2.8)$$

where the function E_a is defined in Eq. (2.6). It should be noted that the order in which the foldings are done is insignificant.

2.1b The interaction between two spheres

The multiple integral (2.1) may be evaluated by the Fourier-Transform Method. The Fourier transform of the Yukawa function (2.2) is given by

$$\mathcal{F}\left[\frac{1}{4\pi a^3}\,\frac{e^{-r/a}}{r/a}\right] = (1 + a^2k^2)^{-1} \tag{2.9}$$

For a sphere, the generating density is given by $\hat{\rho} = \Theta(r - R)$ where R is the radius. The corresponding Fourier transform is

$$\mathcal{F}[\Theta(r - R)] = 4\pi R^2\,\frac{j_1(kR)}{k} \tag{2.10}$$

where j_1 is the spherical Bessel function of order one. For two interacting spheres of radii R_1 and R_2, the insertion of the Fourier transforms into the basic formula (2.1) leads to the expression

$$V(d) = 8C\,R_1^2\,R_2^2\int_0^\infty \frac{j_1(-kR_1)}{1 + a_1^2k^2}\,\frac{j_0(kd)}{1 + a_0^2k^2}\,\frac{j_1(kR_2)}{1 + a_2^2k^2}\,dk \tag{2.11}$$

Here j_0 is the spherical Bessel function of order zero. The three-dimensional $\vec{k}$-integration, which results in the general case, has been reduced to a one-dimensional k-integration due to the spherical symmetry of the objects and the interaction. The subsequent evaluation of this integral may be performed by employing the Residue Theorem.

The result is a sum of three analogous contributions, one for each of the three entering Yukawa functions. This is most easily understood from the formula (2.7), which can be used to reduce the integral (2.1) to a sum of three integrals; each of these integrals is of the

same form as the single integral entering in the Krappe-Nix[1] model. The final formula may thus be written in the form

$$V(d) = -C \sum_{i=0}^{2} \left(1 - \frac{a_j^2}{a_i^2}\right)^{-1} \left(1 - \frac{a_k^2}{a_i^2}\right)^{-1} \Gamma_{a_i}(R_1)\Gamma_{a_i}(R_2)Y_{a_i}(d) \qquad (2.12)$$

This formula is valid in the case where the two generating spheres are positioned outside each other (which is the case of interest in the present study); otherwise more complicated expressions will result (they may be obtained by the same method). Furthermore,

$$\Gamma_a(R) \equiv 4\pi a^3 \left(\frac{R}{a}\cosh\frac{R}{a} - \sinh\frac{R}{a}\right) \qquad (2.13)$$

is a geometrical factor depending on the size of the sphere. It has the limiting forms, $\Gamma \sim \frac{4\pi}{3} R^3$ for $R \ll a$ and $\Gamma \sim 2\pi a^2(R - a)\exp(R/a)$ for $R \gg a$.

The above formula (2.12) applies to the case of different Yukawa ranges, $a_0 \neq a_1 \neq a_2 \neq a_0$. The various degenerate cases can be obtained by taking the appropriate limit. This leads to relatively complicated expressions. However, for most numerical purposes it is satisfactory to use formula (2.12) with a small arbitrary splitting of the degenerate ranges; this has been done in the present study.

It should be noted that if one of the Yukawa ranges tends to zero the corresponding term simply drops out from the formula (2.12).

2.2 Discussion and Comparison of Special Cases

The general formulation given above covers a number of special cases of interest, in particular the case of zero density differences and the case of zero interaction range. In the following, these two

particular models are discussed relative to a more realistic reference case having diffuse densities and a finite interaction range.

The reference case, which we shall henceforth denote as the standard model, is characterized by the following parameter values,

$$\begin{aligned} \gamma &= 1 \text{ MeV/fm}^2 \\ a_0 &= 1 \text{ fm} \\ a_1 &= a_2 = 1/\sqrt{2} \text{ fm} \end{aligned} \tag{2.14}$$

Here γ is the nominal surface energy coefficient defined as minus half the interaction energy per unit area for two parallel semi-infinite systems at contact. It is more convenient to specify γ than the interaction strength C. With the above values for γ and the ranges a_0, a_1 and a_2, the value of the interaction strength follows as $C = 1/(1 - \frac{7}{16}\sqrt{2})$MeV. The value of 1 MeV/fm^2 chosen for the nominal surface energy is realistic.[2] Furthermore, it is noted that the surface diffuseness implied by the values chosen for a_1 and a_2 is given by $b = 1$ fm which is also a realistic value.[3] (The quantity b is the second surface moment.)

2.2a The density diffuseness

The effect of the density diffuseness may be studied by comparing the case of zero density diffuseness to the standard case. In Fig. 1 is shown the interaction potential given by the standard model and the potential that results when the diffuseness ranges are put equal to zero. The comparison is made for a selection of symmetric binary systems ranging from two oxygen nuclei to two superheavy nuclei. As is seen, the main effect of the disappearance of the surface diffuseness is a reduction in the interaction potential by a factor of roughly three. The overall range of the potential is not altered appreciably in the

in the displayed interesting region.

The two models may be compared in an alternative way, namely by prescribing that they reproduce the same value of the nominal surface energy γ and of the effective kernel diffuseness b_{eff}.[†] The effective kernel diffuseness is defined as the surface diffuseness of the density distribution generated by folding the kernel into a sharp semi-infinite distribution. It is given by the relation

$$b_{eff}^2 = b_0^2 + b_1^2 + b_2^2 \tag{2.15}$$

where $b_i^2 = 2a_i^2$, $i = 0,1,2$. For the standard model the effective kernel diffuseness is equal to $b_{eff} = 2$ fm. In the case of zero diffuseness, the demand that γ and b_{eff} remain the same as in the standard model leads to the following set of parameter values:

$$\begin{aligned} \gamma &= 1 \text{ MeV/fm}^2 \\ a_0 &= \sqrt{2} \text{ fm} \\ a_1 &= a_2 = 0 \end{aligned} \tag{2.16}$$

It should be noted that the value of the range parameter a_0 is close to the value of 1.4 fm employed by Krappe and Nix.[1]

In Fig. 2 is shown a comparison of this zero-diffuseness model and the standard model. In order to study the dependence on the asymmetry, the mass ratio of the binary systems has been varied. Results are displayed for the four values 0, 0.25, 0.5, and 0.75 of the asymmetry

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[†]If the folding functions employed were Gaussians rather than Yukawas the two constraints would be sufficient to make the two models identical. This is because the folding of two Gaussians of ranges a and b leads to a third Gaussian of range c given by the relation $c^2 = a^2 + b^2$. Thus, in a folded-Gaussian model there would be total equivalence between the different models discussed for the folded-Yukawa model.

parameter $\alpha = (A_2 - A_1)/(A_2 + A_1)$; this corresponds to mass ratios of 1/1, 3/5, 1/3, and 1/7.

It is seen that while agreeing on the whole quite well when close to contact (which is a consequence of the demand that the nominal surface energies be the same) the zero-diffuseness model gives rise to a longer tail. This is because the long-range behavior is governed by the largest of the three Yukawa ranges entering in each model.

2.2b The interaction range

A similar study has been carried through for the interaction range. First, in Fig. 3, we show the effect of putting the interaction range equal to zero in the standard model. As is seen, this results as well in an overall reduction of the potential as a more rapid fall-off with distance. This is because the long-range behavior in the standard case is governed by the interaction range a_0 and in the zero-range case by the remaining diffuseness parameters a_1 and a_2, which are smaller.

In Fig. 4 is shown the comparison where γ and b_{eff} are held constant. This corresponds to the set of values

$$\begin{aligned} \gamma &= 1 \text{ MeV/fm}^2 \\ a_0 &= 0 \\ a_1 &= a_2 = 1 \text{ fm} . \end{aligned} \tag{2.17}$$

As would be expected, not only the values at contact agree well but also the slopes (the fall-off range).

