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Á. SEBESTYÉN

ON AN EXTENSION OF COVARIANCE

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

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Á. Sebestyén
Theoretical Physics Department
Central Research Institute for Physics
1525 Budapest 114, P.O.B.49 Hungary

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ABSTRACT

An extension of the principle of covariance for internal degrees of freedom is considered. The conditions for a system to be isolated are given. From these the field equations and the equations of motion are deduced. As an application the Keplerian motion and the shift of the perihelion are derived

РЕЗЮМЕ

В данной статье принцип ковариантности распространяется на внутренние степени свободы. Сформулировано определение изолированных систем и показано, что в таких системах возникают потенциальные и типа Лоренца внутренние силы. Выводятся уравнения поля и уравнения движения, и, в качестве применения, обсуждается проблема Кеплера. Указывается и на присутствие Перигелия движения.

KIVONAT

A cikkben a kovariancia elvét terjesztjük ki belső szabadsági fokokra. Megfogalmazzuk a zárt rendszerek definicióját és rámutatunk arra, hogy ilyen rendszerekben Lorentz tipusu és potenciális belső erők ébrednek. Levezetjük a tér- és mozgásegyenleteket és alkalmazásként a Kepler problémát tárgyaljuk. Rámutatunk a periheliummozgás jelenlétére is.

§/1/: INTRODUCTION

In his famous book "Mechanics in New Context" [1] H. Hertz has shown that for a wide class of the mechanical problems a reformulation is possible: the generalized forces can be removed by enlarging the number of degrees of freedom i.e. by introducing new coordinates. In this way one arrives at a problem of free motion: the evolution of the system is described by a geodesic of the manifold of the generalized coordinates, this manifold is curved and has a Riemannian structure with a metric tensor depending on the coordinates only.

The theory of General Relativity in fact reveals how the gravitational forces acting on a test particle can be eliminated by introducing time as a new coordinate. In General Relativity however Einstein's equations serve to determine the structure of space-time, whereas in Hertz's forceless mechanics such equations do not exist; it is simply a reformulation of the classical theory.

In this paper we try to set up a model which is based both on the ideas of the forceless mechanics and on General Relativity and which is invariant under the most general non singular transformations containing both the geometrical and internal coordinates of the system. In §/2/ we give the definition for a system to be isolated. In doing so we assume that the internal forces can be removed by using a sufficient number of generalized coordinates. We assume that the manifold V_n of the coordinates of an isolated system has a Riemannian structure with a metric tensor independent of the velocities. We also write down symmetry requirements for V_n.

In paragraphs /3/ and /4/ we derive the structure of V_n. In §/5/ the geodesics and the field equations resulting from the assumptions for V_n are discussed. In §/6/ the implications of the symmetry requirements are investigated and it is shown how particles of the ordinary sense enter into the model. In §/7/ the gravitational interaction of two particles is derived as an application, and Kepler's laws and the existence of a shift of the perihelion are proved.

§/2/: ISOLATED SYSTEMS

In what follows only systems of finite degrees of freedom will be considered.

First, we assume that the manifold V_n of the coordinates of an isolated system has a Riemannian structure with a metric tensor $g_{ab}(a,b = 1,2,\ldots,n)$ depending only on the coordinates. The evolution or motion of the system is described by a geodesic of V_n by which we suppose to express that external forces are absent and that all internal degrees of freedom and forces due to them are taken into account by introducing a sufficient number of coordinates. Thus the generalized coordinates satisfy the equation

$$\ddot{x}^a + \Gamma^a_{rs}\,\dot{x}^r\,\dot{x}^s = 0 \quad (a,r,s = 1,2,\ldots,n)\ , \qquad /2.1/$$

where Γ^a_{bc} is the Christoffel symbol of the second kind corresponding to g_{ab}. The dot stands for the derivation with respect to an affine parameter playing the role of a proper time. We adopt the usual summation convention.

