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Linearized Gyro-Kinetic Equation

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ABSTRACT

An ordering of the linearized Fokker-Planck equation is performed in which gyroradius corrections are retained to lowest order and the radial dependence appropriate for sheared magnetic fields is treated without resorting to a WKB technique. This description is shown to be necessary to obtain the proper radial dependence when the product of the poloidal wavenumber and the gyroradius is large ($k\rho \gg 1$). A like particle collision operator valid for arbitrary $k\rho$ also has been derived. In addition, neoclassical, drift, finite β (plasma pressure/magnetic pressure), and unperturbed toroidal electric field modifications are treated.

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INTRODUCTION

For a complete evaluation of the growth rates of drift instabilities in sheared magnetic fields, finite gyroradius effects must be carefully retained. In slab geometry finite gyroradius corrections can result in destabilizing contributions to the growth rate.[1-2] The addition of finite β (plasma pressure/magnetic pressure) results in considerable complication even in slab geometry, and for a sheared magnetic field has only been evaluated when the gyroradius correction is small.[3-4] In addition, the extension of finite gyroradius effects to sheared toroidal geometry (Tokamaks) may permit the coupling of a mode localized about a rational surface to modes localized about adjacent rational surfaces. The coupling is expected to occur because the group velocity associated with one of these localized modes is away from the rational surface and, at marginal stability, the mode structure extends to distances where ion Landau damping occurs.[1-4] Present calculations which include finite gyroradius effects assume isolated rational surfaces.[4-6]

In order to treat arbitrary values of the product of poloidal wavenumber k and the ion gyroradius ρ_i in sheared magnetic fields, a generalization of the technique of Rutherford and Frieman[7-8] is employed. The technique retains gyroradius corrections in the lowest order equations for the linearized distribution function. This same gyro-kinetic ordering is employed herein, however, rather than treat the radial and poloidal wave variation via a WKB or eikonal method, an explicit toroidal model with concentric magnetic surfaces is treated. This model permits the toroidal variation to be Fourier decomposed and the derivatives with respect to radial variation to be retained explicitly.

As a result, magnetic shear effects can be properly treated. The result for large $k\rho_i$ is in agreement with previous slab results,[10] but differs from the WKB treatment which is shown to provide only the local result.

The gyro-kinetic formalism also permits neoclassical, finite β, and unperturbed toroidal electric field $\underline{E}_0$ contributions to be retained. The neoclassical corrections are shown to result in negligible modifications. The gyro-kinetic equation valid for $\underline{E}_0 \neq 0$ and finite β is derived.

For completeness, the $k\rho_i$ modification of the ion-ion Fokker-Planck collision operator is considered. The result is quite complex for arbitrary $k\rho_i$; however, for $k\rho_i \gg 1$ it can be reduced to a particularly simple form.

This gyro-kinetic ordering could also be employed to determine the gyroradius modifications of MHD instabilities.

GYRO-KINETIC ORDERING AND EQUATIONS

Working in a toroidal geometry with circular, concentric flux surfaces, the unperturbed magnetic field may be written as $\underline{B}_0 = (B_0R_0/R)\{\hat{\underline{\zeta}} + [\varepsilon/q(r)]\hat{\underline{\theta}}\}$; with $\varepsilon = r/R_0$, $R = R_0(1 + \varepsilon\cos\theta)$, θ and ζ the poloidal and toroidal angle variables, q the safety factor, and r the radial variable measured from the center of the concentric flux circles located at a distance R_0 from the axis of symmetry.

Employing the velocity variables μ, E, and ϕ defined by

$$\mu = v_\perp^2/2B \qquad E = \frac{1}{2}(v_\parallel^2 + v_\perp^2) = \frac{1}{2}v^2$$

$$\underline{v} = v_\parallel\hat{\underline{n}} + v_\perp(\hat{\underline{r}}\cos\phi + \hat{\underline{e}}\sin\phi) = v_\parallel\hat{\underline{n}} + \underline{v}_\perp = v_\parallel\hat{\underline{n}} + v_\perp\hat{\underline{v}}_\perp$$

$$\hat{\underline{e}} = \hat{\underline{n}}\times\hat{\underline{r}} \qquad \hat{\underline{\phi}} = \hat{\underline{n}}\times\hat{\underline{v}}_\perp$$

$$\nabla_v = \underline{v}\frac{\partial}{\partial E} + \underline{v}_\perp\frac{1}{B}\frac{\partial}{\partial\mu} + \hat{\underline{\phi}}\frac{1}{v_\perp}\frac{\partial}{\partial\phi}$$

$$B = |\underline{B}_0| \qquad \hat{\underline{n}} = \underline{B}_0/B \qquad \Omega = ZeB/Mc \quad ,$$

denoting the unperturbed and perturbed distribution functions by F and f, and writing $F = F_M + F_D$ as the sum of Maxwellian F_M and diamagnetic F_D contributions,

$$F_M = F_M(r,E) = N(r)\left[\frac{M}{2\pi T(r)}\right]^{\frac{3}{2}} \exp[-ME/T(r)]$$

$$F_D = F_D(r,\theta,\mu,E,\phi) = \frac{1}{\Omega}\,\underline{v}\times\hat{\underline{n}}\cdot\nabla F_M \quad ,$$

the linearized kinetic or Fokker-Planck equation may be written conveniently by including gyrophase independent portion of F, F_M, inside $\underline{v}\cdot\nabla$ to obtain

$$\frac{\partial f}{\partial t} + [\underline{v}\cdot\nabla - \Omega \frac{\partial}{\partial\phi}][f + \frac{ZeF_M}{T}\Phi] - Zev_\perp\Phi\cos\phi \frac{\partial}{\partial r}\left(\frac{F_M}{T}\right)$$

$$- \frac{Ze}{M}\nabla\Phi\cdot\nabla_v F_D + \dot{\mu}\frac{\partial f}{\partial\mu} + \dot{\phi}\frac{\partial f}{\partial\phi} = C\{f\} , \tag{1}$$

where

$$B\dot{\mu} = -\underline{v}\cdot\nabla B + v_\perp v_\parallel\{v_\parallel[\hat{\underline{n}}\cdot\nabla\hat{\underline{n}}\cdot\hat{\underline{v}}_\perp] - v_\perp[\hat{\underline{r}}\cdot\nabla\hat{\underline{r}}\cdot\hat{\underline{n}}\cos^2\phi$$

$$+ \hat{\underline{e}}\cdot\nabla\hat{\underline{e}}\cdot\hat{\underline{n}}\sin^2\phi + (\hat{\underline{e}}\cdot\nabla\hat{\underline{r}}\cdot\hat{\underline{n}} + \hat{\underline{r}}\cdot\nabla\hat{\underline{e}}\cdot\hat{\underline{n}})\sin\phi\cos\phi]\}$$