For completeness, a corresponding comparison with the zero-diffuseness model is included in Fig. 5. It might be added that a better agreement concerning the slope would be obtained if the maximum value of the range parameters were equal in the two models.

3. PROXIMITY FORMULA

The Proximity Formula[2] reads

$$V_p(s) = 2\pi \bar{R} \int_s^\infty e(s')ds' = 2\pi \bar{R}\, \delta(s) \tag{3.1}$$

Here s is the separation between the effective surfaces which are located at the radii C_1 and C_2. The geometrical factor $\bar{R}$ is given in terms of the effective radii by the relation $\bar{R}^{-1} = C_1^{-1} + C_2^{-1}$. Furthermore, the function $e(s)$ is the interaction energy per unit area between two parallel semi-infinite surfaces with a separation equal to s.

3.1 Derivation of Formula

It is first noted that

$$\iint \frac{1}{4\pi a^3} \frac{e^{-r/a}}{r/a} d^3\vec{r} = \frac{1}{2a} e^{-|x|/a} \tag{3.2}$$

Hence the semi-infinite density distributions are of the form

$$\rho(x_1) = \int \frac{1}{2a} e^{-|x_{12}|/a} \Theta(x_2 - X)dx_2$$

$$= \begin{cases} \tfrac{1}{2} e^{-(x_1 - X)/a}, & x_1 \geq X \\ 1 - \tfrac{1}{2} e^{(x_1 - X)/a}, & x_1 \leq X \end{cases} \tag{3.3}$$

The separation s is related to the semi-infinite surface locations X_i by $s = X_1 - X_2$. Moreover, the interaction energy per unit area $e(s)$ is given by

$$e(s) = -C \iint \rho_1(x_1) \frac{1}{2a_0} e^{-|x_{12}|/a_0} \rho_2(x_2)dx_1dx_2 \tag{3.4}$$

The exponentials may easily be folded together by repeated use of

the formula

$$\int \frac{1}{2a} e^{-|x_{12}|/a} \frac{1}{2b} e^{-|x_{23}|/b} dx_2$$

$$= \left(1 - \frac{b^2}{a^2}\right)^{-1} \frac{1}{2a} e^{-|x_{13}|/a} + \left(1 - \frac{a^2}{b^2}\right)^{-1} \frac{1}{2b} e^{-|x_{13}|/b} \quad (3.5)$$

which is valid for $a \neq b$. Thus one obtains the following expression

$$e(s) = -C \sum_{i=0}^{2} a_i^2 \left(1 - \frac{a_j^2}{a_i^2}\right)^{-1} \left(1 - \frac{a_k^2}{a_i^2}\right)^{-1} \frac{1}{2a_i} e^{-s/a_i} \quad (3.6)$$

The corresponding expression for the integrated function $\&(s)$ is

$$\&(s) = -C \sum_{i=0}^{2} a_i^3 \left(1 - \frac{a_j^2}{a_i^2}\right)^{-1} \left(1 - \frac{a_k^2}{a_i^2}\right)^{-1} \frac{1}{2a_i} e^{-s/a_i} \quad (3.7)$$

As in the preceding section, the proper limit must be taken if some of the Yukawa ranges a_i are equal.

The geometrical factor $\bar{R}$ entering in the Proximity Formula is given in terms of the effective radius C_i which is taken as the average of the density profile location and the potential profile location.[2]

For a *leptodermous* sphere the surface profile radius C is related approximately to the equivalent sharp radius R by $C \approx R - b^2/R$ when b is the second surface moment.[3] For a distribution generated by folding a Yukawa function of range a into a sharp semi-infinite distribution, b and a are related by $b^2 = 2a^2$. Hence, the approximate density profile location is given by $C_\rho \approx R - 2a^2/R$.

The potential is generated from the density by folding with a Yukawa function of range a_0. This merely increases b^2 by $2a_0^2$.

Consequently, the approximate potential profile location is given by $C_V \approx R - (2a^2 + 2a_0^2)/R$.

It thus follows that the effective surface locations C_i to be used in the Proximity Formula (3.1) are given by

$$C_i = R_i - \frac{a_0^2 + 2a_i^2}{R_i} \tag{3.8}$$

The separation s between the effective surfaces is given by $s = d - C_1 + C_2$ where d is the distance between the two centers.

3.2 Comparison with Exact Results

The derived Proximity Formula has been compared numerically with the various exact models discussed in Section 2.

First, in Fig. 6, the comparison is made for the standard model. On the whole the reproduction of the exact results is good, except for systems containing very light nuclei. It should be stressed that the absolute as well as the relative agreement is good, over the wide range of nuclear combinations considered and over distances where the potential changes by more than a factor of a hundred. This result provides a strong support for the applicability of the Proximity Formula[2] to the study of heavy-ion interactions.

A similar comparison has been carried though for the Krappe-Nix[1] zero-diffuseness model (Fig. 7). For this model the Proximity Formula provides a good approximation to the value of the interaction potential at contact but gives rise to an appreciable overestimation for larger separations; typically the proximity potential is 50% above the exact

potential when the surfaces are separated by 5 fm.

At this point it may be added that the Proximity Formula will generally underestimate the range of the interaction potential because the function $\Phi(s)$ is calculated for semi-infinite surfaces. In the simple case of a Yukawa interaction the emerging distance dependence has a factor of d too much. Of course, a simple ad hoc correction for this particular failure could be included for the special case of the zero-diffuseness model.

The Proximity Formula has also been applied to the zero-range model. Some results are displayed in Fig. 8. As would be expected, the quality of agreement with the exact results lies in between that of the standard model (Fig. 6) and the zero-diffuseness model (Fig. 7).

4. CONCLUDING REMARKS

The folded-Yukawa model[1] has been generalized to the case where the two interacting objects have a diffuse surface, generated by folding a Yukawa function into a sharp generating distribution. While only slightly more complicated to treat than the usual folded-Yukawa model, the generalized model is conceptually more appealing. Furthermore, it includes the folded-Yukawa model as well as the δ-interaction model as special cases. This facilitates the analytical study of effects associated with the density diffuseness and the interaction range.

Moreover, the generalized folded-Yukawa model provides a realistic case for testing approximative representations of the interaction potential. In this paper the Proximity Formula[2] was investigated; its quantitative validity was generally supported.

Finally, it should be recalled that the possible applicability of the generalized folded-Yukawa model extends beyond the nuclear interaction potential. For example, the modified-surface-energy prescription suggested by Krappe and Nix[1] can be easily generalized along the same lines.

ACKNOWLEDGMENTS

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FIGURE CAPTIONS

Fig. 1. The effect of the surface diffuseness. The full lines indicate the interaction potentials obtained with the standard model, (2.14), for a selection of symmetrical binary systems (with mass numbers A_1 and A_2 indicated at the beginning of each curve). The abscissa is the distance between centers d (in fermi) and the ordinate is the interaction potential V (in MeV), on a logarithmic scale. The dashed lines indicate the potential resulting from putting the surface diffuseness equal to zero.

Fig. 2. The potential obtained by the standard model (2.14) (full lines) and the zero-diffuseness (2.16) (dashed lines). Figs. 2a-d show the results for four different values of the mass ratio.

Fig. 3. The effect of the interaction range. The standard model (2.14) (full lines) and what results from putting the interaction range equal to zero (dashed lines).