Second, it will be supposed that Einstein's equations hold for the generalized coordinates, in particular for V_n the Ricci tensor vanishes:

$$R_{ab} = 0 \quad (a,b = 1,2,\ldots,n)\ , \qquad /2.2/$$

as the geometry takes care for internal fields and external ones are absent.

Third, it will be supposed that V_n admits E/3/ as a symmetry group of point transformations. This can be visualized; such a system can be rotated and translated with respect to the axes of a Cartesian frame of the fixed stars. Since we do not know a priori anything about the meaning of the coordinates we can only require that for infinitesimal point transformations of this kind a point of V_n with coordinates x^a transform into another point whose coordinates $\bar{x}^a$ satisfy

$$\bar{x}^a = x^a + P_{\alpha|}{}^{a}\varepsilon^{\alpha}\ , \quad \bar{x}^a = x^a + M_{\alpha\beta|}{}^{a}\,\varepsilon^{\alpha\beta}\ ,$$

$$M_{\alpha\beta|}{}^{a} = -M_{\beta\alpha|}{}^{a} \quad (a = 1,2,\ldots,n)\ . \qquad /2.3/$$

The generator vectors $P_{\alpha|}{}^{a}$ and $M_{\alpha\beta|}{}^{a}$ correspond to translations and rotations respectively. The sub- or superscripts preceeding a bar specify the quantity and are not co or contravariant indices. ε^{α} and $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ are the infinitesimal parameters of these transformations.

<u>The greek indices will always take the values 1, 2 and 3.</u>

The necessary and sufficient conditions that V_n admits a group E/3/ of the translations and the rotations of Euclidean three dimensional space are that the Lie brackets of $P_{\alpha|}{}^{a}$ and $M_{\alpha\beta|}{}^{a}$ satisfy the relations [3]

$$\left[P_{\alpha|}, P_{\beta|}\right]^a = P_{\alpha|}{}^{r}\,P_{\beta|}{}^{a}{}_{,r} - P_{\beta|}{}^{r}P_{\alpha|}{}^{a}{}_{,r} = P_{\alpha|}{}^{r}\,P_{\beta|}{}^{a}{}_{;r} - P_{\beta|}{}^{r}P_{\alpha|}{}^{a}{}_{;r} = 0,$$

$$\left[M_{\alpha\beta|}, P_{\gamma|}\right]^a = M_{\alpha\beta|}{}^{r}\,P_{\gamma|}{}^{a}{}_{,r} - P_{\gamma|}{}^{r}\,M_{\alpha\beta|}{}^{a}{}_{,r} = M_{\alpha\beta|}{}^{r}\,P_{\gamma|}{}^{a}{}_{;r} - P_{\gamma|}{}^{r}\,M_{\alpha\beta|}{}^{a}{}_{;r} = \delta_{\alpha\gamma}\,P_{\beta|}{}^{a} - \delta_{\beta\gamma}\,P_{\alpha|}{}^{a}\ , \qquad /2.4/$$

$$\left[M_{\alpha\beta|}, M_{\gamma\delta|}\right]^a = M_{\alpha\beta|}{}^{r}\,M_{\gamma\delta|}{}^{a}{}_{,r} - M_{\gamma\delta|}{}^{r}\,M_{\alpha\beta|}{}^{a}{}_{,r} = M_{\alpha\beta|}{}^{r}\,M_{\gamma\delta|}{}^{a}{}_{;r} - M_{\gamma\delta|}{}^{r}\,M_{\alpha\beta|}{}^{a}{}_{;r} =$$
$$= \delta_{\alpha\gamma}M_{\beta\delta|}{}^{a} + \delta_{\beta\delta}\,M_{\alpha\gamma|}{}^{a} - \delta_{\alpha\delta}\,M_{\beta\gamma|}{}^{a} - \delta_{\beta\gamma}\,M_{\alpha\gamma|}{}^{a}$$

$$(a,r, = 1,2,\ldots,n)\ ,$$

where the comma and the semicolon stands for partial and covariant derivatives respectively. If $M_{\alpha|}{}^{a} = -\frac{1}{2}\sum_{\rho,\sigma}\varepsilon_{\alpha\rho\sigma}\,M_{\rho\sigma|}{}^{a}$ is used instead of $M_{\alpha\beta|}{}^{a}$ the more familiar forms