$$\dot{\phi} = v_\parallel\hat{\underline{n}}\cdot\nabla\hat{\underline{e}}\cdot\hat{\underline{r}} - (v_\parallel^2/v_\perp)\hat{\underline{n}}\cdot\nabla\hat{\underline{n}}\cdot\hat{\underline{\phi}} + v_\parallel[\hat{\underline{r}}\cdot\nabla\hat{\underline{e}}\cdot\hat{\underline{n}}\cos^2\phi$$

$$- \hat{\underline{e}}\cdot\nabla\hat{\underline{r}}\cdot\hat{\underline{n}}\sin^2\phi + (\hat{\underline{e}}\cdot\nabla\hat{\underline{e}}\cdot\hat{\underline{n}} - \hat{\underline{r}}\cdot\nabla\hat{\underline{r}}\cdot\hat{\underline{n}})\sin\phi\cos\phi]$$

$$- v_\perp[\hat{\underline{r}}\cdot\nabla\hat{\underline{r}}\cdot\hat{\underline{e}}\cos\phi - \hat{\underline{e}}\cdot\nabla\hat{\underline{e}}\cdot\hat{\underline{r}}\sin\phi] .$$

In eq. (1), Φ is the perturbed electrostatic potential, N and T are the unperturbed number density and temperature of the species, and C{f} represents the linearized Fokker-Planck collision operator; the quantities e and c are the magnitude of the charge on an electron and speed of light, while M and Z are the species mass and charge in units of e ($Z = -1$ for electrons).

Taking f and Φ to be of the form

$$\begin{bmatrix} f \\ \Phi \end{bmatrix} = \begin{bmatrix} \hat{f}(r,\theta,\underline{v}) \\ \hat{\Phi}(r,\theta) \end{bmatrix} \exp(-i\omega t + im\theta - i\ell\zeta) , \tag{2}$$

where the fast θ dependence is explicitly indicated ($m \gg 1$) and the hatted quantities contain only slow θ dependence, then neglecting the distinction between R and R_0 except under derivatives results in

$$\nabla[\hat{Q}(r,\theta)\exp(im\theta - i\ell\zeta)] \approx \exp(im\theta - i\ell\zeta)\left[\hat{\underline{r}}\frac{\partial}{\partial r} + \hat{\underline{\theta}}\left(\frac{im}{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\right) - \hat{\underline{\zeta}}\frac{i\ell}{R_0}\right]\hat{Q}$$

$$\approx \exp(im\theta - i\ell\zeta)[\nabla_f + \nabla_s]\hat{Q} ,$$

with

$$\nabla_f = \hat{\underline{r}}\frac{\partial}{\partial r} + i\underline{\theta}\,\underline{\hat{n}}\cdot\underline{\hat{\zeta}}\left(\frac{m}{r} + \frac{\ell\varepsilon}{qR_0}\right) \simeq \hat{\underline{r}}\frac{\partial}{\partial r} + \hat{\underline{e}}ik$$

$$\nabla_s = \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r} + i\underline{\hat{n}}\,\underline{\hat{n}}\cdot\underline{\hat{\zeta}}\left(\frac{m - \ell q}{qR_0}\right) \approx \hat{\underline{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\underline{n}}ik_\parallel \tag{3}$$

and

$$k = m/r \qquad k_\parallel = (m - \ell q)/qR_0 \quad .$$

If the distinction between R and R_0 is retained, then in order to describe perturbed electric fields that vanish at a rational surface ($\oint_c d\underline{r}\cdot\underline{E} = 0$ along a closed field) a sum over poloidal wavenumbers is required. This complication is a limitation of the concentric magnetic surface model, but could be removed by employing a more sophisticated coordinate system. For simplicity and ease of presentation the concentric magnetic surface model is employed with order ε corrections neglected. Order $\varepsilon^{1/2}$ corrections, which are associated with the distinction between trapped and circulating particles, are retained.

With the notation of eqs. (2) and (3), a gyro-kinetic ordering of eq. (1) may be performed conveniently. This ordering proceeds by assuming that in eq. (1) the $[\underline{v}\cdot\nabla_f - \Omega\partial/\partial\phi][\hat{f} + (ZeF_M\hat{\Phi}/T)]$ terms dominate, rather than just the $\Omega\partial/\partial\phi[\hat{f} + (ZeF_M\hat{\Phi}/T)]$ term of the more familiar drift-kinetic ordering. Formally, the gyro-kinetic ordering corresponds to an expansion in the small parameters $|\omega/(\Omega\partial\,\ell n\,\hat{Q}/\partial\phi)|$ and $|\underline{v}\cdot\nabla_s\hat{Q}/\underline{v}\cdot\nabla_f\hat{Q}|$.

In order to obtain a particularly simple representation it is desirable to retain certain small terms to lowest order. Expanding $\hat{f}$ and $\hat{\Phi}$ in powers of the small parameters, $\hat{f} \to \hat{f} + \delta\hat{f} + \ldots$ and $\hat{\Phi} \to \hat{\Phi} + \delta\hat{\Phi} + \ldots$; and retaining these small terms, the lowest order equation is taken to be

$$\left\{[\tilde{v}_\perp\hat{\underline{r}}\cos\phi + v_\perp\hat{\underline{e}}\sin\phi]\cdot\nabla_f - \Omega\frac{\partial}{\partial\phi} + i\tilde{v}_\perp\cos\phi\frac{\partial L}{\partial r}\right\}[\hat{f} + \frac{ZeF_M}{T}\hat{\Phi}] = 0 \quad , \tag{4}$$

where $L \equiv (kv_\perp/\Omega)\cos\phi$ and $\tilde{v}_\perp = v_\perp/[1 + \sin\phi\, \partial(v_\perp/\Omega)/\partial r]$. The particular virtue of this lowest order form is that the change of variables[9] $r' = r + (v_\perp/\Omega)\sin\phi$, $\theta' = \theta$, $\phi' = \phi$, $\mu' = \mu$, and $E' = E$ applied to eq. (4) results in

$$\frac{\partial}{\partial\phi'}\left\{[\hat{f} + \frac{ZeF_M}{T}\hat{\Phi}]\exp(iL)\right\} = 0 \ .$$

Consequently,

$$\begin{aligned}\hat{f} + \frac{ZeF_M}{T}\hat{\Phi} &= \hat{g}'\exp(-iL)\\ &= \hat{g}(r + \frac{v_\perp}{\Omega}\sin\phi,\theta,\mu,E)\exp(-iL) \ ,\end{aligned} \tag{5}$$

that is, $\hat{g}'$ is independent of ϕ'.

