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IFA-FT-117-1975

RENORMALIZATION GROUP EQUATIONS WITH MULTIPLE COUPLING CONSTANTS

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Summary

The main purpose of this paper is to study the renormalization group equations of a renormalizable field theory with multiple coupling constants. A method for the investigation of the asymptotic stability is presented. This method is applied to a gauge theory with Yukawa and self-quartic couplings of scalar mesons in order to find the domains of asymptotic freedom. An asymptotic expansion for the solutions which tend to the origin of the coupling constants is given.

1. Introduction

The purpose of this paper is to analyse the renormalization group equations for theories with more than one coupling constant [1]. This study is not trivial since the number of differential equations increases, involving many Callan-Symanzik functions β_i which depend on all the couplings.

We shall investigate the behaviour of the solutions of the renormalization group equations in the vicinity of the critical points. The method is quite general, but we shall exemplify it on a gauge theory involving scalar mesons with Yukawa and self-quartic couplings. Whe shall investigate the asymptotic behaviour of the solution in the vicinity of the origin in the space of the coupling constants.

We are interested to investigate the asymptotic behaviour of the theory, having in view the stability in various directions around the origin of the coupling constants. In particular, we find that a gauge theory with scalar and Yukawa coupling constants is not asymptotically stable in the usual sense [2]. But there are some domains in the space of coupling constants which contain trajectories which tend asymptotically to the origin of couplings

The method is quite simple, involving only elementary calculations. On the other hand, it exhausts all the domains and directions of asymptotic freedom (asymptotic stability).

The idea of the calculations is, in a way, similar to that from paper [3] where the behaviour of some ratios of coupling constants is investigated. But in the present formalism we are able to find all the domains and directions of asymptotic freedom giving their characterization and dimensionality. In this way some additional particular directions of asymptotic freedom found in the literature [4] are obtained in a unified manner.

We mention also that the formalism can be applied to the study of the infrared behaviour of the solutions of the renormalization group equations

Finally we note that the present method offers some information concerning the asymptotic expansions of the solutions which tend to the origin in the space of the coupling constants. These expansions give more than the usual first terms and can be useful in some physical applications.

2. Multiple coupling constants

Let us consider a general renormalizable gauge theory with more than one coupling constant. In addition to the gauge coupling g there are scalar quartic couplings λ ($\lambda_{ijkl}\ \varphi_i\varphi_j\varphi_k\varphi_l$) and Yukawa couplings h ($h_i\ \bar{\psi}\psi\varphi_i$).

The renormalization group equations for the effective coupling constants are of the form [3]:

$$\frac{dg}{dt} = \beta_g = -\frac{1}{2} b_0 g^3 + \text{higher order terms in all couplings}$$

$$\frac{dh}{dt} = \beta_h = P_3(h) + Chg^2 + \text{higher order terms in all couplings}$$

$$\frac{d\lambda}{dt} = \beta_\lambda = P_2(\lambda) + C'\lambda g^2 + C''g^4 + P_2'(h)\lambda + P_4(h) + \text{higher order terms in all couplings} \quad (1)$$

where $t \equiv -\ln s$ and s is the scale by which the renormalization point is changed. C, C', C'' are constants, P_2, P_2' are quadratic forms, P_3 is a cubic form and P_4 is a quartic form in the arguments. Here h and λ can be any h_i and λ_{ijkl} respectively and the corresponding sumation on the indices is understood.

We are interested in studying the behaviour of the solutions of the system (1) in the vicinity of the critical point $g = h = \lambda = 0$. Generally, such a study is quite difficult since the number of differential equations is great and the first terms in the right hand side are not linear in the coupling constants.