$$\left[M_{\alpha|}, P_{\beta|}\right]^a = -\sum_{\rho}\varepsilon_{\alpha\beta\rho}\,P_{\rho|}{}^{a},$$

$$\left[M_{\alpha|}, M_{\beta|}\right]^a = -\sum_{\rho}\varepsilon_{\alpha\beta\rho}\,M_{\rho|}{}^{a}$$

obtain for the second and third of /2.4/, however it turns out to be easier to work with $M_{\alpha\beta|}{}^{a}$.

If we assume that the Lagrangian $L = \frac{1}{2}\,g_{rs}\,\dot{x}^r\,\dot{x}^s\ (r,s,=1,2,\ldots,n)$ leading to /2.1/ is invariant under the transformations /2.3/ i.e. that possible motions go over into possible motions we get

$$P_{\alpha|\,a;b} + P_{\alpha|\,b;a} = 0,$$

$$M_{\alpha\beta|\,a;b} + M_{\alpha\beta|\,b;a} = 0 \qquad /2.5/$$

thus $P_{\alpha|}{}^{a}$ and $M_{\alpha\beta|}{}^{a}$ must be Killing vectors of V_n.

Summarizing we have the following definition:

A system is isolated if the manifold V_n of it's coordinates has a Riemannian structure and /a/ the line representing the motion is a geodesic /condition /2.1/ /, /b/ the Ricci tensor of V_n vanishes /condition/2.2//, /c/ V_n admits a group E/3/ of Killing transformations /conditions/2.4/ and /2.5//.

We determine now the generator vectors that satisfy /2.4/.

§/3/: THE GENERATOR VECTORS

In what follows we shall consider the case when the matrix $P_{\alpha\beta}$ of the scalar products of the vectors $P_{\alpha}{}^{a}$ is not singular:

$$P_{\alpha\beta} = P_{\alpha}{}^{r} P_{\beta}{}^{s} g_{rs}, \quad \det (P_{\alpha\beta}) \neq 0 \quad (r,s = 1,2,\dots,n) \qquad /3.1/$$

leaving the singular case for a later work.

The vectors $P_{\alpha}{}^{a}$ are linearly independent for if $\lambda^{\alpha} P_{\rho}{}^{a} = 0$ then contracting this with $P_{\alpha a}$ one obtains $\lambda^{\rho} P_{\rho\alpha} = 0$ which according to /3.1/ yields $\lambda^{\alpha} = 0$. Hence in consequence of the first of /2.4/ it is possible to find a system of coordinates such that [2,3]

$$P_{\alpha}{}^{a} = \delta_{\alpha}^{a} \quad (a = 1,2,\dots,n) . \qquad /3.2/$$

Thus the first of /2.4/ is satisfied identically. On substituting /3.2/ in the second of /2.4/ we get

$$M_{\alpha\beta}{}^{a}{}_{,\gamma} = \delta_{\alpha\gamma}\delta_{\alpha}^{a} - \delta_{\alpha\gamma}\delta_{\beta}^{a} \quad (a = 1,2,\dots,n) ,$$

this when integrated yields

$$M_{\alpha\beta}{}^{a} = x_{\beta}\delta_{\alpha}^{a} - x_{\alpha}\delta_{\beta}^{a} + \bar{M}_{\alpha\beta}{}^{a} ,$$

$$\bar{M}_{\alpha\beta}{}^{a}{}_{,\gamma} = 0 \quad (a = 1,2,\dots,n) , \qquad /3.3/$$

where $x_{\alpha} \equiv \delta_{\alpha\rho} x^{\rho}$. It is seen that

$$M_{\alpha\beta}{}^{a} = \bar{M}_{\alpha\beta}{}^{a} \quad (a = 4,5,\dots,n) .$$