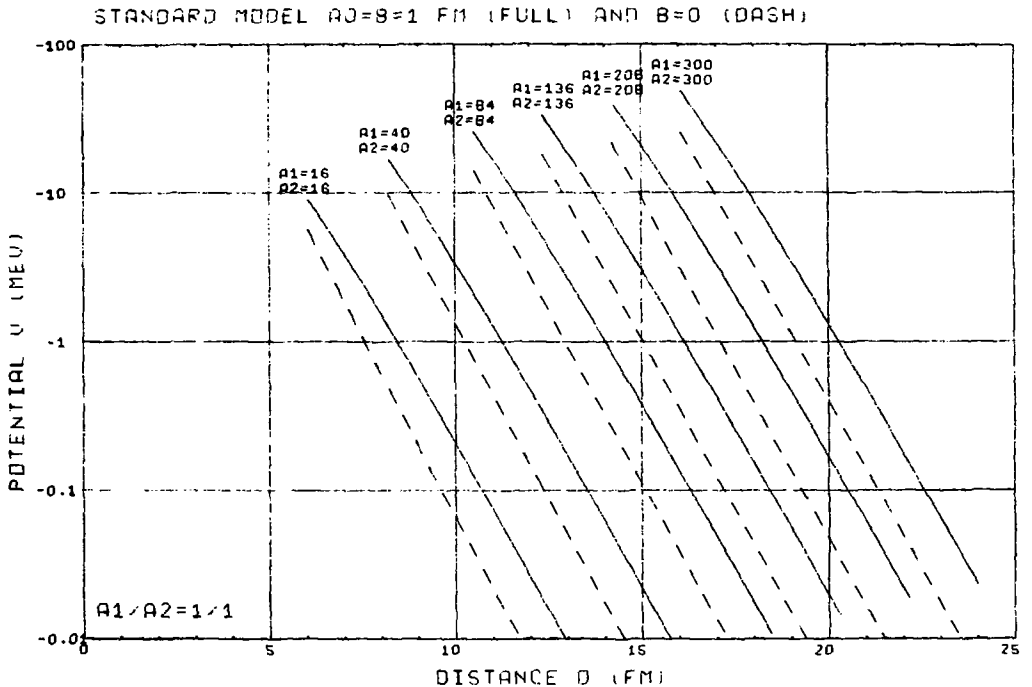
Fig. 4. The standard model (2.14) (full lines) and the zero-range model (2.17) (dashed lines), for two different mass ratios.

Fig. 5. The zero-diffuseness model (2.16) (full lines) and the zero-range model (2.17), for two different mass ratios.

Fig. 6. Standard model (2.14). The exact interaction potential is given by the full curves while the proximity expression is indicated by the dashed curves. Results for four different mass ratios are displayed.

Fig. 7. Zero-diffuseness model (2.16) for four mass ratios. Full curves: exact potential, dashed curves: proximity potential.

Fig. 8. Zero-range model (2.17), for two mass ratios. Full curves, exact potential; dashed curves, proximity potential.



XBL 759-8283

Figure 1

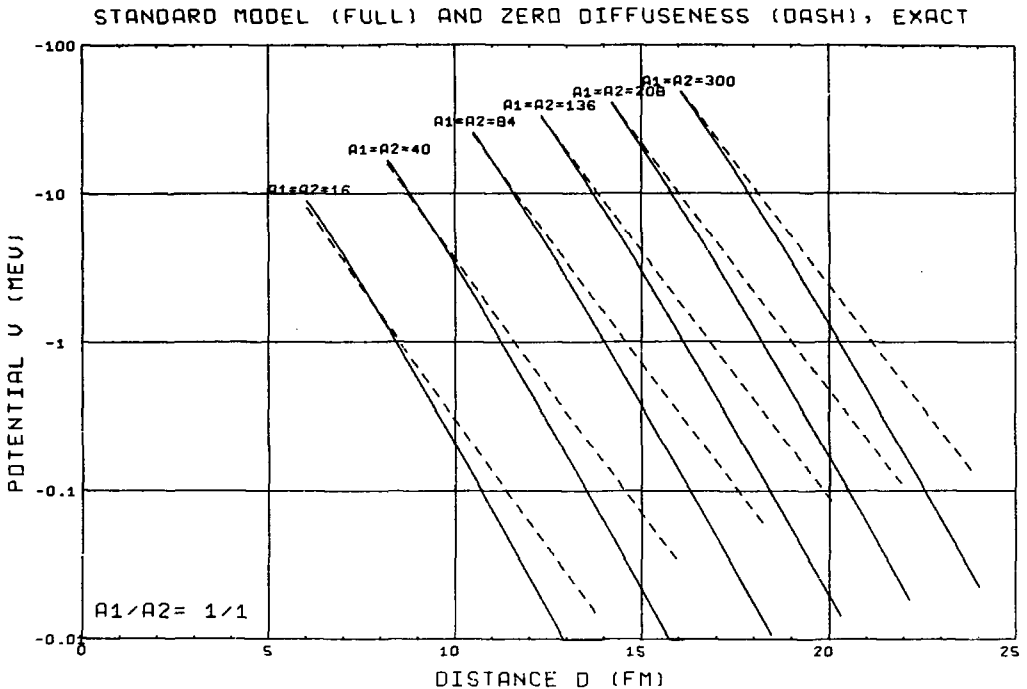


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Figure 2a.

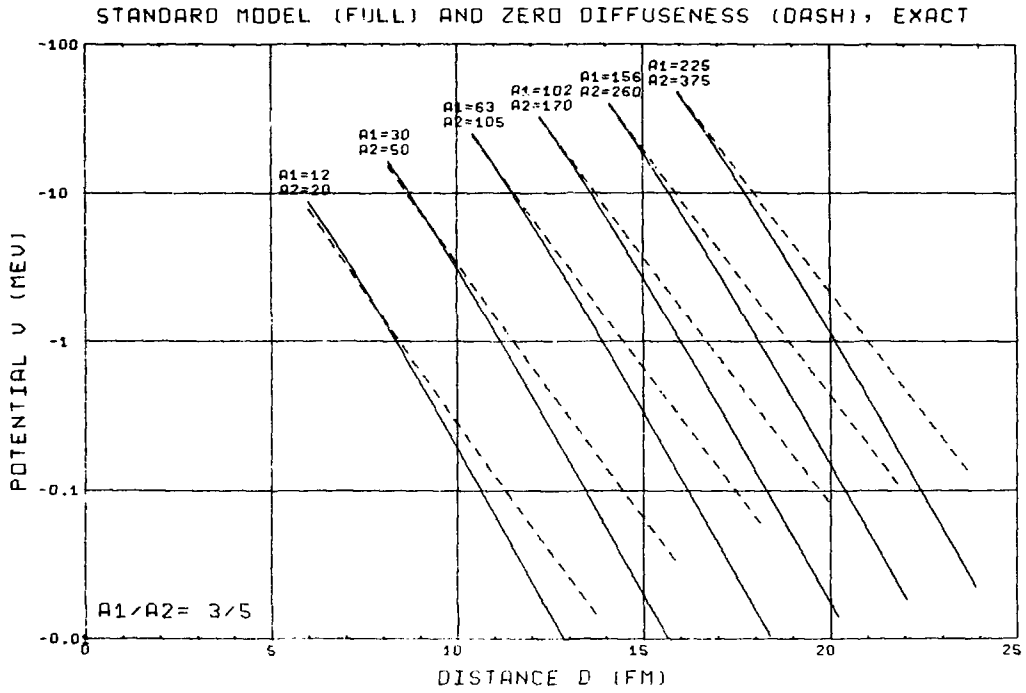

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Figure 2b.