But one can notice that the lower order terms from Eqs. (1) can be written as homogeneous polynomials X_m of a given degree m such that the system of differential equations becomes in vector form

$$\frac{dx}{dt} = X_m(x) + f(x) \quad (2)$$

where x, X_m, f are n-vectors and $f(x)$ is a sum of homogeneous polynomials $P_k(x)$ of degree k higher than m

$$f(x) = \sum_{k \geq m+1} P_k(x) \quad (3)$$

For the first two equations from (1) the degree m is 3, but for the complete system the equivalent system (2) can be obtained through a change of variables

$$\tilde{g} = g^2 , \quad \tilde{h} = h^2$$

and in this way we get equations for $\tilde{g}$, $\tilde{h}$ and λ with $m = 7$.

In that follows we shall study the asymptotic stability of the system (2). In order to achieve this task a method is to reduce the system (2) to a linear one in the first approximation [5,6]. For this purpose we consider the algebraic equations

$$x + X_m(x) = 0 \qquad (4)$$

which can have real solutions. Let us assume that such a real solution x_0 exists with x_0 not identically zero. In this case we can make the following change of functions

$$x(t) = z(t)[x_0 + y(t)] \qquad (5)$$

where $z(t)$ is a positive solution of the equation

$$\frac{dz}{dt} = - z^m \qquad (6)$$

From Eqs. (2) and (5) we obtain

$$\frac{dx}{dt} = \frac{dz}{dt}[x_0 + y] + z\frac{dy}{dt} = X_m[z(x_0 + y)] + 6[z(x_0 + y)] \quad . \qquad (7)$$

Using Eqs. (6) and (3) we get

$$- z^m[x_0 + y] + z\frac{dy}{dt} = z^m X_m(x_0 + y) + \sum_{k\geq m+1} z^k P_k(x_0 + y) \, . \qquad (8)$$

Making a power expansion around $y = 0$ we obtain

$$\frac{dy}{dt} = z^{m-1}\{x_0 + y + X_m(x_0) + \frac{\partial X_m}{\partial x}(x_0).y + \ldots.$$

$$+ \sum_{k\geq m+1} z^{k-m}[P_k(x_0) + \frac{\partial P_k}{\partial x}(x_0)\, y + \ldots]\} \quad . \qquad (9)$$

Taking into account the definition of x_0 as a solution of of Eq. (9) we have

$$\frac{dy}{dz} = z^{m-1} \left(\left[I + \frac{\partial X_m}{\partial x} (x_0) \right] y + z P_{m+1} (x_0) + \dots \right) . \qquad (10)$$

Eq. (6) can be easily solved with the initial condition $z(0) = z_0$ as

$$z^{m-1} = z_0^{m-1} \cdot \left[1 + (m-1) z_0^{m-1} t \right]^{-1} \qquad (11)$$

With the aid of Eq.(11) a linearization of Eq. (10) can be done making the substitution

$$\tau = \frac{1}{m-1} \ln \left[\frac{1}{z_0^{m-1}} + (m-1) t \right] = - \ln z \qquad (12)$$

obtaining finally the system

$$\frac{dy(\tau)}{d\tau} = \left[I + \frac{\partial X_m}{\partial x} (x_0) \right] y(\tau) + z P_{m+1} (x_0) + \dots \qquad (13)$$

$$\frac{dz}{d\tau} = - z$$

which is linear in the first approximation.

For such a system the asymptotic behaviour of the solutions is very well known [6,7]. Let ℓ characteristic roots of the matrix $\left[I + \frac{\partial X_m}{\partial x} (x_0) \right]$ have negative real parts and $n-\ell$ have positive real parts. Then there exists a family of solutions, depending on ℓ parameters, which tend to zero for $\tau \to \infty$ (i.e. $t \to \infty$). This means that there is a ℓ dimensional manifold containing the origin such that any solution of Eqs. (13) with the initial values on this manifold will tend to zero for $\tau \to \infty$. On the other hand, if the initial value of a solution is not on this manifold, the corresponding trajectory cannot remain for any τ near the origin.