If /3.2/ and /3.3/ are substituted in the third of /2.4/ we get equations for $\bar{M}_{\alpha\beta}{}^{a}$ $(a = 1,2,\dots,n)$:

$$\delta_{\gamma}^{\mu} \bar{M}_{\alpha\beta}{}^{\rho} \delta_{\rho\delta} - \delta_{\delta}^{\mu} \bar{M}_{\alpha\beta}{}^{\rho}\delta_{\rho\gamma} + \delta_{\beta}^{\mu} \bar{M}_{\gamma\delta}{}^{\rho}\delta_{\rho\alpha} -$$

$$-\delta_{\alpha}^{\mu}\bar{M}_{\gamma\delta}{}^{\rho}\delta_{\rho\beta} + M_{\alpha\beta}{}^{r}\bar{M}_{\gamma\delta}{}^{\mu}{}_{,r} - M_{\gamma\delta}{}^{r}\bar{M}_{\alpha\beta}{}^{\mu}{}_{,r} =$$

$$= \delta_{\alpha\gamma}\bar{M}_{\beta\delta}{}^{\mu} - \delta_{\beta\gamma} \bar{M}_{\alpha\delta}{}^{\mu} + \delta_{\beta\delta} \bar{M}_{\alpha\gamma}{}^{\mu} - \delta_{\alpha\delta} \bar{M}_{\beta\gamma}{}^{\mu} ,$$

$$M_{\alpha\beta}{}^{r} M_{\gamma\delta}{}^{a}{}_{,r} - M_{\gamma\delta}{}^{r} M_{\alpha\beta}{}^{a}{}_{,r} = \qquad /3.4/$$

$$= \delta_{\alpha\gamma} M_{\beta\delta}{}^{a} - \delta_{\beta\gamma} M_{\alpha\delta}{}^{a} + \delta_{\beta\delta} M_{\alpha\gamma}{}^{a} - \delta_{\alpha\delta} M_{\beta\gamma}{}^{a}$$

$$(a,r = 4,5,\dots,n) ,$$

where according to the second of /3.3/ the summation for r actually runs from 4 to n.

The forms of /3.2/, /3.3/ and /3.4/ are invariant under the transformations

$$x'^{\alpha} = x^{\alpha} + F^{\alpha} (x^{4},\dots,x^{n}) ,$$

$$x'^{a} = F^{a} (x^{4},\dots,x^{n}) \quad (a = 4,5,\dots,n) . \qquad /3.5/$$

Thus convenient choices of the functions F^{α} and F^{a} in /3.5/ can serve to simplify the expressions for $\bar{M}_{\alpha\beta}{}^{\gamma}$ and $M_{\alpha\beta}{}^{a}$ $(a = 4,5,\dots,n)$.

From the second of /3.4/ it is seen that the $M_{\alpha\beta}{}^{a}$ $(a = 4,5,\dots,n)$ generate a group SO(3) on the manifold of the coordinates x^{a} $(a = 4,5,\dots,n)$. For the solution of the second of /3.4/ we have three cases [4]. The first is the trivial one

$$M_{\alpha\beta}{}^{a} = 0 \quad (a = 4,5,\dots,n). \qquad /3.6/$$

The second can be written by means of a suitable transformation of the coordinates x^{a} $(a = 4,5,\dots,n)$ in the from

$$M_{12\prime}{}^{4} = 0\,, \qquad M_{12\prime}{}^{5} = -1\,,$$

$$M_{23\prime}{}^{4} = -\sin x^5, \qquad M_{23\prime}{}^{5} = \operatorname{tg} x^4 \cos x^5\,, \qquad /3.7/$$

$$M_{31\prime}{}^{4} = \cos x^5, \qquad M_{31\prime}{}^{5} = \operatorname{tg} x^4 \sin x^5\,,$$

$$M_{\alpha\beta\prime}{}^{a} = 0\,, \quad a = (6,7,\dots,n)\,.$$

In the third case the vectors $M_{\alpha\beta\prime}{}^{a}$ $(a = 4,5,\dots,n)$ span a three dimensional space. This leads to three particle systems in the sense discussed in §/5/, the investigation of this solution is left to a later work.