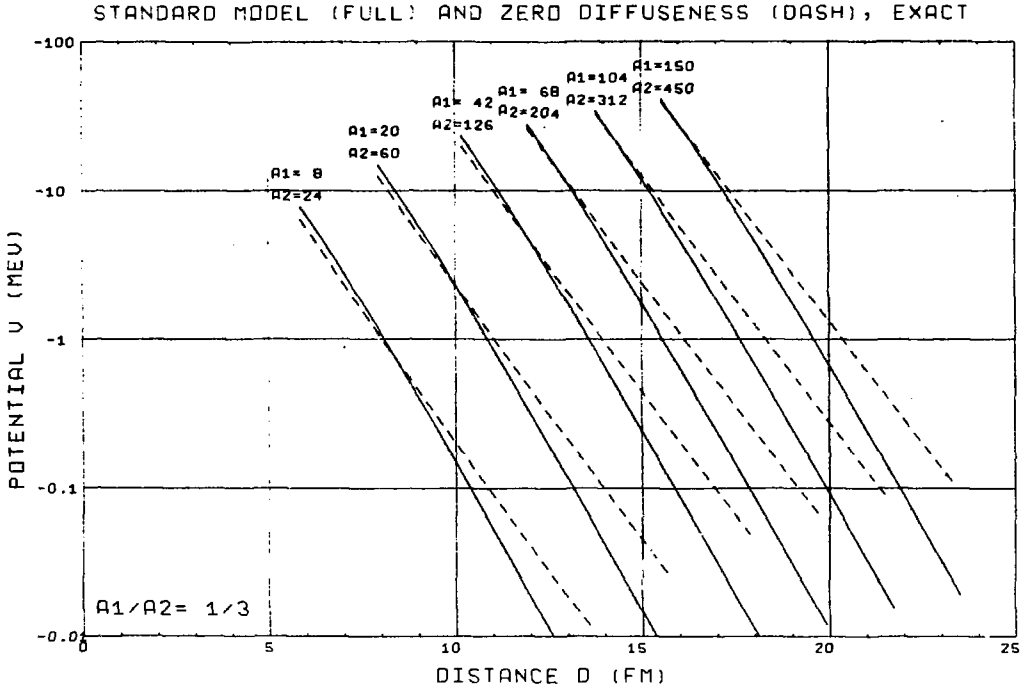

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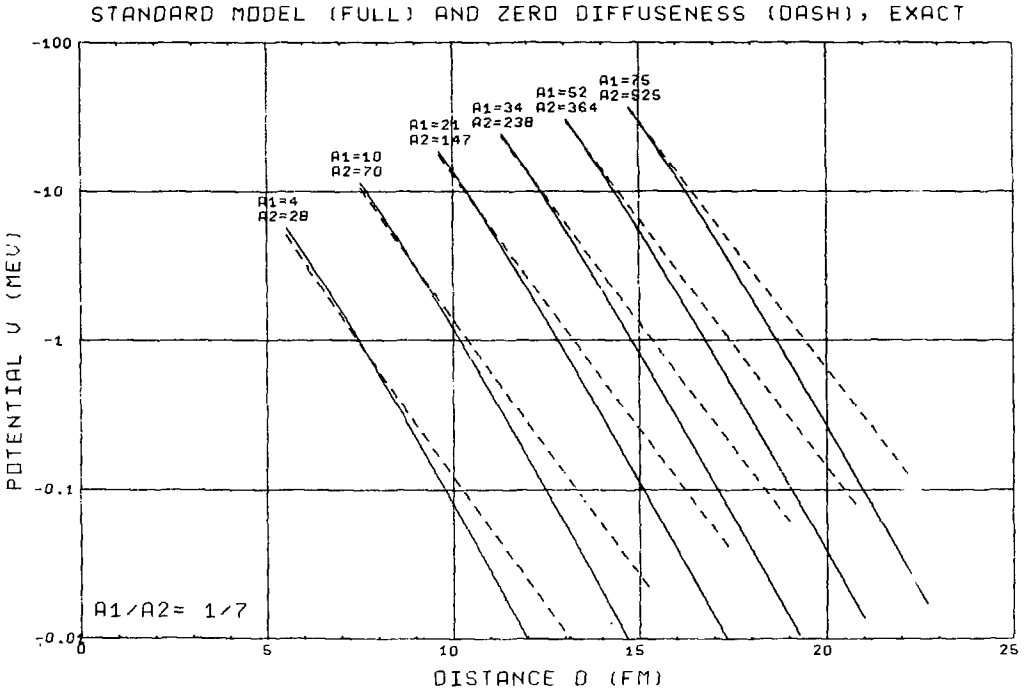
Figure 2c.



XBL 759-8238

Figure 2d.

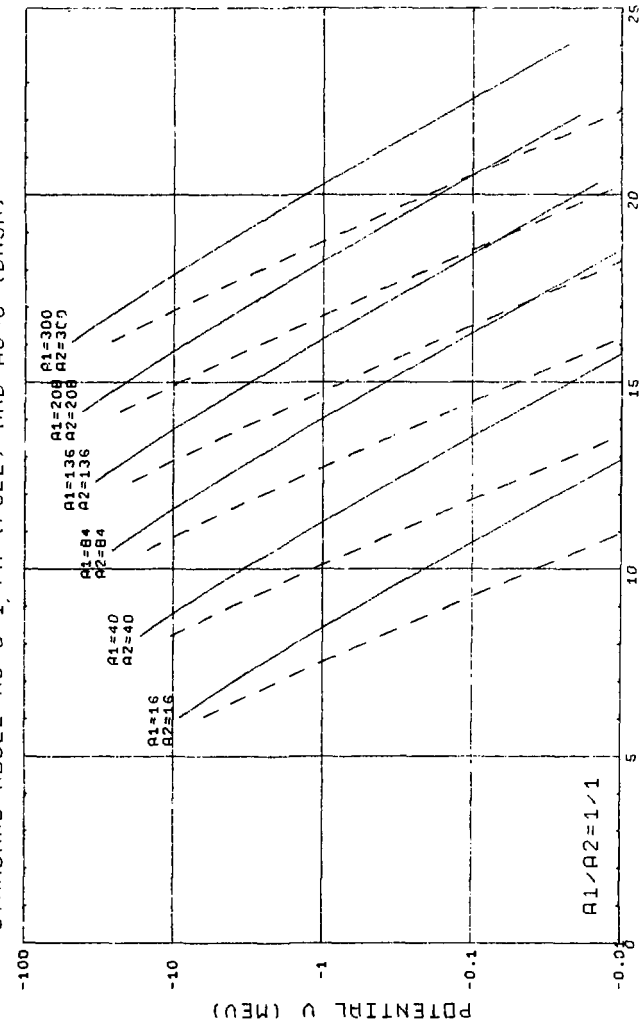


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Figure 3.

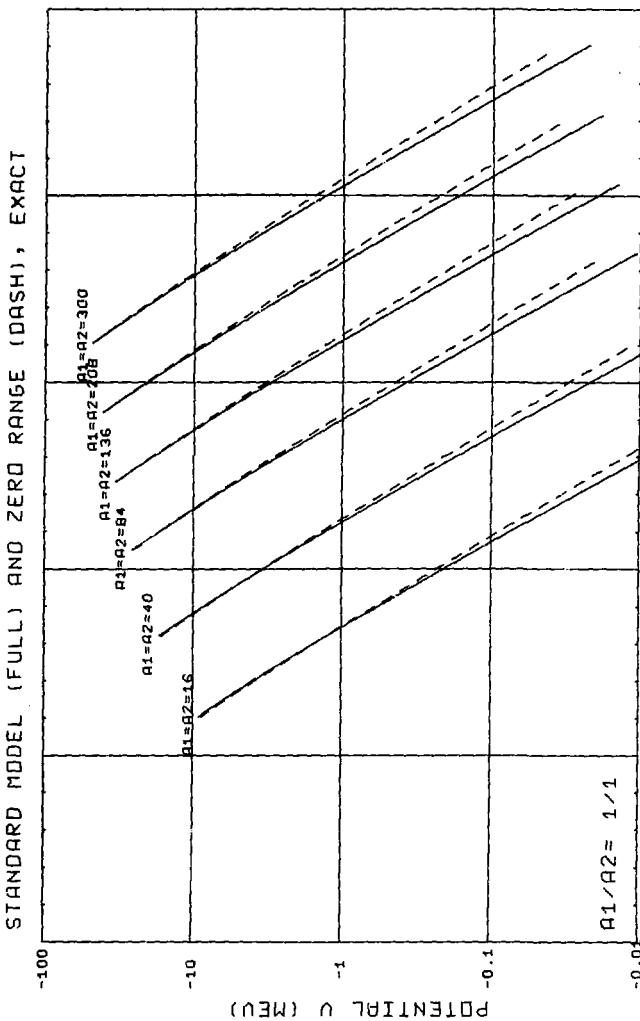

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Figure 4a.