Unfortunately, we are not able to construct this manifold using only the lower terms from the right hand side of Eqs.(13). From a physical point of view, the first terms of the perturbation series in the coupling constants are not enough to describe completely this manifold. For example, a one loop calculation will determine only the linear terms in Eqs. (13). Keeping only the linear terms and neglecting the higher ones the attractive manifold is spanned by ℓ eigenvectors of the matrix $\left[I + \frac{\partial X_m}{\partial x}(x_0)\right]$ corresponding to the characteristic roots with negative real parts.

A number of comments concerning the asymptotic stability are in order.

Assuming that all the real parts of the characteristic roots of the matrix $\left[I + \frac{\partial X_m}{\partial x}(x_0)\right]$ are different from zero, then any trajectory of the system will tend to the origin or will move away from it and the existence of a limit cycle around the origin is imposible.

We shall also observe that there is the possibility to have many real solutions x_0 of Eq. (4). Every solution x_0 is a "ray" in the space of coupling constants in the sense that

$$\lim_{t\to\infty} \frac{x}{t} = x_0$$

and around it we have an attractive manifold.

These manifolds can have different dimensions corresponding to the number ℓ of the characteristic roots with negative real parts of the matrix $\left[I + \frac{\partial X_m}{\partial x}\right]$ evaluated at the corresponding ray x_0.

We must emphasize that even if $\ell = m$ for a given ray x_0 this does not mean necessarily that we have a complete asymptotic stability, but only a partial one. This means there are initial values for the coupling constants which define trajectories that cannot tend to the origin. This fact is due to the change of functions (5) which translates the asymptotic stability of the system (13) eventually in a partial asymptotic stability for the initial system (2). This partial asymptotic stability is connected with the fact that, generally, the set of attractive manifolds does not fill the neighbourhood of the origin in the space of the coupling constants.

This point was recognised in some cases by different authors [1,6] who putting some relations between the initial values of the coupling constants constrained the trajectories to be situated on the attractive manifolds. We must mention that these conditions on the initial values (renormalized coupling constants defined as the effective couplings at $t = 0$) cannot guarantee that the trajectories are located on the attractive manifolds since we are not able to know completely these manifolds but only approximately in the neighbourhood of the origin.

A necessary condition to have asymptotic stability is the absence of the trajectories which tend to zero at $\tau \to -\infty$ (i.e. $t \to -\infty$). The study of the asymptotic limit $t \to -\infty$ (infrared asymptotic) can be done as in case $t \to +\infty$ (ultraviolet asymptotic) with some minor changes. Eqs. (4) and (6) become

$$- x + X_m (x) = 0 \qquad (4')$$

$$\frac{dz}{dt} = + z^m \qquad (6')$$

and we shall analyze the characteristic roots of the matrix $\left[- I + \frac{\partial X_m}{\partial x} (x_o)\right]$ where now x_o is a real solution of Eq. (4').

In fact, we shall observe that a gauge theory with Yukawa and scalar couplings is partially asymptotic stable, showing that Eq. (4') has solutions, i.e. there are trajectories which tend to zero at $t \to -\infty$.

Since we are interested to find at least one such trajectory it is enough to consider a subsystem of two equations projecting the remaining coupling constants to zero. In the case of two equations, the attractive manifold can be onedimensional or bidimensional corresponding to the number of the roots with negative real parts of the matrix $\left[1 + \frac{\partial X_m}{\partial x}(x_0)\right]$. More precisely, a onedimensional attractive manifold is an isolated trajectory which tends to the origin and a bidimensional one is a sector in the plane with trajectories tending to the origin.

Let us consider first a subsystem consisting of a gauge coupling constant and a Yukawa one [3]:

$$\frac{dg}{dt} = -\frac{1}{2} b_0 g^3 + \text{higher order terms}$$

$$\frac{dh}{dt} = Ah^3 - Bg^2 h + \text{higher order terms} \tag{14}$$

with A, B, b_0 some positive constants. Simple calculations show that

$$g_0 = 0\,, \quad h_0 = \pm\sqrt{\frac{1}{A}}$$

are solutions of Eq.(14') defining a trajectory which tends to the origin for $t \to -\infty$ (infrared limit). On the other hand Eq.(14) has a solution

$$h_0 = 0\,, \quad g_0 = \pm\sqrt{\frac{2}{b_0}} \tag{15}$$

which defines an attractive manifold in the plane of the coupling constants around the axis $h = 0$ for which we have a sector of partially asymptotic stability. This attractive manifold is bidimensional for $2B/b_0 - 1 > 0$ and onedimensional for $2B/b_0 - 1 < 0$.

