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ON THE AXIALLY SYMMETRIC EQUILIBRIUM OF A MAGNETICALLY CONFINED PLASMA

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ABSTRACT

The axially symmetric equilibrium of a magnetically confined plasma is reconsidered, with the special purpose of studying high-beta schemes with a purely poloidal magnetic field. A number of special solutions of the pressure and magnetic flux functions are shown to exist. The obtained results may form starting-points in a further analysis of physically relevant configurations.

1. Introduction

The equilibrium problem of a magnetically confined plasma has been treated by several authors during the last twenty years. In particular, it was found by Biermann et al. [1] and by Biermann and Schlüter [2] that a toroidal current component becomes necessary for an axially symmetric equilibrium to exist. Further, Meyer and Schmidt [3] showed that there are also equilibria in absence of such a current, in the case of specially shaped bumpy torus configurations.

In this report part of the earlier analysis by Shafranow [4] and others is reconsidered and extended to some detail, with the purpose of studying high-beta plasmas confined in a purely poloidal magnetic field. Thus, some special solutions of the pressure and poloidal magnetic flux distributions are deduced which may form starting-points in further investigations of physically relevant configurations.

2. Basic Equations

The present analysis is limited to plasmas having a scalar pressure p and particle density n and being confined by a magnetic field $\underline{B}$ which is generated by currents in external regions as well as by plasma currents of density $\underline{j}$. The basic equations of static equilibrium become

$$\underline{B} = \mathrm{curl}\underline{A} \quad ; \qquad \mathrm{div}\underline{B} = 0 \tag{1}$$

$$\mathrm{curl}\underline{B} = \mu_0 \underline{j} \quad ; \qquad \mathrm{div}\underline{j} = 0 \tag{2}$$

$$\underline{\nabla} p = \underline{j}\times\underline{B} = -\underline{\nabla}(B^2/2\mu_0) + (\underline{B}\cdot\underline{\nabla})\underline{B}/\mu_0 \tag{3}$$

$$\underline{\nabla}\phi = (\underline{\nabla} p - 2\underline{j}\times\underline{B})/2en = -(1/2en)\underline{\nabla} p \tag{4}$$

where $\underline{A}$ is the magnetic vector potential and $\underline{E} = -\underline{\nabla}\phi$ the electric field. It is possible to add the gradient of an arbitrary scalar quantity to $\underline{A}$ and a constant to ϕ in a gauge transformation without changing eqs. (1)-(4).

The volume force $\underline{j}\times\underline{B}$ in eq. (3) is divided into one part originating from the gradient of a "magnetic pressure" $B^2/2\mu_0$ and one part $(\underline{B}\cdot\underline{\nabla})\underline{B}/\mu_0$ being due to additional stresses of the field $\underline{B}$. The latter cannot always be expressed as the gradient of a scalar function, i.e.

$$\mathrm{curl}(\underline{j}\times\underline{B}) = \mathrm{curl}\left[(\underline{B}\cdot\underline{\nabla})\underline{B}/\mu_0\right] = 0 \tag{5}$$

becomes a necessary condition for equilibrium. In addition, eq. (3)

yields the conditions

$$\underline{B}\cdot\underline{\nabla}p = 0 \quad (6)$$

$$\underline{j}\cdot\underline{\nabla}p = 0 \quad (7)$$

Finally, equation (4) is satisfied when

$$\underline{\nabla}n\times\underline{\nabla}p = 0 \quad ; \qquad \underline{\nabla}\phi\times\underline{\nabla}p = 0 \quad (8)$$

which implies that $n=n(p)$ and $\phi=\phi(p)$, i.e. the surfaces of constant pressure, density, and electric potential all have to coincide in (r,z)-space [5].