In obtaining the next higher order equation from eq. (1), small terms are again added in so that the transformation to the primed variables results in a total ϕ' derivative of $\delta\hat{f} + (ZeF_M\delta\hat{\Phi}/T)$. In addition, the small terms added in to lowest order must be corrected for by subtracting them out in this order. The resulting equation is then multiplied by $\Omega^{-1}\exp(iL)$ and a gyrophase average over the primed gyrophase ϕ' performed,

$$< \cdot \cdot \cdot > = \frac{1}{2\pi}\int_0^{2\pi} d\phi'(\ \cdot \cdot \cdot\ ) \ .$$

The result is

$$\begin{aligned}&\left\langle \frac{1}{\Omega}\left\{-i\omega + \left[\underline{v}\cdot\nabla_s - i\frac{v_\perp}{\Omega}\sin\phi\frac{\partial L}{\partial\theta}\right] + \dot{\mu}\left[\frac{\partial}{\partial\mu} - i\frac{\partial L}{\partial\mu}\right] + \dot{\phi}\left[\frac{\partial}{\partial\phi} - i\frac{\partial L}{\partial\phi}\right]\right.\right.\\ &\qquad \left.\left. - \left[i\tilde{v}_\perp\cos\phi\frac{\partial L}{\partial r} + (\tilde{v}_\perp - v_\perp)\cos\phi\left(\frac{\partial}{\partial r} - i\frac{\partial L}{\partial r}\right)\right]\right\}\right\rangle \hat{g}'\\ &- \left\langle \frac{1}{\Omega}\exp(iL)C\{\hat{g}'\exp(-iL)\}\right\rangle = -\left\langle \frac{iZeF_M}{\Omega T}(\omega - \omega_*^T)\hat{\Phi}\exp(iL)\right\rangle \ ,\end{aligned} \tag{6}$$

where

$$\omega_*^T \equiv \frac{kT}{M\Omega F_M}\frac{\partial F_M}{\partial r} = \frac{kT}{M\Omega N}\frac{\partial N}{\partial r}\left[1 + \eta\left(\frac{ME}{T} - \frac{3}{2}\right)\right]$$

$$\eta = d\,\ell n\,T/d\,\ell n\,N \ ;$$

$\nabla_f\hat{\Phi}\cdot\nabla_v F_D$ has been rewritten by employing

$$\nabla_v F_D = \Omega^{-1}\left[\underline{e}\,\frac{\partial F_M}{\partial r} - Mv_\perp \underline{v}\sin\phi\,\frac{\partial}{\partial r}\left(\frac{F_M}{T}\right)\right] \quad ,$$

and

$$[\hat{\Phi}\cos\phi - \Omega^{-1}\sin\phi\underline{v}\cdot\nabla_f\hat{\Phi}] \approx \frac{\partial}{\partial\phi'}\,[\hat{\Phi}\sin\phi\exp(iL)]$$

has been employed, so that contributions from $\partial(F_M/T)/\partial r$ terms may be safely neglected.

To evaluate the ϕ' integrals in eq. (6) the integrands must be expressed in terms of the primed variables. Only the r dependence presents a problem; however, the usual assumption[1-6] that the gyroradius is small compared to scale lengths and radial wavelengths permits any function of r, Q(r), to be expanded about r' with the result that $Q(r) = Q(r') + (r' - r)\partial Q/\partial r'\cdots$.

Evaluating the integrals in the first angular brackets on the left hand side of eq. (6) by neglecting corrections of order $v_\perp/\Omega r$ compared to one results in the familiar curvature and gradient B drifts plus the parallel velocity correction. The curvature drift is recovered precisely (because it contains no slow θ dependence), while the gradient B drift and parallel velocity correction are reproduced to within terms of order m^{-1}. Again by neglecting corrections of order $v_\perp/\Omega r$ the right hand side of eq. (6) can be evaluated by retaining $\partial^2\hat{\Phi}/\partial r'^2$ terms from the gyroradius expansion of $(\omega - \omega_*^T)\hat{\Phi}$ and employing $\langle\exp(iL)\rangle = J_0(kv_\perp/\Omega)$ and $\langle\sin^2\phi\exp(iL)\rangle = (\Omega/kv_\perp)J_1(kv_\perp/\Omega)$.

As a result of the preceding, eq. (6) yields

$$\left\{-i\omega + [v_\parallel + \frac{\mu B}{\Omega}\,\underline{\hat{n}}\cdot\nabla\times\underline{\hat{n}}]\underline{\hat{n}}\cdot\nabla_s + \underline{\hat{n}}\times\left[\frac{v_\parallel^2}{\Omega}\,\underline{\hat{n}}\cdot\nabla\underline{\hat{n}} + \frac{\mu}{\Omega}\nabla B\right]\cdot\nabla_f\right\}\hat{g}' - \left\langle \exp(iL)C\{\hat{g}'\exp(-iL)\}\right\rangle = -\frac{iZeF_M}{T}(\omega - \omega_*^T)\left\langle\hat{\Phi}\exp(iL)\right\rangle \quad , \tag{7}$$

where $\underline{\hat{n}}\cdot\nabla\times\underline{\hat{n}} = [\partial(\varepsilon/q)/\partial r] + (1/qR_0)$ and

$$\begin{aligned}\left\langle\hat{\Phi}(r)\exp(iL)\right\rangle &= J_0(kv_\perp/\Omega)\hat{\Phi}(r') + J_1(kv_\perp/\Omega)\,\frac{v_\perp}{2k\Omega}\,\frac{\partial^2\hat{\Phi}}{\partial r'^2} \\ &= J_0(kv_\perp/\Omega)\left[\hat{\Phi}(r) + \frac{v_\perp}{\Omega}\sin\phi\,\frac{\partial\hat{\Phi}}{\partial r} + \frac{v_\perp^2}{2\Omega^2}\sin^2\phi\,\frac{\partial^2\hat{\Phi}}{\partial r^2}\right] \\ &\quad + J_1(kv_\perp/\Omega)\,\frac{v_\perp}{2k\Omega}\,\frac{\partial^2\hat{\Phi}}{\partial r^2} \quad .\end{aligned} \tag{8}$$