Another solution is

$$h_0^2 = \frac{2B - b_0}{Ab_0}\,, \qquad g_0 = \pm\sqrt{\frac{2}{b_0}} \tag{16}$$

which for $2B/b_0 - 1$ defines a onedimensional attractive manifold around the rays

$$\frac{h}{g} = \pm\sqrt{\frac{2B - b_0}{2A}}$$

A phase portrait of the trajectories in the neighbourhood of the origin in the (g,h) plane is given in Fig.(1). If $2B/b_0-1$ is negative, solution (16) is complex and will be disregarded.

Now let us consider a theory with a gauge coupling and a scalar one [3]:

$$\frac{d\tilde{g}}{dt} = -b_0\tilde{g}^2 + \text{higher order terms}$$
$$\frac{d\lambda}{dt} = C\lambda^2 - D\lambda\tilde{g} + E\tilde{g}^2 + \text{higher order terms} \tag{17}$$

where $\tilde{g} = g^2 \geq 0$ and b_0, C, D, E are some positive constants.

The infrared trajectory is given by the solution of Eq. (4') which is $\tilde{g} = 0$, $\lambda = \frac{1}{C} > 0$. The ultraviolet asymptotic trajectories are given by the solution of Eq. (4) which is

$$\tilde{g} = 0 \quad , \quad \lambda = -\frac{1}{C} < 0 \tag{18}$$

that defines a onedimensional altractive manifold and

$$\tilde{g} = \frac{1}{b_0} \quad , \qquad \lambda = \lambda_1, \lambda_2 \tag{19}$$

where λ_1, λ_2 are the solutions of the equation

$$C\lambda^2 + \left(1 - \frac{D}{b_0}\right)\lambda + \frac{E}{b_0^2} = 0 \tag{20}$$

If these solutions are real, they will be both positive or negative since $\frac{E}{Cb_0^2} > 0$. Let us assume $1 - D/b_0 < 0$ and in this case $0 < \lambda_1 \leq \lambda_2$ (for negative λ_1, λ_2 the discussion in similar). A simple calculation of the roots of the matrix $\left[1 + \frac{\partial X_a}{\partial x}\right]$ for the solution (19) shows that

$$\tilde{g}/\lambda = \frac{1}{b_0\lambda_1}$$

is a ray around which we have a bidimensional attractive manifold and

$$\tilde{g}/\lambda = \frac{1}{b_0\lambda_2}$$

is an isolated trajectory which tends also to zero for $t \to \infty$. This case is illustrated in Fig.2. If Eq.(20) does not have real solutions, the above rays disappear, remaining only the ray (18).

These two examples illustrated in figures 1 and 2 show that in the domains of (partial) asymptotic stability, a physical theory is asymptotically free. On the other hand there are regions of initial values for which the corresponding trajectories cannot reach the origin for $t \to \infty$.

We must emphasize that all real solutions of Eq. (4) define all rays around which we have attractive sectors (asymptotic stable trajectories) of different dimensions. The asymptotic freedom requirement determines some domains of permitted renormalized coupling constants or some relations between them depending on the dimensionality of the attractive manifold. The so called particular solutions of the renormalization group equations [4] are in fact onedimensional asymptotic atractive manifolds (isolated trajectories) and the various coupling constants are functions of the gauge coupling alone. In figures 1 and 2 one can identify such isolated trajectories which tend asymptotically to zero.

3. Asymptotic expansions

The asymptotic behaviour of the Green's functions depends on the asymptotic behaviour of the effective couplings.