3. The General Axially Symmetric Case

In an axially symmetric configuration described by the cylindrical coordinates (r, φ, z) the magnetic field $\underline{B}=\underline{B}_p + \underline{B}_t$ generally consists of a poloidal part $\underline{B}_p = (B_r, 0, B_z) = \text{curl } \underline{A}_t$ and a toroidal part $\underline{B}_t = (0, B_\varphi, 0) = \text{curl } \underline{A}_p$ where $\underline{A}_t = (0, A_\varphi, 0)$, $\underline{A}_p = (A_r, 0, A_z)$, and all quantities are independent of φ. Defining the scalar functions $\psi = \psi(r,z) = rA_\varphi$ and $\chi = \chi(r,z) = rB_\varphi$ as well as the toroidal and poloidal current densities $\underline{j}_t = (0, j_\varphi, 0)$ and $\underline{j}_p = (j_r, 0, j_z)$, eqs. (1)-(3) then yield

$$\frac{\partial p}{\partial r} = (j_\varphi / r)\frac{\partial \psi}{\partial r} - (\chi/\mu_o r^2)\frac{\partial \chi}{\partial r} \tag{9}$$

$$0 = \frac{\partial \psi}{\partial r}\frac{\partial \chi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \chi}{\partial r} \tag{10}$$

$$\frac{\partial p}{\partial z} = (j_\varphi / r)\frac{\partial \psi}{\partial z} - (\chi/\mu_o r^2)\frac{\partial \chi}{\partial z} \tag{11}$$

Further, conditions (6) and (7) reduce to

$$\frac{\partial \psi}{\partial z}\frac{\partial p}{\partial r} - \frac{\partial \psi}{\partial r}\frac{\partial p}{\partial z} = 0 \tag{12}$$

$$\frac{\partial \chi}{\partial z}\frac{\partial p}{\partial r} - \frac{\partial \chi}{\partial r}\frac{\partial p}{\partial z} = 0 \tag{13}$$

Eqs. (10), (12), and (13) are satisfied when

$$\psi = \psi(p) \; ; \qquad \chi = \chi(p) \tag{14}$$

Consequently, the pressure balance expressed by eqs. (9)-(11) becomes possible for a non-vanishing ψ when

$$\mu_o j_\varphi = \mu_o r\frac{dp}{d\psi} + \frac{\chi}{r}\frac{d\chi}{d\psi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} \qquad (15)$$

as obtained by Shafranov [4] in a somewhat different way.

4. Confinement in Magnetic Fields Including a Toroidal Component

For magnetic fields with a non-vanishing toroidal component the pressure gradient can in general become balanced by the sum of the forces $j_t \times B_p$ and $j_p \times B_t$. There are two cases of special interest here:

(i) If the poloidal current j_p and the corresponding pinch force $j_p \times B_t$ are allowed to decay, the toroidal magnetic field will be given by $\chi = rB_\varphi = \text{const.}$ All terms including χ then vanish from eqs. (9)-(15), and the poloidal field provides the only force which balances the plasma pressure. The equilibrium problem thus becomes equivalent to that of a plasma confined in a purely poloidal magnetic field. This situation prevails in Tokamak equilibria when the poloidal beta value is close to unity.

(ii) In the case of a purely toroidal magnetic field, we have $\psi = 0$ in eqs. (9)-(15). According to eq. (9) this leads to $d\chi^2/dp = -2\mu_0 r^2$ being a singular result of minor physical interest. Eq. (5) can then be satisfied only by $\partial\chi/\partial z = 0$, i.e. $\chi = \chi(r)$, $p = p(r)$ and the system degenerates into a straight cylinder configuration. This is easily understood from the fact that all charged particles then drift along the axial direction in the toroidal vacuum field $B_\varphi = \text{const.}/r$.

5. Confinement in a Purely Poloidal Magnetic Field

The treatment is now concentrated on confinement in a purely poloidal field. There already exist theoretical and experimental proofs for an equilibrium to exist in low-beta systems of this type, whereas high-beta systems require further analysis. In any case, orbit theory shows that the charged particles should be confined in the (r,z)-plane within equivalent potential "troughs", having a width of the order of a Larmor radius and being extended along the poloidal magnetic field lines.

5.1. General Considerations

With $X = 0$ the equilibrium problem reduces to studies of eq. (15) in the form

$$D\psi \equiv r \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2} = - \mu_o r^2 \frac{dp}{d\psi} \qquad (16)$$

A large class of pressure distributions $p(\psi)$ related to field geometries of the form $\psi(r,z)$ can be described by

$$p = P_{oo} \ln\psi + \sum_{s=-\infty}^{+\infty} P_s \psi^s \quad ; \quad \frac{dp}{d\psi} = P_{oo}\psi^{-1} + \sum_{s=-\infty}^{+\infty} sP_s \psi^{s-1} \qquad (17)$$

where P_{oo} and P_s are constants. In this connection it has to be observed that a gauge transformation can be applied to eq. (1) in the sense that an arbitrary constant ψ_o is added to ψ, yielding no controbution to the field $\underline{B}$, because $\mathrm{curl}(0,\psi_o/r,0) \equiv 0$. This implies that both zero points of the scales labelling the surfaces ψ =const. and p =const. can be chosen arbitrarily.