The latter form of eq. (8) follows upon changing from the prime variables back to the unprimed variables. This final form for $\langle\hat{\Phi}\exp(iL)\rangle$ agrees with the result obtained from slab models in sheared magnetic fields[10] but differs from the result obtained from a WKB or eikonal treatment of the radial variation of $\hat{\Phi}$.[8] Such WKB treatments give $\langle\hat{\Phi}\exp(iL)\rangle = J_0(k_\perp v_\perp/\Omega)\hat{\Phi}$ where $k_\perp^2$ is often interpreted as $k^2 - \partial^2/\partial r^2$. Consequently, the proper result is obtained from a WKB treatment only in the local (shearless) limit. Note also that the factor exp(-iL) in eq. (5) results in additional Bessel functions when moments of $\hat{f}$ are formed. In particular,

$$\int d^3v F_M \left\langle\hat{\Phi}\exp(iL)\right\rangle \exp(-iL) = N\exp(-b)I_0(b)\left\{\hat{\Phi} + \left[1 - \frac{\partial}{\partial b}\ln I_0(b)\right]\frac{v_\perp^2}{\Omega^2}\,\frac{\partial^2\hat{\Phi}}{\partial r^2}\right\} \quad ,$$

where $b \equiv k^2 v_T^2/\Omega^2$ and $v_T = (T/M)^{1/2}$. When $\hat{\Phi}J_0(k_\perp v_\perp/\Omega)$ is employed in the preceding integral the result is $N\exp(-k_\perp^2 v_T^2/\Omega^2)I_0(k_\perp^2 v_T^2/\Omega^2)$ which reduces to the proper result for $k_\perp^2 v_T^2/\Omega^2 < 1$ if the mnemonic $k_\perp^2\hat{\Phi} \rightarrow (k^2 - \partial^2/\partial r^2)\hat{\Phi}$ is employed.[4]

In appendix A, a full electromagnetic derivation of the gyro-kinetic equations is presented which also retains an unperturbed electric field $\underline{E}_0 = E_0\hat{\underline{\zeta}}$ and neoclassical, F_N, contributions to $F = F_M + F_D + F_N$. The vector potential $\underline{A}$ is written as $\underline{A} = A_\parallel\hat{\underline{n}} + A_r\hat{\underline{r}} + A_e\hat{\underline{e}}$ and no gauge is assumed. Employing $F_N/F_M << 1$, the neoclassical modifications are found to result in only small corrections to the Maxwellian contributions. As a result, eq. (5) is found to remain valid and eqs. (7) and (8) are replaced by

$$\Big\{-i\omega + [v_\parallel + \frac{\mu B}{\Omega}\,\hat{\underline{n}}\cdot\nabla\times\hat{\underline{n}}]\hat{\underline{n}}\cdot\nabla_s + \hat{\underline{n}}\times[\frac{v_\parallel^2}{\Omega}\,\hat{\underline{n}}\cdot\nabla\hat{\underline{n}} + \frac{\mu}{\Omega}\,\nabla B]\cdot\nabla_f$$

$$+ \frac{Ze}{M}\,v_\parallel E_0\,\frac{\partial}{\partial E}\Big\}\hat{g}' - \Big\langle \exp(iL)C\{\hat{g}'\exp(-iL)\}\Big\rangle$$

$$= -\frac{iZeF_M}{T}[\omega - \omega_*^T - \frac{iZe}{T}\,E_0 v_\parallel]\Big\{J_0\left(\frac{kv_\perp}{\Omega}\right)\left[\hat{\Phi} - \frac{v_\parallel}{c}\,\hat{A}_\parallel\right. \tag{9}$$

$$\left. + \frac{v_\perp}{\Omega}\sin\phi\frac{\partial\hat{\Phi}}{\partial r} + \frac{v_\perp^2}{2\Omega^2}\,\sin^2\phi\,\frac{\partial^2\hat{\Phi}}{\partial r^2}\right] + J_1\left(\frac{kv_\perp}{\Omega}\right)\left[\frac{v_\perp}{2k\Omega}\,\frac{\partial^2\hat{\Phi}}{\partial r^2} + \frac{v_\perp}{kc}\,\frac{\partial\hat{A}_e}{\partial r} - \frac{iv_\perp}{c}\,\hat{A}_r\right]\Big\}$$

$$-\frac{Z^2e^2F_M}{T^2}\,v_\parallel E_0\left[\frac{v_\parallel}{c}\,\hat{A}_\parallel J_0\left(\frac{kv_\perp}{\Omega}\right) - \left(\frac{v_\perp}{k\Omega}\,\frac{\partial\hat{A}_e}{\partial r} - \frac{iv_\perp}{c}\,\hat{A}_r\right)J_1\left(\frac{kv_\perp}{\Omega}\right)\right] \quad .$$

By employing eqs. (5) and (9) for drift waves in a sheared magnetic field the extension of the finite β equations of references 3 and 4 to arbitrary $kv_\perp/\Omega$ could be carried out by employing a Coulomb gauge for $\underline{A}$ in order to eliminate $\hat{A}_e$. A system of three coupled differential equations is obtained from Poisson's equation and the parallel and $\hat{\underline{r}}$ components of Ampere's law. The generality of the subsequent reduction of these finite β equations to a system of two coupled second order differential equations in $\hat{\Phi}$ and $\hat{A}_\parallel$ could be checked.

Equation (5), and (7) - (8) or (9), plus an appropriate collision operator are the basic equations that result from the gyro-kinetic ordering. In Appendix B collision operators valid for arbitrary $kv_\perp/\Omega$ are obtained for like and unlike particle collisions.

For collisions of electrons with ions a Lorentz or pitch angle operator, eq. (B6), may be employed.[11-12] Changing to μ, E, ϕ variables and gyro-averaging results in

$$\left\langle \exp(iL)C_{ei}\{\hat{g}_e'\exp(-iL)\}\right\rangle = \frac{4\pi Z_i^2 e^4 N_i \ln\Lambda}{m_e^2 v^3}\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\left[\mu v_\parallel \frac{\partial\hat{g}_e'}{\partial\mu}\right] . \tag{11}$$

For the collisions of ions with electrons $C_{ie} \approx 0$ may be employed because $C_{ie} \sim (m_e/M_i)^{1/2}C_{ii}$ and the order m_e/M_i momentum corrections from C_{ie} may be neglected. In obtaining the gyro-averaged collision operators the distinction between the primed and unprimed variables may be ignored.