The study of the asymptotic stability of the solutions of the renormalization group equations from the previous section can offer some information concerning the asymptotic expression on the effective couplings.

The asymptotic expansions of the solutions of a system of differential equations are known since a long time [7,9]. Let us assume that we found a solution x_0 of Eq. (4) which leads to an

-dimensional domain of attraction towards the origin of the coupling constants. The problem is to write an asymptotic expansion for the solutions of Eqs. (13) which correspond to ℓ characteristic roots of the matrix $\left[I + \frac{\partial X_m}{\partial x}(x_0)\right]$ with negative real parts. Let us denote these roots as $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ to which we must add the root (-1) corresponding to the equation for the function z. We can simply verify that one of the roots $(\lambda_1, \ldots, \lambda_\ell)$ is $(1-m)$ where m is the order of homogeneity of the system (2) [6].

The family of solutions corresponding to the roots $(\lambda_1, \ldots, \lambda_\ell)$ which tend to zero for $\tau \to \infty$ is given by the following formal series

$$y_s = \sum_{m+m_1+\ldots+m_\ell \geq 1} L_s^{(m, m_1, \ldots, m_\ell)}(\tau)\, a_1^{m_1}\, a_2^{m_2} \ldots a_\ell^{m_\ell} \, . \, e^{(-m-\sum_{i=1}^{\ell} m_i \mu_i)\tau} , \tag{21}$$

$$\mu_i = -\mathrm{Re}\lambda_i \; ; \qquad i = 1, \ldots \ell \; ; \qquad s = 1, \ldots n \; .$$

where $L_s^{(m, m_1, \ldots m_\ell)}(\tau)$ are polynomials in τ, m, $m_1, \ldots m_\ell$ are positive intege numbers and $a_1, \ldots a_\ell$ are ℓ arbitrary constants connected with the ℓ-dimensionality of the attractive manifold

Taking into account the relation between the coupling constants x and the functions y and using Eq. (12) we get

$$x_s = z\left[x_{0s} + \sum_{m+m_1+\ldots+m_\ell \geq 1} L_s^{(m, m_1, \ldots m_\ell)}(\ln z) \, . \, a_1^{m_1}\, a_2^{m_2} \ldots a_\ell^{m_\ell} \, . \, z^{m + \sum_{i=1}^{\ell} m_i \mu_i}\right] \tag{22}$$

$$s = 1, \ldots n \, .$$

This is a formal series, for which the convergency problem is a delicate question connected with the convergency of the perturbation series in field theory.

If we admit that the series (3) represent holomorphic functions in a given domain around the origin, then the series (22) are convergent for z and a_i sufficiently small [5].

We must note that for the renormalization group equa the polynomials L_s cannot be reduced to constants. This is due to the the fact that at least one of the roots is an integer as mentioned above and this implies that there are relations of the form

$$m + \sum_{i=1}^{l} m_i \lambda_i = \lambda_j$$

with m, m_i integers

$$m \geq 0, \quad m_i \geq 0, \quad m + \sum_i m_i \geq 1$$

But in this case, introducing solution (22) into Eqs.(2) one can easily see that in general L_s are not constants but polynomials in $(\ln z)$.

Expansion (22) shows that in the case of multiple coupling constants it is possible to have some small u_i and consequently we cannot have a good asymptotic approximation for the coupling constants retaining only the first term. In such a case higher loop calculations for the β functions are necessary.

Finally we mention that the expansion (22) generalizes to the multiple coupling constant case the usual asymptotic expansion for one effective coupling constant. The fact that we have many coupling constants determines an expansion in power of z depending on the characteristic roots of the entire system of renormalization group equations. But on the other hand, the expansion (22) is still simple to permit the evaluation of the asymptotic form of t the Green's functions.

Acknowledgements

The authors thank Professor A.Halanay of the University of Bucharest, Mathematics Department, for valuable discussions on the asymptotic stability of the differential equations.