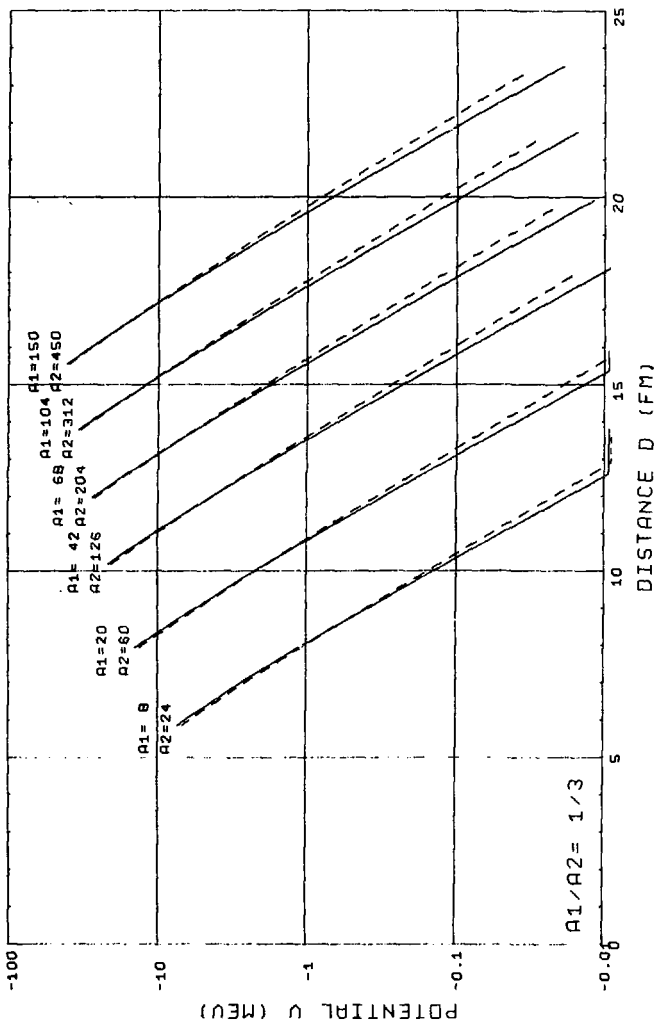

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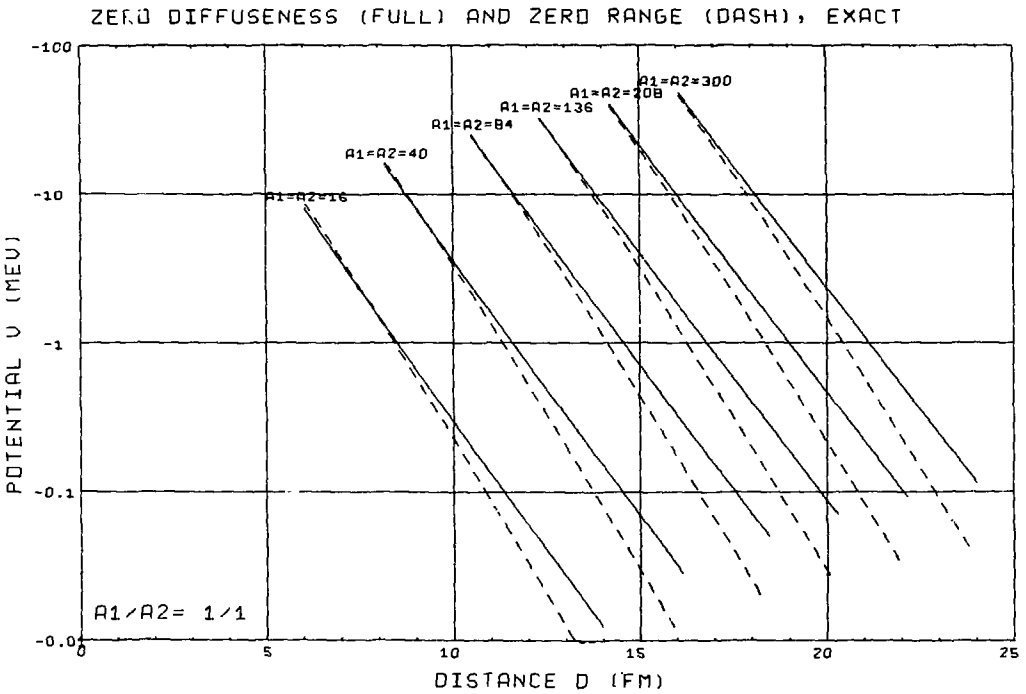
Figure 4b.



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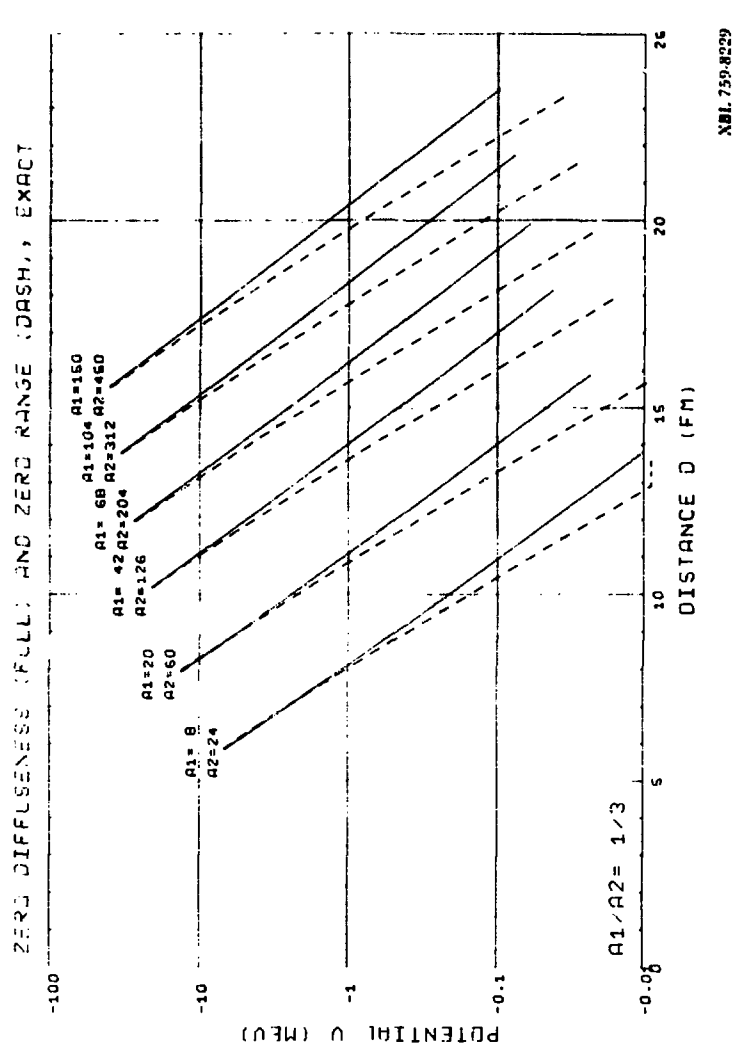

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Figure 5b.