Eq. (11) does nor contain $kv_\perp/\Omega$ corrections because only the perturbed electron distribution function enters. For ion-ion encounters, however, finite gyroradius corrections can be extremely important for $kv_\perp/\Omega >> 1$.[13] Transforming the first form of (B7) to μ, E, ϕ and gyro-averaging, the like particle collision operator appropriate for arbitrary $kv_\perp/\Omega$ is found to be

$$\left\langle \exp(iL)\{C_\ell\{\hat{g}'\exp(-iL)\} - P_\ell F_M\}\right\rangle =$$

$$\frac{4\pi Z^4 e^4 N\ln\Lambda}{M^2 v^3}\Bigg\{[\mathrm{Erf}(x) - \psi(x)]\,\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}[\mu v_\parallel \frac{\partial\hat{g}'}{\partial\mu}]$$

$$+ 4E^2\psi(x)\left[\frac{\partial^2 g'}{\partial E^2} + \frac{\mu}{E}\frac{\partial\hat{g}'}{\partial\mu}\right] + 2v^3\left(\frac{M}{2T}\right)^{3/2}[\mathrm{Erf}'(x)]\hat{g}'$$

$$- \frac{k^2 E}{\Omega^2}\,[\mathrm{Erf}(x) - \psi(x) - \frac{\mu B}{2E}\,(\mathrm{Erf}(x) - 3\psi(x))]\Bigg\} , \tag{12}$$

where $x = v(M/2T)^{1/2}$; Erf, Erf', and ψ are defined in appendix B; and from eqs. (B9),

$$\left\langle \exp(iL)P_\ell F_M \right\rangle = \frac{8\pi Z^4 e^4 F_M \ell n\Lambda}{T^2} \left\{ v_\parallel J_0^2\left(\frac{kv_\perp}{\Omega}\right) \int d^3v \, \frac{v_\parallel}{v} \, \psi(x)\hat{g}' \right.$$

$$+ v_\perp J_1^2\left(\frac{kv_\perp}{\Omega}\right) \int d^3v \, \frac{v_\perp}{v} \, \psi(x)\hat{g}'$$

$$\left. + \frac{1}{3}\left(E - \frac{3T}{2M}\right) J_0^2\left(\frac{kv_\perp}{\Omega}\right) \int d^3v \, \frac{\hat{g}'}{v} \left[\mathrm{Erf}(x) - 2x\mathrm{Erf}'(x)\right] \right\} . \qquad (13)$$

For large $k^2T/M\Omega^2$, eqs. (12) and (13) simplify considerably so that for ion-ion collisions

$$\left\langle \exp(iL)C_{ii}\{\hat{g}_i'\exp(-iL)\} \right\rangle = - \frac{4\pi Z_i^4 e^4 N_i \ell n\Lambda}{M_i^2 v^3} \frac{k^2 E}{\Omega_i^2} \left[\mathrm{Erf} - \psi - \frac{\mu B}{2E}(\mathrm{Erf} - 3\psi)\right]$$

$$\approx - \frac{2\pi Z_i^4 e^4 N_i \ell n\Lambda}{M_i^2 v} \frac{k^2}{\Omega_i^2} \mathrm{Erf}(x) \quad .$$

The last step follows by noting that to within factors of two $\mathrm{Erf} - \psi - (\mu B/2E)(\mathrm{Erf} - \psi) \approx \mathrm{Erf}$. For $k^2T/M\Omega^2 \lesssim 1$ no similar simplification is apparent. For trapped particles a simplification results when the orderings $\partial/\partial E \sim 1/E$ and $\partial/\partial\mu \sim 1/\mu\sqrt{\varepsilon}$ are employed, where ε is the inverse aspect ratio. Then eqs. (12) and (13) reduce to

$$\left\langle \exp(iL)C_\ell\{g'\exp(-iL)\} \right\rangle =$$

$$\frac{4\pi Z^4 e^4 N \ell n\Lambda}{M^2 v^3} \left\{ \left[\mathrm{Erf}(x) - \psi(x)\right] \frac{v_\parallel}{B} \frac{\partial}{\partial\mu} \left[\mu v_\parallel \frac{\partial \hat{g}'}{\partial\mu}\right] + 4\mu^2 \psi(x) \frac{\partial^2 \hat{g}'}{\partial\mu^2} \right\} \quad .$$

This form is appropriate for the collisions of trapped particles with particles of the same species which are either trapped or circulating.

DISCUSSION

By employing an ordering in which finite gyroradius corrections are retained to lowest order in the linearized Fokker-Planck equation, a description based on the notions of drift-kinetic theory is developed which is appropriate in sheared magnetic fields. The equation that results from this gyro-kinetic ordering is valid for an arbitrary ratio of poloidal wavelength to gyroradius. The result agrees with previous slab results[2] and indicates that WKB or eikonal gyro-kinetic treatments[8-9] must be employed with care for sheared magnetic fields. In addition, the full electromagnetic generalization of the gyro-kinetic equation is obtained with the unperturbed toroidal electric field modifications. Neoclassical corrections to the unperturbed distribution function have been retained also, but are shown to result in negligible corrections.

In order to obtain a like particle (ion-ion) collision operator valid for arbitrary value of poloidal wavenumber times gyroradius, the appropriate gyro-average of the Fokker-Planck collision operator is evaluated. The resulting expression is rather involved, but for large values of the gyroradius over poloidal wavelength is extremely simple. Other simplifications of the like particle collision operator may also be possible in certain limits; an additional example appropriate for trapped particles is noted.

APPENDIX A

Retaining the vector potential $\underline{A}$, the neoclassical modification of the unperturbed distribution function $F_N = F_N(r,\theta,\mu,\mathcal{E})$, and the unperturbed electric field $\underline{E}_0 = E_0\hat{\underline{\zeta}}$ replaces eq. (1) by

$$\frac{\partial}{\partial t}\left[f - \frac{Ze}{Mc}\underline{A}\cdot\nabla_v F\right] + \left[\underline{v}\cdot\nabla - \Omega\frac{\partial}{\partial\phi}\right]\left\{f - \frac{Ze\Phi}{M}\frac{\partial}{\partial\mathcal{E}}(F_M + F_N) - \frac{Ze}{MB}\frac{\partial F_N}{\partial\mu}\left(\Phi - \frac{v_\parallel}{c}A_\parallel\right)\right\}$$

$$- \frac{Ze}{M\Omega}\frac{\partial F_M}{\partial r}\,\hat{\underline{e}}\cdot\nabla\Phi - Zev_\perp\left[\Phi\cos\phi - \frac{1}{\Omega}\sin\phi\,\underline{v}\cdot\nabla\Phi\right]\frac{\partial}{\partial r}\left(\frac{F_M}{T}\right)$$

$$+ \frac{Zev_\parallel}{MB}\frac{\partial F_N}{\partial\mu}\,\hat{\underline{n}}\cdot\nabla\left(\Phi - \frac{v_\parallel}{c}A_\parallel\right) + \frac{Ze}{M}\Phi\underline{v}\cdot\nabla\left(\frac{\partial F_N}{\partial\mathcal{E}} + \frac{1}{B}\frac{\partial F_N}{\partial\mu}\right) - \frac{Zev_\parallel}{Mc}A_\parallel\underline{v}\cdot\nabla\,\frac{1}{B}\left(\frac{\partial F_N}{\partial\mu}\right)$$

$$+ \frac{Ze}{Mc}\left(\nabla\underline{A}\cdot\underline{v} - \underline{v}\cdot\nabla\underline{A}\right)\cdot\nabla_v F_D - \frac{Ze}{Mc}\left[v_\parallel\hat{\underline{n}}\cdot\nabla\underline{A}\cdot\underline{v}_\perp + \underline{v}_\perp\cdot\nabla(\hat{\underline{n}}v_\parallel)\cdot\underline{A}\right]\frac{1}{B}\frac{\partial F_N}{\partial\mu}$$

$$+ \frac{Ze}{M}\hat{\underline{\zeta}}\cdot\underline{v}E_0\frac{\partial f}{\partial\mathcal{E}} + \left[\dot{\mu} + \frac{Ze}{MB}\hat{\underline{\zeta}}\cdot\underline{v}_\perp E_0\right]\frac{\partial f}{\partial\mu} + \left[\dot{\phi} + \frac{Ze}{Mv_\perp}\hat{\underline{\zeta}}\cdot\hat{\underline{\phi}}E_0\right]\frac{\partial f}{\partial\phi} = C\{f\}\ . \qquad \text{(A1)}$$