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FIGURE CAPTIONS

Fig.1. Phase portrait corresponding to Eq. (14)

Fig.2. Phase portrait corresponding to Eq. (17)

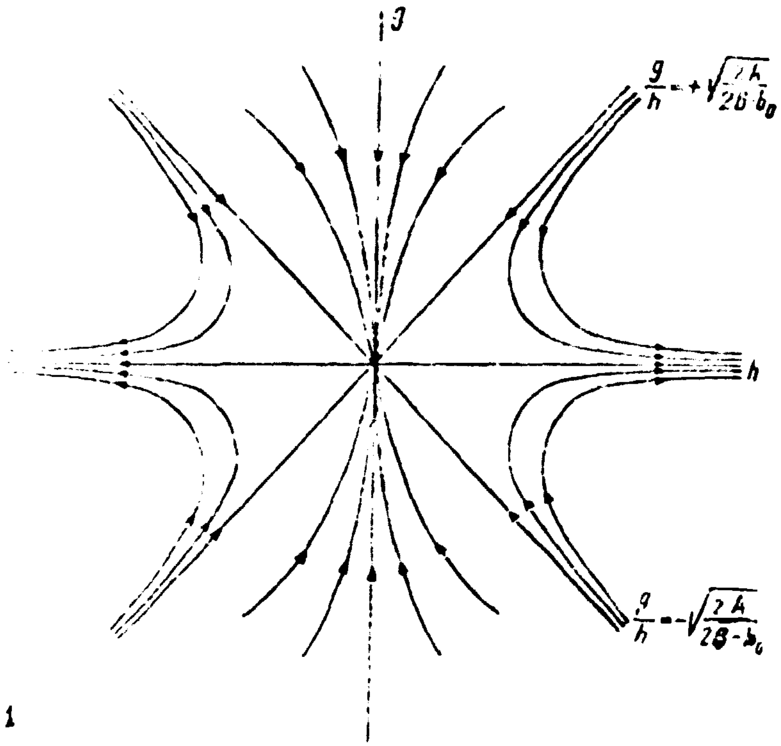
g
h
$\frac{g}{h} = +\sqrt{\frac{2A}{2B-b_0}}$
$\frac{g}{h} = -\sqrt{\frac{2A}{2B-b_0}}$

Fig 1

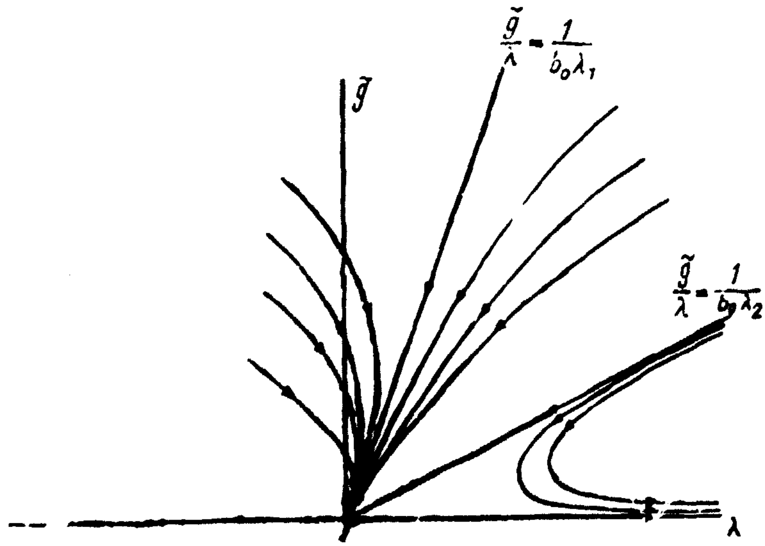
$\tilde{g}$
$\frac{\tilde{g}}{\lambda} = \frac{1}{b_0\lambda_1}$
$\frac{\tilde{g}}{\lambda} = \frac{1}{b_0\lambda_2}$
λ

Fig. 2