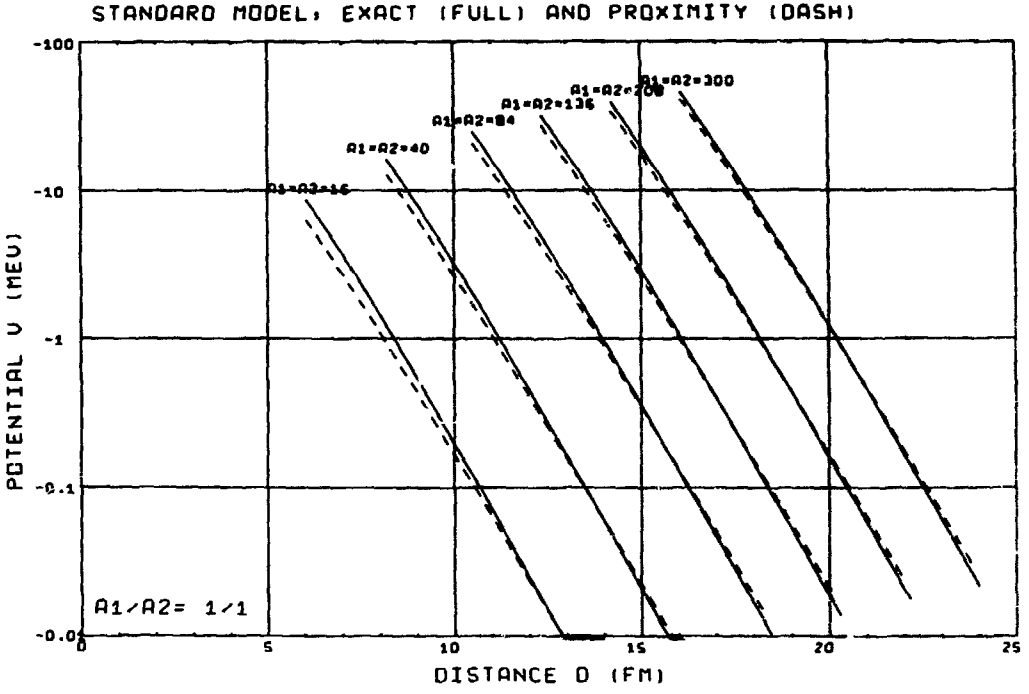

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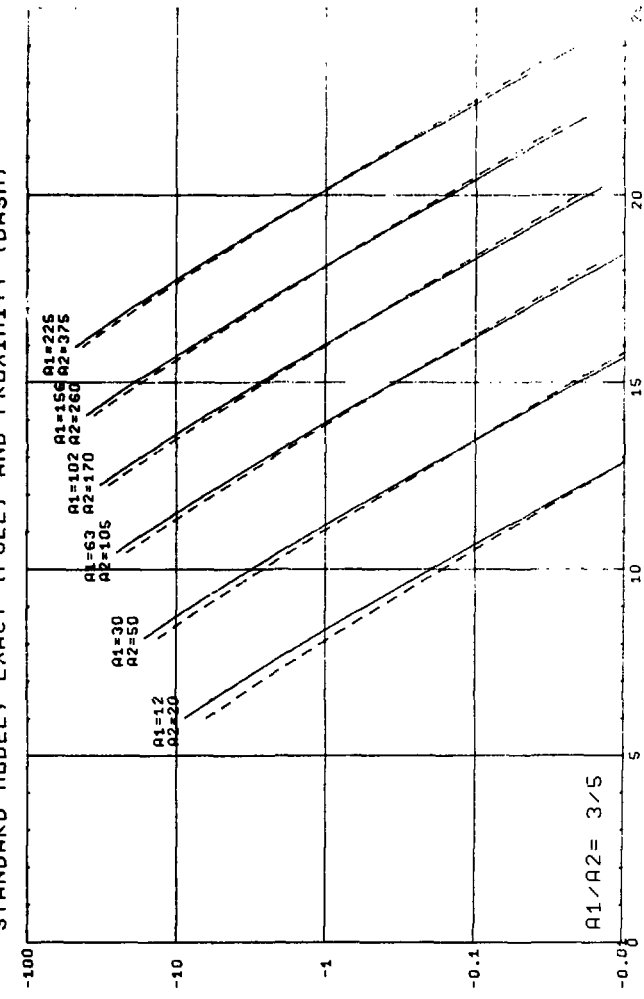
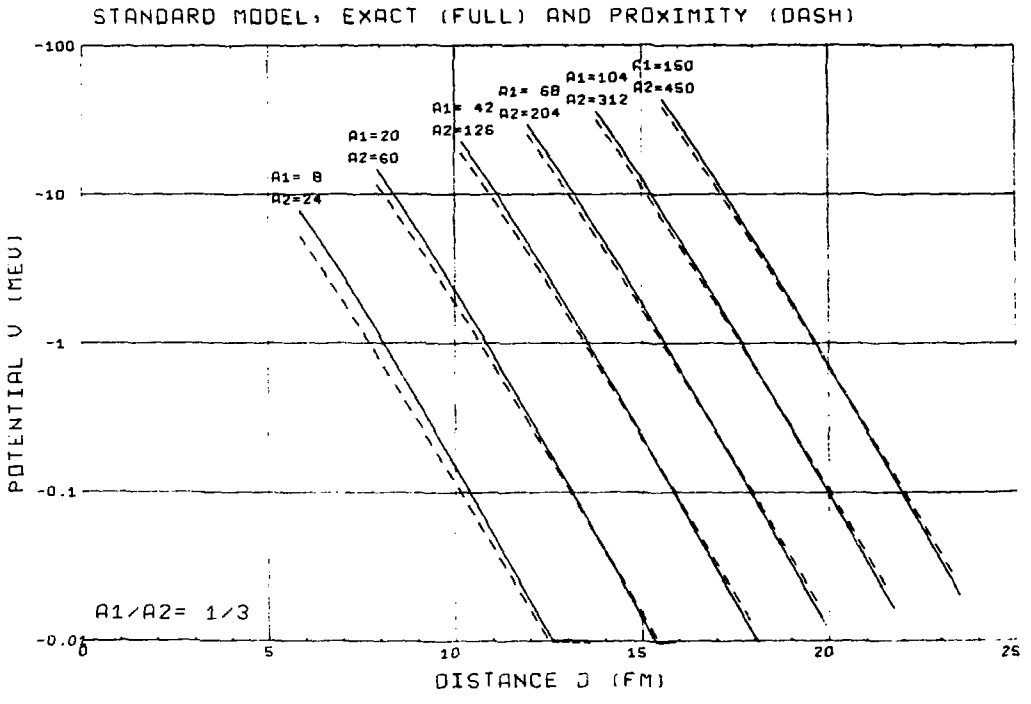
Figure 6a.