When studying the relation between the pressure $p(\psi)$ and the poloidal magnetic flux ψ for various types of magnetic field geometries, physically relevant values of the functions p and ψ and their derivatives within the plasma and at the boundaries have to be adopted. This puts some constraints on the possible forms of the expansion (17) in particular, as well as on $p = p(\psi)$ in general.

In Figs. 1 and 2 some examples on the relations between the spatial distributions of p and ψ have been outlined. Thus Fig.1 gives an example of an internal ring system "INTRAP" where the pressure becomes negligible at the inner and outer boundaries of a "hollow" confinement region, as compared to the maximum value p_c in the hot plasma interior. In Fig.2 a corresponding example is given for a suggested external ring system "EXTRAP" [6]. In both figures the zero levels of the flux and pressure scales can be changed by an arbitrary constant. The current density j_φ is assumed to be finite within the shaded areas of Figs. 1 and 2.

5.2. Special Solutions

The pressure gradient introduces a force which affects the pattern of the "frozen-in" poloidal field. At the same time there is a degree of freedom in the choice of the pressure distribution with respect to the direction across the surfaces of constant magnetic flux $\psi(r,z)$. Thus, for given values $p = p_{b1}, p_{b2}$ and $p = p_c$ at the plasma boundaries and at the plasma core defined by $\psi(r,z) = \psi_{b1}$, ψ_{b2} and $\psi(r,z) = \psi_c$, a manifold of distributions with "steep" or "flat" shapes have to be expressed by the expansion (17). To treat situations of experimental interest, we have at least to include a number of the terms contained in this expansion. Examples on some special solutions of eq. (16) will be presented in the coming subsections.

5.2.1. Linear Pressure-Flux Relationship

One elementary case is represented by letting all coefficients in eq. (17) vanish except P_o and P_1. Then eq. (16) reduces to

$$D\psi + \mu_o P_1 r^2 = 0 \tag{18}$$

This linear distribution of p with respect to ψ can only apply to single-boundary configurations like those of Fig.2.

With the ansatz $\psi = F(r) + G(r)\cdot Z(z)$ it is thus required that

$$F'' - \frac{1}{r} F' + \mu_o P_1 r^2 + H = 0 \qquad (19)$$

where

$$H = (G'' - \frac{1}{r} G')Z + GZ'' \qquad (20)$$

can be made independent of z by the condition

$$(G'' - \frac{1}{r} G')Z' + GZ''' = 0 \qquad (21)$$

This leads to two cases:

(i) Each term of eq. (21) vanishes separately when

$$Z''' = 0 \; ; \qquad Z = z_c + z_1 z + z_2 z^2 \qquad (22)$$

$$G'' - \frac{1}{r} G' = 0 \; ; \qquad G = g_o + g_2 r^2 \qquad (23)$$

where z_s and g_s are constants. Thus, eq. (19) reduces to

$$r\frac{d}{dr}(F'/r) + (2z_2 g_2 + \mu_o P_1)r^2 + 2z_2 g_o = 0 \qquad (24)$$

which can be int. grated to yield the solution

$$\psi = f_o + f_2 r^2 - g_o z_2 r^2 \ln r - (2g_2 z_2 + \mu_o P_1) r^4/8 +$$

$$+ (g_o + g_2 r^2)(z_1 z + z_2 z^2) \quad (25)$$

where also f_s are constants of integration. The special case where $f_o = 0$, $g_o = 0$, $z_1 = 0$ has earlier been considered by Shafranov [4].