First we consider /3.6/. If this is substituted in the first of /3.4/ then on contracting μ and γ we get

$$\bar{M}_{\alpha\beta\prime}{}^{\delta} = \frac{1}{2}\ (\delta^{\delta}_{\beta}\,\bar{M}_{\alpha\prime} - \delta^{\delta}_{\alpha}\,\bar{M}_{\beta\prime})$$

where $\bar{M}_{\alpha\prime} = \bar{M}_{\alpha\rho\prime}{}^{\rho}$ and $M_{\alpha\prime,\beta} = 0$. Performing a transformation /3.5/ with $F^{\alpha} = -\frac{1}{2}\,\delta^{\alpha\rho}\,\bar{M}_{\rho\prime}$ and $F^{a} = x^{a}$ and omitting the prime in the new system we get

$$P_{\alpha\prime}{}^{a} = \delta^{a}_{\alpha}\,,\ M_{\alpha\beta\prime}{}^{a} = x_{\beta}\,\delta^{a}_{\alpha} - x_{\alpha}\,\delta^{a}_{\beta}\,, \quad (a = 1,2,\dots,n),$$

$$x_{\alpha} = \delta_{\alpha\rho}x^{\rho}, \qquad /3.8/$$

showing that in this case the transformations /2.3/ affect the coordinates x^{α} only.

Next we consider /3.7/. In order to cast the calculations in a concise form we introduce the three quantities $N_{\alpha\prime}$:

$$N_{1\prime} = \cos x^4 \cos x^5, \quad N_{2\prime} = \cos x^4 \sin x^5, \quad N_{3\prime} = \sin x^4 \qquad /3.9/$$

If we define $N^{\alpha\prime} \equiv \delta^{\alpha\rho}N_{\rho\prime}$ then

$$N^{\alpha\prime}N_{\rho\prime} = 1. \qquad /3.10/$$

<u>In what follows capital Latin indices will always take the values 4, 5.</u>

We have $N^{\alpha\prime}{}_{,A} = \delta^{\alpha\rho}N_{\rho\prime,A}$. The functions $N_{\alpha\prime}$, $N_{\alpha\prime,A}$ are orthogonal

$$N^{\rho\prime}N_{\rho\prime,A} = 0 \qquad /3.11/$$

The matrix

$$\gamma_{AB} = N_{\rho\prime,A}\,N^{\rho\prime}{}_{,B} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 x^4 \end{pmatrix} \qquad /3.12/$$

is not singular.

<u>In what follows the γ_{AB} of /3.12/ will be used to raise and lower the capital Latin indices</u> thus e.g.

$$N^{\alpha\prime}{}_{,}{}^{A} = \gamma^{AR}\,N^{\alpha\prime}{}_{,R}\,,$$

$$N_{\alpha\prime,}{}^{A} = \gamma^{AR}\,N_{\alpha\prime,R}.$$

We also have

$$\delta_{\alpha\beta} = N_{\alpha\prime}\,N_{\beta\prime} + N_{\alpha\prime,R}\,N_{\beta\prime,}{}^{R}\,, \qquad /3.13/$$

which is a completeness relation, and

$$N_{\alpha\prime;AB} = -\gamma_{AB}\,N_{\alpha\prime}\,. \qquad /3.14/$$

<u>In /3.14/ and what follows a semicolon followed by capital Latin indices will stand for covariant derivatives with respect to the γ_{AB} of /3.12/.</u>