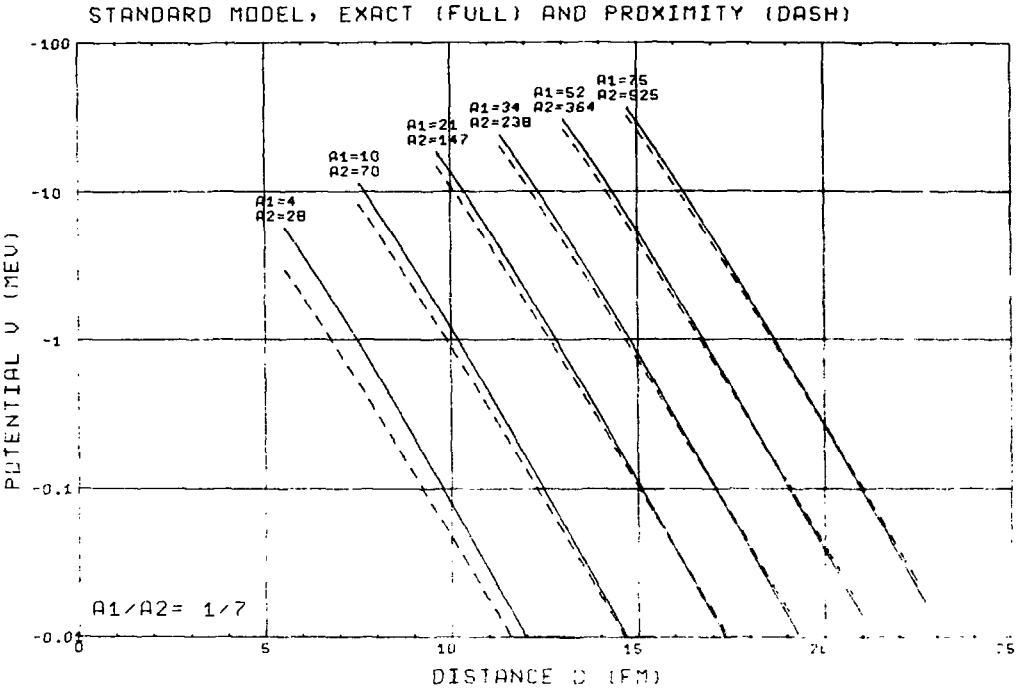
XBL 759-8236

Figure 6b.



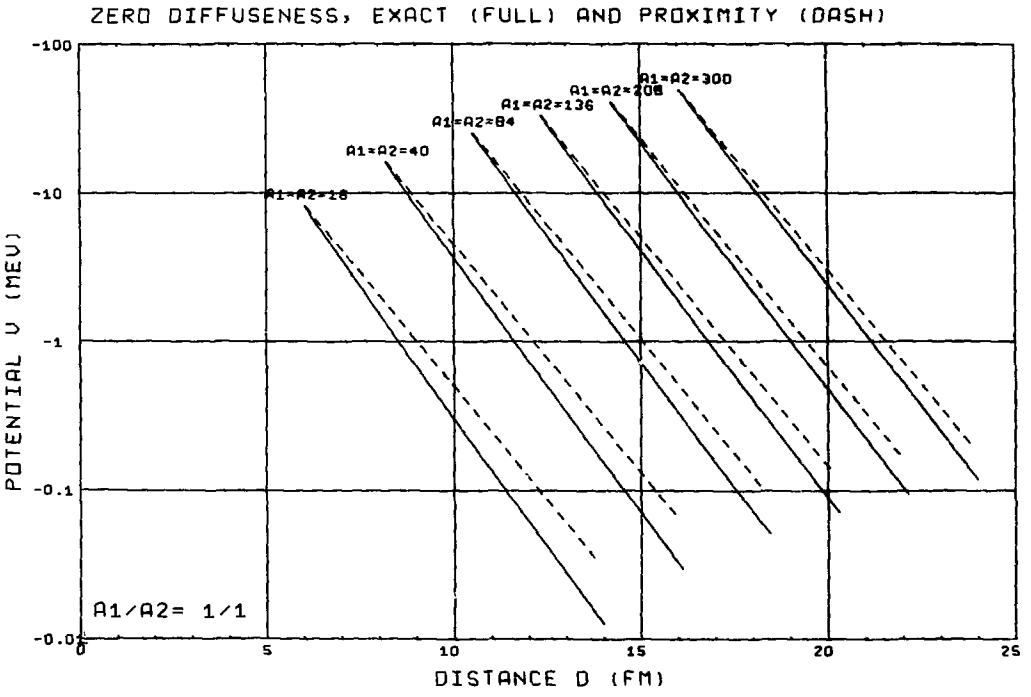
XBL 759-8227

Figure 6c.



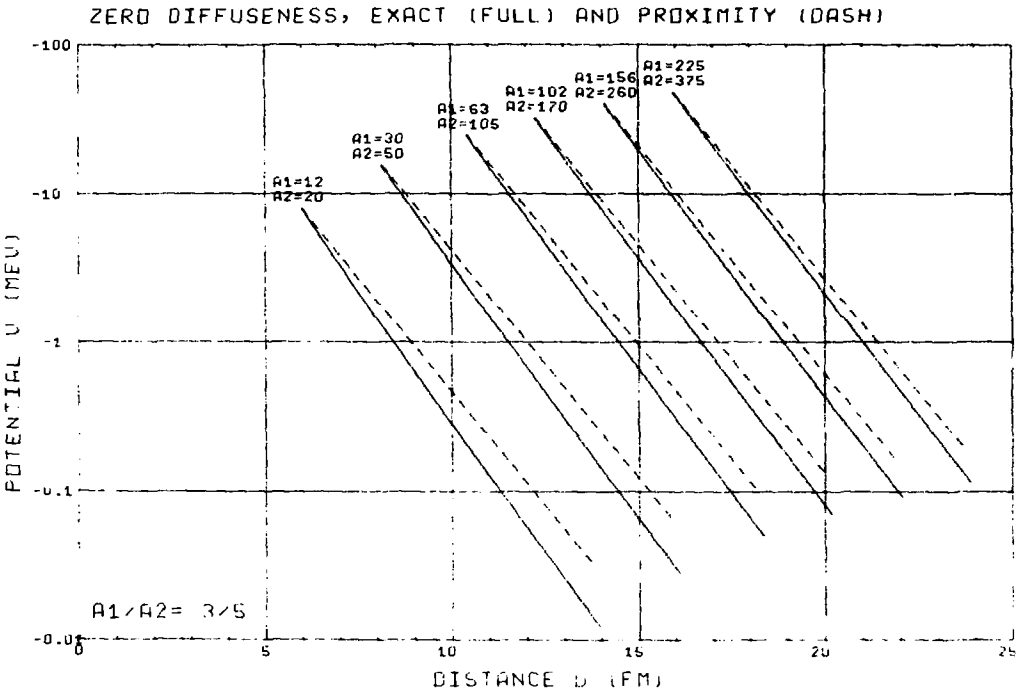
XBL 759-8233

Figure 6d.



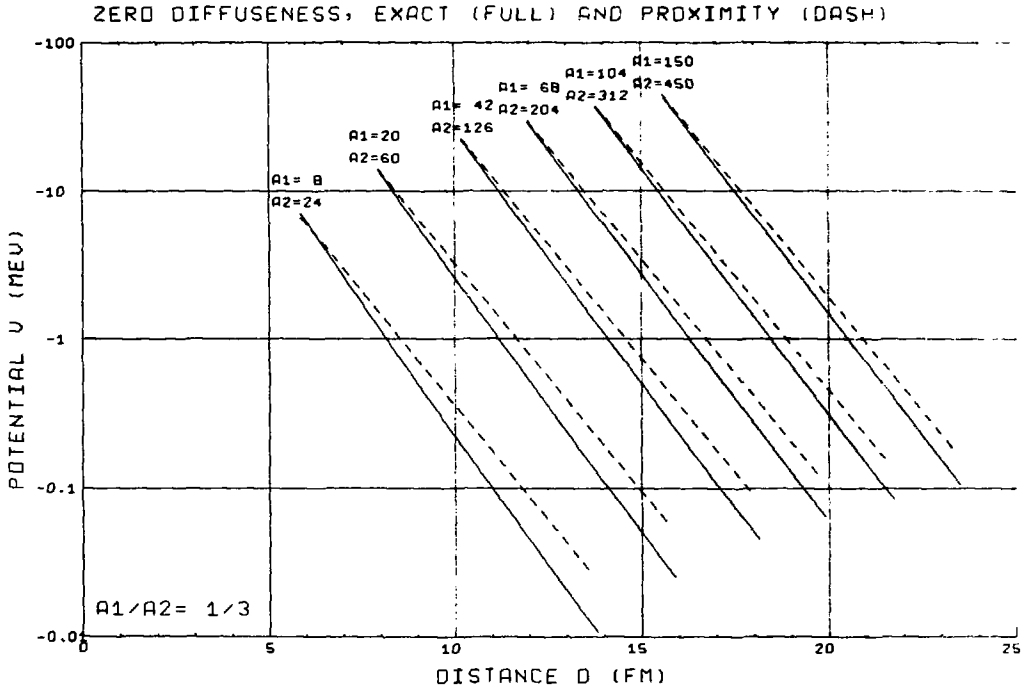
XBL 759-8222

Figure 7a.