(ii) The variables of eq. (21) are separated by

$$Z = c_\gamma \cos\gamma z + s_\gamma \sin\gamma z \quad (26)$$

where c_γ, s_γ are constants and

$$rG'' - G' - \gamma^2 rG = 0 \quad (27)$$

which has the solution

$$G = A_\gamma r I_1(\gamma r) + B_\gamma r K_1(\gamma r) \quad (28)$$

Here I_1 and K_1 stand for hyperbolic Bessel functions when γ is real, and A_γ, B_γ are constants. The function I_1 is finite at the origin but infinite at infinity whereas K_1 is infinite at the origin and zero at infinity. With expressions (26) and (28) we further obtain $H = 0$ and integration of eq. (19) yields the solution

$$\psi = f_o + f_2 r^2 - \mu_o P_1 r^4/8 +$$

$$+ \sum_\gamma \left[A_\gamma r I_1(\gamma r) + B_\gamma r K_1(\gamma r)\right] \cdot (c_\gamma \cos\gamma z + s_\gamma \sin\gamma z) \qquad (29)$$

where the sum over different values of γ depends on the particular boundary conditions to be imposed. Thus, for a fixed value of r, the boundary values $\psi(r,z) = \psi_b$ introduce certain restrictions on the series of γ-values to be adopted in eq. (29).

5.2.2. Quadratic Pressure-Flux Relationship

With all coefficients vanishing in eq. (17) except P_o and P_2, and putting $\psi_c = 0$, $p_b = 0$, a pressure distribution

$$p = p_c \left[1 - (\psi/\psi_b)^2\right] \qquad (30)$$

is obtained. This distribution could apply to the single-boundary configuration of Fig.2, as well as to the double-boundary case of Fig.1 in the particular case $\psi_{b1} = -\psi_{b2}$ where the pressure distribution $p(\psi)$ is symmetric with respect to $\psi = \psi_c$.

With the ansatz (30) the balance equation (16) reduces to

$$D\psi - P^2 r^2 \psi = 0 \ ; \qquad P^2 = 2\mu_o p_c/\psi_b^2 \qquad (31)$$

being a linear differential equation for which the variables can be separated by introducing $\psi = R(r)\cdot Z(z)$. This yields a function Z of the form (26) and

$$r^2 R'' - rR' - r^2(\gamma^2 + P^2 r^2)R = 0 \qquad (32)$$

With $x = Pr^2$ the latter relation reduces to Whittaker's equation

$$4x^2 R''/R = x^2 + (\gamma^2/P)x \tag{33}$$

and with the substitution $R(x) = xf(x)\exp(-x/2)$ the confluent hypergeometric equation

$$xf'' + (2 - x)f' - [1 + (\gamma^2/4P)]f = 0 \tag{34}$$

is obtained. Whittaker's equation (33) has the general solution

$$R = A_\gamma \mathcal{M}_{\varkappa,1/2}(x) + B_\gamma W_{-\varkappa,1/2}(x) \tag{35}$$

where A_γ, B_γ are constants, $\varkappa = -\gamma^2/4p$, and the two independent solutions $\mathcal{M}_{\varkappa,1/2}(x)$ and $W_{-\varkappa,1/2}(x)$ have been specified in detail by Buchholtz [7] and others. Consequently, a general solution is obtained, having the form

$$\psi = \sum_\gamma [A_\gamma \mathcal{M}_{\varkappa,1/2}(P^2 r) + B_\gamma W_{-\varkappa,1/2}(P^2 r)] \cdot$$

$$\cdot (c_\gamma \cos\gamma z + s_\gamma \sin\gamma z) \tag{36}$$

6. Conclusions

(i) In axially symmetric systems where there is no poloidal plasma current producing a pinch force, the pressure balance has to be provided solely by a poloidal magnetic field component.

(ii) The present investigations have shown that there are at least a number of special equilibrium solutions of a plasma confined in a purely poloidal field. This result is a first step in the analysis of the conditions for equilibria to exist at high beta values in such a field.

(iii) The general problem of plasma equilibrium in a purely poloidal field needs further investigation, especially in the high-beta case. In particular, pressure distributions of a more general shape have to be considered. It also has to be found out under what circumstances special solutions, such as those obtained in this paper, can be matched to the boundary conditions in geometries of physical interest. These questions also become related to the problem of transforming known results with their boundary conditions to other geometries by means of conformal mapping [8].