In obtaining eq. (A1), $\nabla_v F_D$ from the text and

$$\underline{v}\times(\nabla\times\underline{A})\cdot\hat{\underline{n}}\,\frac{v_\parallel}{B}\frac{\partial F_N}{\partial\mu} = v_\parallel A_\parallel\underline{v}\cdot\nabla\left(\frac{1}{B}\frac{\partial F_N}{\partial\mu}\right) - \underline{v}\cdot\nabla\left(\frac{v_\parallel A_\parallel}{B}\frac{\partial F_N}{\partial\mu}\right)$$

$$+ \left[v_\parallel\hat{\underline{n}}\cdot\nabla(v_\parallel A_\parallel) + v_\parallel\hat{\underline{n}}\cdot\nabla\underline{A}\cdot\underline{v}_\perp + \underline{v}_\perp\cdot\nabla(\hat{\underline{n}}v_\parallel)\cdot\underline{A}\right]\frac{1}{B}\frac{\partial F_N}{\partial\mu}$$

have been employed.

In addition to the small terms retained in eq. (4) it is now convenient to retain certain neoclassical terms to lowest order, with the result that

$$\left\{[\tilde{v}_\perp\hat{\underline{r}}\cos\phi + v_\perp\hat{\underline{e}}\sin\phi]\cdot\nabla_f - \Omega\frac{\partial}{\partial\phi} + i\tilde{v}_\perp\cos\phi\frac{\partial L}{\partial r}\right\}\left\{\hat{f} - \frac{Ze\hat{\Phi}}{M}\frac{\partial}{\partial\mathcal{E}}(F_M + F_N)\right.$$

$$\left. - \frac{Ze}{MB}\frac{\partial F_N}{\partial\mu}\left(\hat{\Phi} - \frac{v_\parallel}{c}\hat{A}_\parallel\right)\right\} \approx - \frac{Zev_\parallel}{MB}\frac{\partial F_N}{\partial\mu}\,\hat{\underline{n}}\cdot\nabla_f\left(\hat{\Phi} - \frac{v_\parallel}{c}\hat{A}\right) \equiv 0\ . \qquad \text{(A2)}$$

Changing eq. (A2) to the primed variables results in

$$\hat{f} - \frac{Ze\hat{\Phi}}{M}\frac{\partial}{\partial E}(F_M + F_N) - \frac{Ze}{MB}\frac{\partial F_N}{\partial \mu}(\hat{\Phi} - \frac{v_\parallel}{c}\hat{A}_\parallel) = \hat{g}'\exp(-iL) , \tag{A3}$$

where again $\hat{g}'$ is independent of ϕ'.

The next order equation is obtained from eq. (A1) in the same manner as the electrostatic case. Upon carrying out the primed gyrophase average the resulting equation has the left hand side shown in eq. (9). Retaining only terms that result in non-negligible contributions the right hand side gives

$$\frac{iZe}{M}\Big\langle\Big\{\Big[\omega\frac{\partial}{\partial E}(F_M+F_N)-\omega_*^T\frac{\partial F_M}{\partial E}\Big]\hat{\Phi}+\frac{iZe}{M}\underline{\hat{\xi}}\cdot\underline{v}E_0\frac{\partial}{\partial E}\Big[\hat{\Phi}\frac{\partial}{\partial E}(F_M+F_N)+(\hat{\Phi}-\frac{v_\parallel}{c}\hat{A}_\parallel)\frac{1}{B}\frac{\partial F_N}{\partial \mu}\Big]$$

$$+\frac{1}{B}\frac{\partial F_N}{\partial \mu}[(\omega+iv_\parallel\underline{\hat{n}}\cdot\nabla_s)(\hat{\Phi}-\frac{v_\parallel}{c}\hat{A}_\parallel)]+i\hat{\Phi}\underline{v}\cdot\nabla\left(\frac{\partial F_N}{\partial E}+\frac{1}{B}\frac{\partial F_N}{\partial \mu}\right)-\frac{iv_\parallel}{c}\hat{A}_\parallel\underline{v}\cdot\nabla\left(\frac{1}{B}\frac{\partial F_N}{\partial \mu}\right)$$

$$-\frac{\omega}{c}\underline{\hat{A}}\cdot\nabla_vF+\frac{i}{c}[\nabla_f\underline{\hat{A}}\cdot\underline{v}-\underline{v}\cdot\nabla_f\underline{\hat{A}}]\cdot\nabla_vF_D\Big\}\exp(iL)\Big\rangle =$$

$$\frac{iZe}{M}\Bigg\{\Bigg[\omega\frac{\partial}{\partial E}(F_M+F_N)-\omega_*^T\frac{\partial F_M}{\partial E}+\frac{iZev_\parallel E_0}{MB}\frac{\partial}{\partial E}\left(\frac{\partial F_N}{\partial \mu}\right)\Bigg][\langle\hat{\Phi}\exp(iL)\rangle-\frac{v_\parallel}{c}\hat{A}_\parallel J_0]$$

$$+\frac{iZe}{M}v_\parallel E_0\left[\frac{\partial^2}{\partial E^2}(F_M+F_N)\right]\langle\hat{\Phi}\exp(iL)\rangle \tag{A4}$$

$$+\left[\omega\left(\frac{\partial F_M}{\partial E}+\frac{\partial F_N}{\partial E}+\frac{1}{B}\frac{\partial F_N}{\partial \mu}\right)-\omega_*^T\frac{\partial F_N}{\partial E}\right]\left[\frac{v_\perp}{kc}\frac{\partial \hat{A}_e}{\partial r}-\frac{iv_\perp}{c}\hat{A}_r\right]J_1$$

$$+\frac{1}{B}\frac{\partial F_N}{\partial \mu}[\omega(\langle\hat{\Phi}\exp(iL)\rangle-\frac{v_\parallel}{c}\hat{A}_\parallel J_0)+J_0iv_\parallel\underline{\hat{n}}\cdot\nabla_s(\hat{\Phi}-\frac{v_\parallel}{c}\hat{A}_\parallel)]$$

$$-v_\perp J_1\left[\hat{\Phi}\frac{\partial}{\partial r}\left(\frac{\partial F_N}{\partial E}\right)+(\hat{\Phi}-\frac{v_\parallel}{c}\hat{A}_\parallel)\frac{\partial}{\partial r}\left(\frac{1}{B}\frac{\partial F_N}{\partial \mu}\right)\right]\Bigg\} .$$