By means of /3.9/ we can rewrite /3.7/:

$$M_{\alpha\beta\prime}{}^{A} = N_{\alpha\prime,}{}^{A}\,N_{\beta\prime} - N_{\beta\prime,}{}^{A}\,N_{\alpha\prime}\,,$$

$$M_{\alpha\beta\prime}{}^{a} = 0 \ (a = 6,7,\dots,n)\,, \qquad /3.15/$$

Equations /3.10/-/3.15/ can be proved by elementary calculations. We proceed to calculate $\bar{M}_{\alpha\beta\prime}{}^{\gamma}$ which due to the relations /3.10/-/3.13/ can be written uniquely in the form

$$\bar{M}_{\alpha\beta\prime}{}^{\gamma} = (P^{R}\,N^{\gamma\prime} + S^{RT}\,N^{\gamma\prime}{}_{,T})\,(N_{\alpha\prime,R}\,N_{\beta\prime} - N_{\beta\prime,R}\,N_{\alpha\prime}) +$$

$$+ (W^{RT}\,N^{\gamma\prime} + Z^{RTS}\,N^{\gamma\prime}{}_{,S})\,N_{\alpha\prime,R}\,N_{\beta\prime,T}\,, \qquad /3.16/$$

where acording to /2.3/ $W^{AB} = -W^{BA}$ and $Z^{ABC} = -Z^{BAC}$. If /3.16/ is contracted by suitable products of $N_{\alpha|}$ and $N_{\alpha|,A}$ and /3.10/-/3.12/ are used we get the equations defining the quantities P^A, S^{AB}, W^{AB} and Z^{ABC}, thus e.g.

$$P^A \quad N_{\rho} N^{\tau|} N^{\sigma|}{}_{,A} \bar{M}_{\sigma\tau|}{}^{\rho} .$$

We substitute /3.15/ and /3.16/ in the first of /3.4/ contract the later by suitable products of $N_{\alpha|}$ and $N_{\alpha|,A}$ and use /3.10/-/3.14/ to get

$$W_{AB} = P_{A;B} - P_{B;A},$$

$$Z_{ABC} = S_{AC;B} - S_{BC;A},$$

$$W_{AB;C} = 0,$$

$$Z_{BCD;A} = \gamma_{DC} S_{AB} - \gamma_{BD} S_{AC} , \qquad /3.17/$$

$$\gamma_{AC} W_{BD} - \gamma_{BC} W_{AD} + \gamma_{BD} W_{AC} - \gamma_{AD} W_{BC} = 0,$$

$$\gamma_{EC} Z_{ABD} - \gamma_{ED} Z_{ABC} + \gamma_{EB} Z_{CDA} - \gamma_{EA} Z_{CDB} =$$

$$= \gamma_{AC} Z_{BDE} - \gamma_{BC} Z_{ADE} + \gamma_{BD} Z_{ACE} - \gamma_{AD} Z_{BCE}$$

The second, fourth and sixth of /3.17/ can be solved:

$$S_{AB} = Q_{B;A}$$

$$Z_{ABC} = \gamma_{BC} Q_A - \gamma_{AC} Q_B \qquad /3.18/$$

where $Q_A = Z_{AR}{}^{R}$. The fifth of /3.17/ is an identity since the indices take two values 4, 5 and W_{AB} is antisymmetric. If we perform a transformation /3.5/ with $F^{\alpha} = - Q^R N^{\alpha|}{}_{,R}$ and $F^a = x^a$ ($a = 4,5,\dots,n$) and substitute /3.18/ in /3.16/ we get for the transform

$$\bar{M}'_{\alpha\beta|}{}^{\gamma} = P^R N^{\gamma|} (N_{\alpha|,R} N_{\beta} - N_{\beta|,R} N_{\alpha}) + W^{RT} N_{\alpha|,R} N_{\beta|,T} N^{\gamma|} .$$