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8. References

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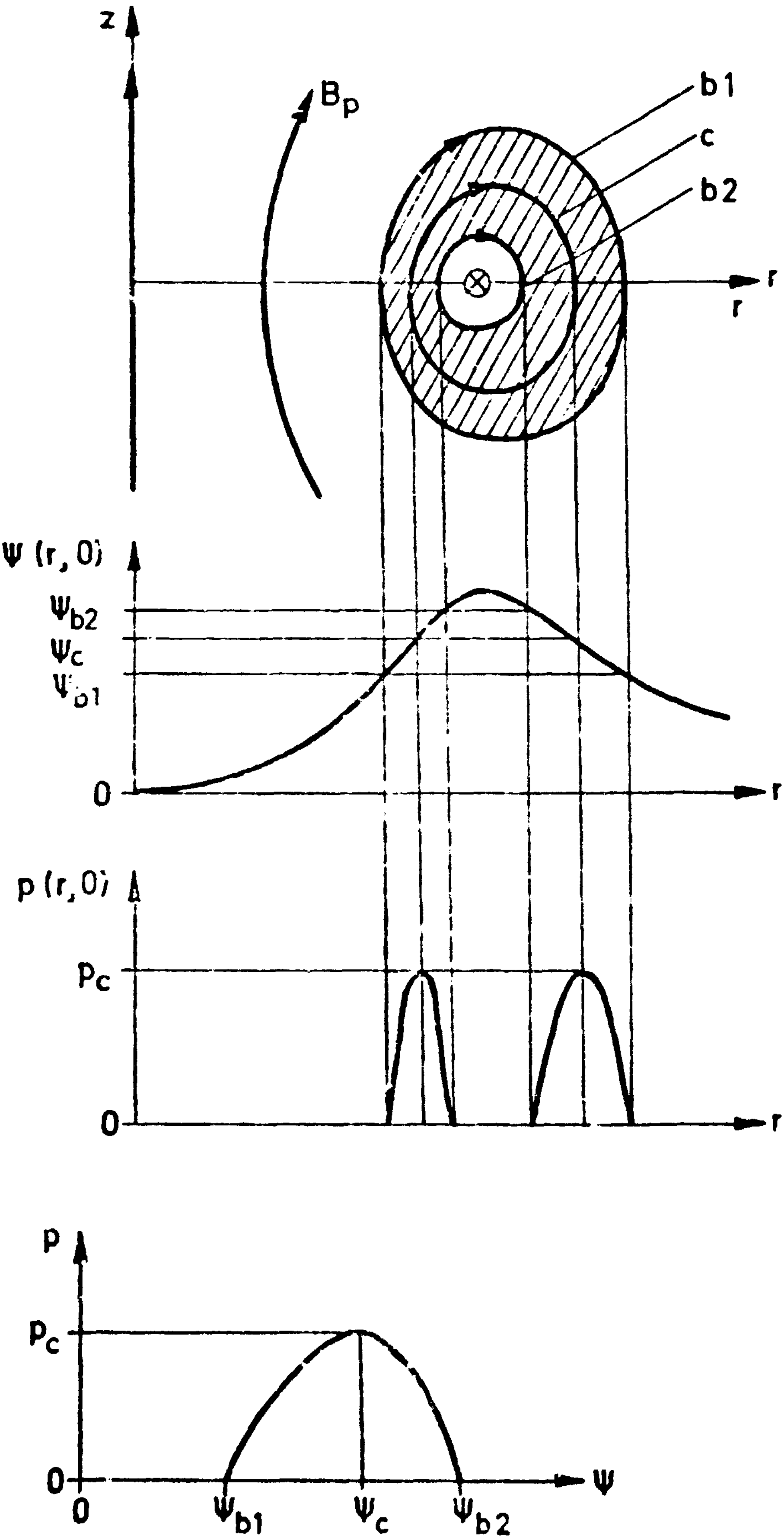
Figure Captions

Fig.1. Outline of the spatial distributions of pressure p and poloidal magnetic flux ψ of an "INTRAP" system. The shaded area indicates the confinement region. The symbols c, b1, b2 are related to the magnetic surfaces at which the pressure has its maximum p_c, and to the surfaces defining the boundaries at which the pressure becomes negligible as compared to p_c.