From eqs. (A3) and (A4) the importance of the neoclassical terms can be estimated. Employing $F_N/F_M \sim (q/\varepsilon)(v_\perp/\Omega)|N^{-1}\partial N/\partial r| << 1$ and noting that the $(\omega + i\underline{\hat{n}}\cdot\nabla_s)(\hat{\Phi} - v_\parallel c^{-1}\hat{A}_\parallel)$ terms of (A4) can be combined with the terms on the left hand side, the neoclassical terms are seen to result in small corrections to the Maxwellian contributions. As a result, eq. (A3) reduces to eq. (5) and eq. (A4) becomes

$$\frac{iZe}{M}\left\{\frac{\partial F_M}{\partial E}(\omega - \omega_*^T)\left[\langle\hat{\Phi}\exp(iL)\rangle - \frac{v_\parallel}{c}A_\parallel J_o + \left(\frac{v_\perp}{kc}\frac{\partial\hat{A}_e}{\partial r} - \frac{iv_\perp}{c}\hat{A}_r\right)J_1\right.\right.$$
$$\left.+\frac{iZeE_o}{M}v_\parallel\frac{\partial^2 F_M}{\partial E^2}\langle\hat{\Phi}\exp(iL)\rangle\right\} \quad . \tag{A5}$$

Expression (9) then follows by substituting eq. (8) into (A5).

APPENDIX B

The pitch angle scattering collision operator of Rosenbluth, Hazeltine, and Hinton[11] and/or Rosenbluth, Ross and Kostomarov[12] can no longer be safely employed because operations on the exp(-iL) in $C\{\hat{g}'\exp(-iL)\}$ can result in terms which must be retained for $kv_\perp/\Omega \gtrsim 1$. As a result, an appropriate collision operator will be derived by starting with the full Fokker-Planck operator[14]

$$\left.\frac{d\mathcal{F}}{dt}\right|_{coll} = \sum{}' \Gamma' \nabla_v \cdot \left\{ \nabla_v \cdot [\tfrac{1}{2}\mathcal{F}\nabla_v\nabla_v G] - \mathcal{F}\nabla_v H \right\}$$

with

$$H = H(\underline{v}) = \frac{M + M'}{M'} \int d^3v' \frac{\mathcal{F}'}{|\underline{v} - \underline{v}'|}$$

$$\nabla_v^2 H = -4\pi \frac{M + M'}{M'} \mathcal{F}'$$

$$G = G(\underline{v}) = \int d^3v' |\underline{v} - \underline{v}'| \mathcal{F}' \qquad \text{(B1)}$$

$$\nabla_v^2 G = \frac{2M'}{M + M'} H , \qquad \nabla_v^4 G = -8\pi\mathcal{F}'$$

$$\Gamma' = (4\pi Z^2 Z'^2 e^4/M^2)\ell n\,\Lambda ,$$

where $\mathcal{F}$ is the total (unperturbed plus perturbed) distribution function and primes denote the species integrated over all velocities. The sum is over all primed species and must include like particle collisions; $\ell n\Lambda$ is the Coulomb logarithm.

Linearizing by writing $\mathcal{F} = F_{MB} + h$, where

$$F_{MB} = F_M \exp(-Ze\Phi/T) \simeq F_M[1 - \frac{Ze}{T}\Phi]$$

$$h = \hat{g}'\exp(-iL) \; ,$$

results in

$$C\{h\} = \sum{}'\exp\left(-\frac{Z'e\Phi}{T'}\right)[\Gamma'K' + P'(\underline{v})F_M]$$

$$\approx \sum{}'[\Gamma'K' + P'(\underline{v})F_M] \tag{B2}$$

with

$$K' = \nabla_v\cdot\{\nabla_v\cdot[\tfrac{1}{2}\, h\nabla_v\nabla_v G] - h\nabla_v H\}$$

$$H = \frac{N'}{v}\,\frac{(M + M')}{M'}\,\mathrm{Erf}\left[v\left(\frac{M'}{2T'}\right)^{1/2}\right]$$

$$G = N'\left\{(v + \frac{T'}{M'v})\mathrm{Erf}\left[v\left(\frac{M'}{2T'}\right)^{1/2}\right] + \left(\frac{T'}{2M'}\right)^{1/2}\mathrm{Erf}'\left[v\left(\frac{M'}{2T'}\right)^{1/2}\right]\right\} \tag{B3}$$

$$\mathrm{Erf}(x) = \frac{2}{\pi^{1/2}}\int_0^x dt\,\exp(-t^2)$$

$$\mathrm{Erf}'(x) = \frac{2}{\pi^{1/2}}\frac{d}{dx}\int_0^x dt\,\exp(-t^2) = \frac{2}{\pi^{1/2}}\exp(-x^2) \quad ,$$

where H and G have been evaluated using F_M' and $v = |\underline{v}|$. The functional of h', $P'F_M$, is in general proportional to $\nabla_v\cdot\{\nabla_v\cdot[\frac{1}{2} F_M\nabla_v\nabla_v G] - F_M\nabla_v H$ with G and H evaluated using h'; however, rather than use such a complicated form is it convenient to employ a simpler P' determined by the conservation of momentum and energy constraints. The model of eq. (B2) may also[11] be thought of as somewhat similar to a Bhatnagar-Gross-Krook model in that the momentum and energy removed by the K' term is replenished in a Maxwellian distribution by the $P'F_M$ term.

Employing the properties of G and H listed in eqs. (B1), K' may be rewritten as

$$K' = \frac{1}{2}\nabla_v\nabla_v G:\nabla_v\nabla_v h + \frac{M' - M}{M' + M}\nabla_v H\cdot\nabla_v h + 4\pi\frac{M}{M'} hF'_M \quad .$$

For $\mathcal{F}' = F'_M$, G and H are functions of v only, as indicated in eqs. (B3). As a result K' may be rewritten once again as

$$K' = \left\{\left[\frac{\underline{\underline{I}}v^2 - 3\underline{v}\,\underline{v}}{2v^3}\right]\frac{\partial G}{\partial v} + \frac{M'H}{M + M'}\frac{\underline{v}\,\underline{v}}{v^2}\right\} : \nabla_v\nabla_v h$$

$$+ \frac{M' - M}{M' + M}\frac{1}{v}\frac{\partial H}{\partial v}\underline{v}\cdot\nabla_v h + 4\pi\frac{M}{M'}F'_M h \quad , \qquad \text{(B4)}$$

where $\underline{\underline{I}}$ is the unit dyadic.