Thus by a suitable choice of the coordinates the generator vectors assume the forms

$$P_{\alpha|}{}^{a} = \delta^a_{\alpha} \ (a = 1,2,\dots,n) ,$$

$$M_{\alpha\beta|}{}^{\gamma} = x_{\beta} \delta^{\gamma}_{\alpha} - x_{\alpha} \delta^{\gamma}_{\beta} + P^R N^{\gamma|} (N_{\alpha|,R} N_{\beta|} - N_{\beta|,R} N_{\alpha|}) +$$

$$+ W^{RT} N^{\gamma|} N_{\alpha|,R} N_{\beta|,T} , \ x_{\alpha} \equiv \delta_{\alpha\rho} x^{\rho} ,$$

$$M_{\alpha\beta|}{}^{A} = N_{\alpha|,}{}^{A} N_{\beta|} - N_{\beta|,}{}^{A} N_{\alpha|} , \qquad /3.19/$$

$$M_{\alpha\beta|}{}^{a} = 0 \ (a = 6,7,\dots,n)$$

and the quantities P^A and W^{AB} are still subject to the first and third of /3.17/. It is seen from /3.19/ that in this case the transformations /2.3/ affect the coordinates x^{α} and x^A only.

Having constructed the generator vectors /3.8/ and /3.19/ we proceed to determine the two metric tensors corresponding to them.

§/4/: THE METRIC TENSORS OF V_n

We exploit now condition /2.5/. Writing the first of this in the form

$$g_{ar} P^r_{\alpha|,b} + g_{br} P^r_{\alpha|,a} + g_{ab,r} P_{\alpha|}{}^{r} = 0$$

$$(a,b,r, = 1,2,\dots,n)$$

and substituting $P_{\alpha|}{}^{a}$ given by /3.2/ we see that

$$g_{ab,\alpha} = 0 \ (a,b = 1,2,\dots,n) \qquad /4.1/$$

and this property is shared by both of the metric tensors corresponding to /3.8/ and /3.19/. It remains to investigate the second of /2.5/ which may be written as

$$g_{ar} M_{\alpha\beta|}{}^{r}{}_{,b} + g_{br} M_{\alpha\beta|}{}^{r}{}_{,a} + g_{ab,r} M_{\alpha\beta|}{}^{r} = 0 \qquad /4.2/$$

$$(a,b,r, = 1,2,\dots,n)$$

Consider first the set /3.8/. If $M_{\alpha\beta|}{}^{a}$ given by the second of /3.8/ is substituted in /4.2/ and use is made of /4.1/ a simple calculation yields

$$g_{\alpha\beta} = t\delta_{\alpha\beta}, \ t_{,\alpha} = 0$$

$$\left.\begin{array}{l} g_{\alpha a} = 0 \\ g_{ab,\alpha} = 0 \end{array}\right\} \ (a,b = 4,5,\dots,n) \qquad /4.3/$$

where t and g_{ab} are functions of $x^4, x^5, \dots, x^n$ such that t and $\det (g_{ab})$ $(a,b = 4,5,\dots,n)$ do not vanish identically as the metric tensor of V_n must not be singular.

Consider next the set /3.19/. We substitute $M_{\alpha\beta'}{}^a$ given by the second, third and fourth of /3.19/ and set $a = \gamma$, $b = \delta$ in /4.2/. A somewhat lengthy calculation using /3.10/-/3.13/ gives

$$g_{\alpha\beta} = (h-f)\delta_{\alpha\beta} + fN_{\alpha'}N_{\beta'}$$

$$h_{,\alpha} = h_{,A} = f_{,\alpha} = f_{,A} = 0, \qquad /4.4/$$

where the functions h and f depend on x^a $(a = 6,7,\dots,n)$ only.

It can be shown using /4.4/ and the first of /3.19/ that the condition /3.1/ is equivalent to

$$h \neq 0, \quad h - f \neq 0. \qquad /4.5/$$

The calculation of $g_{\alpha A}$ needs a more detailed consideration. Using /3.10/-/3.13/ this can uniquely be written in the form

$$g_{\alpha A} = k_A N_{\alpha'} + n_A{}^R N_{\alpha',R}. \qquad /4.6/$$

If /4.6/ is contracted by $N^{\alpha'}$ or $N^{\alpha'}{}_{,B}$ and use is made of /3.10/-/3.12/ we get the equations defining k_A and n_{AB}