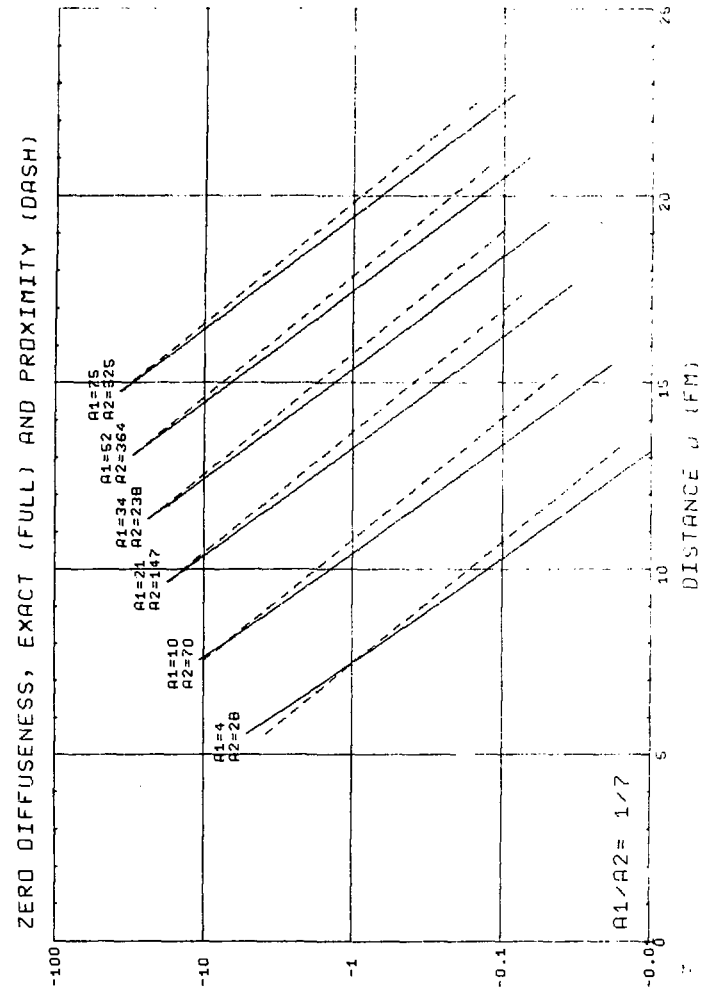
XBL 759-8235

Figure 7b.



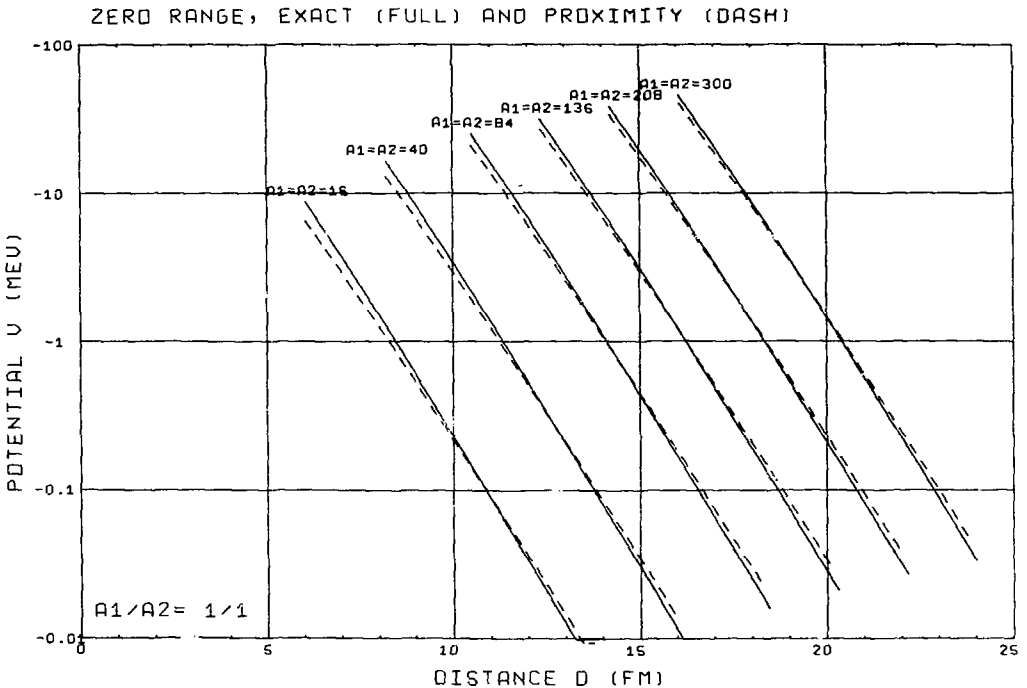
XBL 759-8231

Figure 7c.



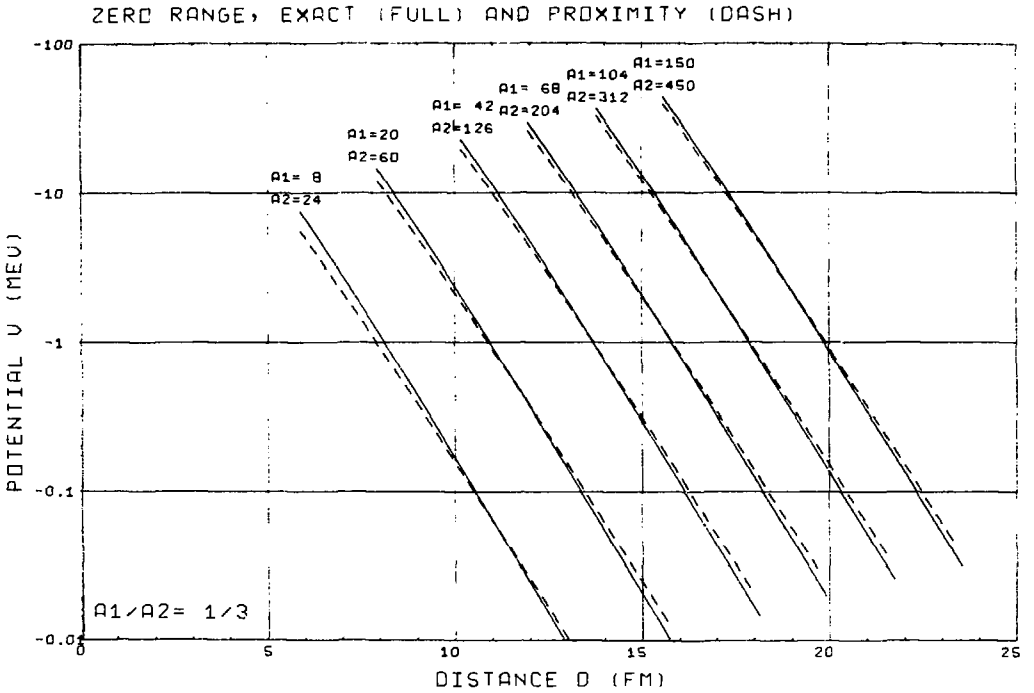
XBL 759-8234

Figure 7d.



XBL 759-8224

Figure 8a.



XBL 759-8232

Figure 8b.