To simplify K' further, spherical velocity variables $v = |\underline{v}|$, α, and ϕ defined with respect to $\underline{B}_0 = B\hat{\underline{n}}$ are employed. The angles α and ϕ are the pitch angle $\tan^{-1}(v_\perp/v_\parallel)$ and the gyrophase, respectively; the unit vectors form a right handed orthogonal system in which $\hat{\underline{v}}\times\hat{\underline{\alpha}} = \hat{\underline{\phi}}$. Because of the form of eq. (B4), only terms in $\hat{\underline{v}}\,\hat{\underline{v}}$, $\hat{\underline{\alpha}}\,\hat{\underline{\alpha}}$, and $\hat{\underline{\phi}}\,\hat{\underline{\phi}}$ need be retained when forming $\nabla_v\nabla_v h$; all other terms give zero when double dotted into $\underline{v}\,\underline{v}$ and $\underline{\underline{I}} = \hat{\underline{v}}\,\hat{\underline{v}} + \hat{\underline{\alpha}}\,\hat{\underline{\alpha}} + \hat{\underline{\phi}}\,\hat{\underline{\phi}}$. Carrying out the transformation to spherical velocity variables results in

$$K' = \frac{1}{2v^3}\frac{\partial G}{\partial v}\left\{\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{\partial h}{\partial\alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial^2 h}{\partial\phi^2}\right\} + 4\pi\frac{M}{M'}F'_M h$$

$$\text{(B5)}$$

$$+ \left[\frac{M'H}{M + M'} - \frac{1}{v}\frac{\partial G}{\partial v}\right] v\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial h}{\partial v}\right) + \left[\frac{M'H}{M + M'} + \frac{M' - M}{M' + M}v\frac{\partial H}{\partial v}\right]\frac{1}{v}\frac{\partial h}{\partial v} \,.$$

At this point is it convenient to consider unlike and like particle collisions separately. In particular, for the collisions of electrons with ions a Lorentz collision operator will be employed. As a result, collisions of electrons with ions will automatically conserve number and energy, and the electrons may transfer as much momentum to the ions as they wish. Consequently, for these electron-ion collisions the P' of eq. (B2) is zero to the order of interest. In addition, for the collisions of ions with electrons $C_{ie} = 0$ may be employed because $C_{ie} \sim (m_e/M_i)^{1/2} C_{ii}$.

For the collisions of electrons ($M \to m_e$, $Z \to -1$, $T \to T_e$, $h \to h_e$) with ions ($N' \to N_i$, $M' \to M_i$, $Z' \to Z_i$, $T' \to T_i$, $F_M \to F_{Mi}$), $H \approx N'/v$, $\partial H/\partial v \approx -N'/v^2$ and $\partial G/\partial v \approx N'$ may be employed to obtain the Lorentz gas collision operator ($P' \to 0$)

$$C_{ei} \approx \frac{2\pi Z_i^2 e^4 N_i \ln\Lambda}{m_e^2 v^3}\left[\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\,\frac{\partial h_e}{\partial\alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial^2 h_e}{\partial\phi^2}\right] \quad , \tag{B6}$$

where mass ratio corrections are neglected. From the form of eq. (B6), conservation of number and energy are apparent.

For like particle collisions, employing the expressions for G and H from eqs. (B3) allows eq. (B5) to be written as

$$\begin{aligned} C_\ell - P_\ell F_M &= \frac{N\Gamma}{2v^3}\,[\mathrm{Erf}(x) - \psi(x)]\left[\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\,\frac{\partial h}{\partial\alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial^2 h}{\partial\phi^2}\right] \\ &\quad + N\Gamma\,\left[\psi(x)\,\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial h}{\partial v}\right) + \mathrm{Erf}(x)\,\frac{1}{v^2}\frac{\partial h}{\partial v} + 2\left(\frac{M}{2T}\right)^{3/2} h\,\mathrm{Erf}'(x)\right] \\ &= \frac{N\Gamma}{2v^3}\,[\mathrm{Erf}(x) - \psi(x)]\left[\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\,\frac{\partial h}{\partial\alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial^2 h}{\partial\phi^2}\right] \\ &\quad + \frac{N\Gamma}{v^2}\frac{\partial}{\partial v}\left\{v\psi(x)\,\frac{\partial h}{\partial v} + [\mathrm{Erf}(x) - 2\psi(x)]h - v\,\frac{\partial\psi(x)}{\partial v}\,h\right\} \quad , \end{aligned} \tag{B7}$$

where $x = v(M/2T)^{1/2}$ and $\psi(x) \equiv (1/2x^2)[\mathrm{Erf}(x) - x\mathrm{Erf}'(x)]$. The second form of equation (B7) follows by noting that

$$\frac{\partial}{\partial v}\left[v\,\frac{\partial\psi(x)}{\partial v}\right] - \frac{\partial}{\partial v}\,[\mathrm{Erf}(x) - 2\psi(x)] + 2\,\frac{x^3}{v}\,\mathrm{Erf}'(x) \equiv 0\ .$$

This latter form of eq. (B7) can then be rewritten in a form more convenient for determining conservation properties, namely,

$$C_\ell - P_\ell F_M = \nabla_v\cdot\left\{\frac{2\pi Z^4 e^4 N\ell n\Lambda}{M^2}\left[\nabla_v\left([\mathrm{Erf}(x) - \psi(x)]\,\frac{h}{v}\right)\right.\right.$$
$$\left.\left. + \frac{\hat{v}}{v^2}\left(\frac{\partial}{\partial v}\,\{v[3\psi(x) - \mathrm{Erf}(x)]h\} + 8x^2\psi(x)h\right)\right]\right\}\ . \qquad \text{(B8)}$$

In this form conservation of number is automatic, and P_ℓ is determined from conservation of momentum and energy. Taking $P_\ell = \underline{v}\cdot\underline{p}_\ell + \lambda_\ell[(Mv^2/2T) - \frac{3}{2}]$, p_ℓ and λ_ℓ are determined from conservation of momentum and energy, respectively,

$$\underline{p}_\ell = \frac{8\pi Z^4 e^4 \ell n\Lambda}{T^2}\int d^3v\,\underline{\hat{v}}\,\psi(x)h$$

$$\text{(B9)}$$

$$\lambda_\ell = \frac{8\pi Z^4 e^4 \ell n\Lambda}{3MT}\int d^3v\,\frac{h}{v}\,[\mathrm{Erf}(x) - 2x\mathrm{Erf}'(x)]\ .$$

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