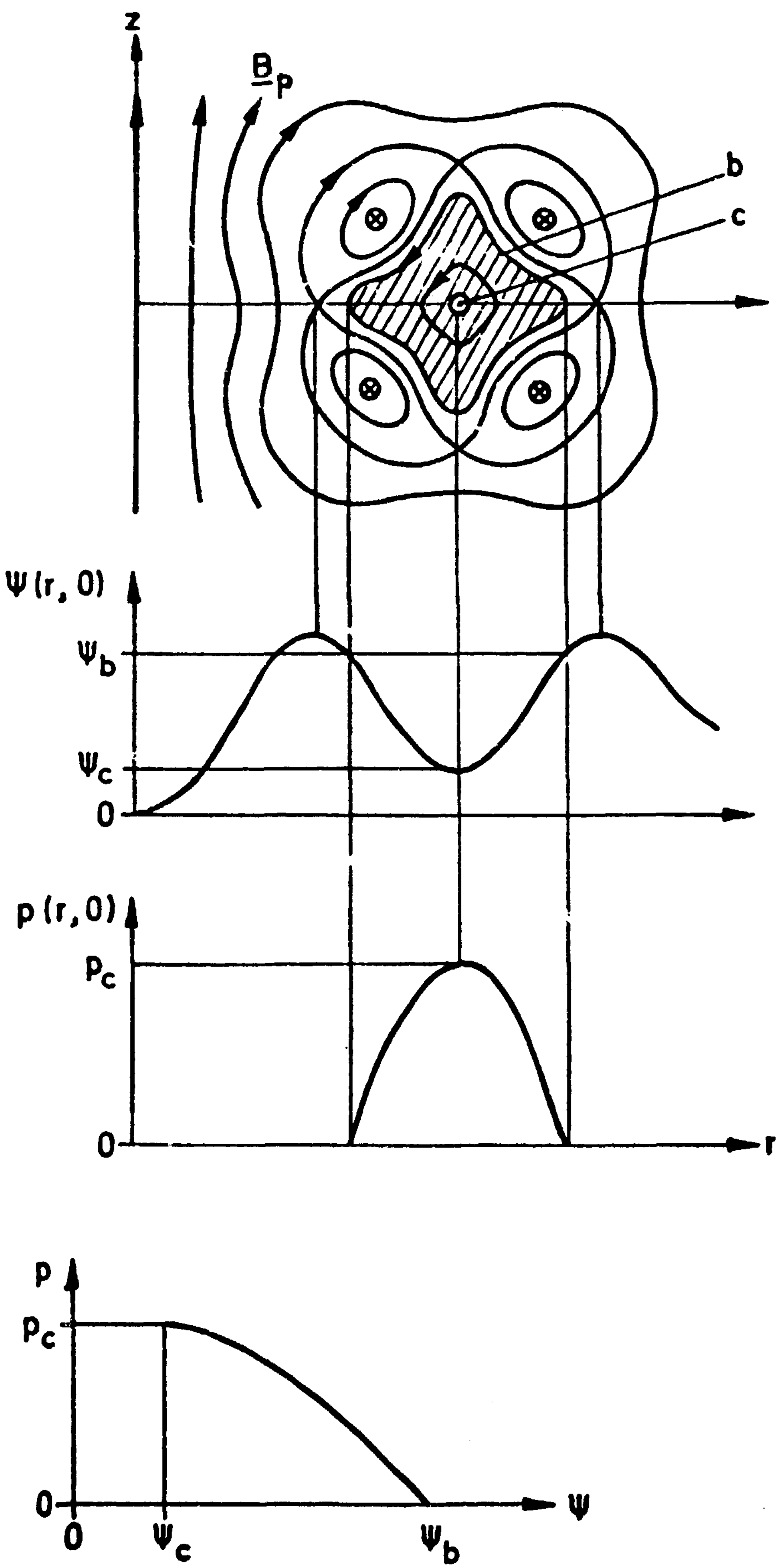
Fig.2. Same as Fig.1, but for an "EXTRAP" system where there is no inner boundary and $b1 = b2 \equiv b$.

Fig. 1



z
B_p
b1
c
b2
r
r
$\Psi(r, 0)$
Ψ_{b2}
Ψ_c
Ψ_{b1}
0
r
$p(r, 0)$
p_c
0
r
p
p_c
0
0
Ψ_{b1}
Ψ_c
Ψ_{b2}
Ψ

Fig. 2



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The axially symmetric equilibrium of a magnetically confined plasma is reconsidered, with the special purpose of studying high-beta schemes with a purely poloidal magnetic field. A number of special solutions of the pressure and magnetic flux functions are shown to exist. The obtained results may form starting-points in a further analysis of physical relevant configurations.

Key words Magnetic confinement, axially symmetric fields, equilibria.