$$k_A = N^{\rho'} g_{\rho A},$$

$$n_{AB} = N^{\rho'}{}_{,B}\, g_{\rho A}.$$

We set $a = \gamma$, $b = A$ in /4.2/ and substitute /4.4/, /4.6/ and $M_{\alpha\beta'}{}^a$ given by /3.19/. If the resulting equation is multiplied by $N^{\gamma'}{}_{,B} N^{\alpha'}{}_{,C} N^{\beta'}{}_{,D}$ and contracted for α, β and γ we get using /3.10/-/3.14/

$$(h - f)\,\gamma_{AB} W_{CD} + \gamma_{AD} n_{CB} - \gamma_{AC} n_{DB} + \gamma_{BD} n_{AC} - \gamma_{BC} n_{AD} = 0.$$

Contracting this by γ^{AB} we obtain $(h - f)\, W_{CD} = 0$ which due to the second of /4.5/ gives

$$W_{AB} = 0. \qquad /4.7/$$

Now we return to the first of /3.17/ for a moment. This shows that P_A is of the form $P_A = \phi_{,A}$. If we perform a transformation /3.5/ with $F^\alpha = -\phi N^{\alpha'}$ and $F^a = x^a$ $(a = 4,5,\dots,n)$, which as it can be shown does not affect the components $g_{\alpha\beta}$ of /4.4/, then we get using /3.10/-/3.14/ for the transforms of $P_{\alpha'}{}^a$ and $M_{\alpha\beta'}{}^a$ given by /3.19/ in the new system of coordinates

$$P_{\alpha'}{}^a = \delta^a_\alpha \quad (a = 1,2,\dots,n)$$

$$M_{\alpha\beta'}{}^\gamma = x_\beta\, \delta^\gamma_\alpha - x_\alpha \delta^\gamma_\beta\,, \quad x_\alpha \equiv \delta_{\alpha\rho}\, x^\rho,$$

$$M_{\alpha\beta'}{}^A = N_{\alpha',}{}^A N_{\beta'} - N_{\beta',}{}^A N_{\alpha'}, \qquad /4.8/$$

$$M_{\alpha\beta'}{}^a = 0 \quad (a = 6,7,\dots,n),$$

where we have omitted the primes of the new coordinates. From now on we shall use the system /4.8/ instead of /3.19/; thus we obtain the system of differential equations for $g_{\alpha\beta}$, $g_{\alpha b}$ $(b = 6,7,\dots,n)$, g_{AB}, g_{Ab} $(b = 6,7,\dots,n)$, g_{ab} $(a,b = 6,7,\dots,n)$ by setting $a = \alpha, b = B$; $a = \alpha, b = 6,7,\dots,n$; $a = A, b = B$; $a = A, b = 6,7,\dots,n$ and $a,b = 6,7,\dots,n$ respectively in /4.2/ and by substituting $M_{\alpha\beta'}{}^a$ given by the second, third and fourth of /4.8/ and using /4.4/. The solution of this system of equations can easily be found by applying again the method of expanding any triplet of functions in terms of $N_{\alpha'}$ and $N_{\alpha',A}$ and using /3.10/-/3.14/. We get

$$g_{\alpha A} = dN_{\alpha',A} + e\, \epsilon_A{}^R N_{\alpha',R}\,,$$

$$\left.\begin{aligned} g_{\alpha a} &= u_a N_{\alpha'}, \\ g_{AB} &= q\gamma_{AB}, \\ g_{Aa} &= 0, \\ g_{ab,A} &= 0 \end{aligned}\right\} \quad (a,b = 6,7,\dots,n) \qquad /4.9/$$

where the functions d, e, u_a $(a = 6,7,\dots,n)$, q and g_{ab} $(a,b = 6,7,\dots,n)$ depend on x^a $(a = 6,7,\dots,n)$ only; ϵ_{AB} is the completely antisymmetric tensor in the indices 4,5 corresponding to γ_{AB} of /3.12/.

We summarize our results /3.8/, /4.3/, /4.4/, /4.8/, /4.9/ for the generator vectors and the corresponding metric tensors.

In the first case we have the solution /3.8/ of condition /2.4/ for the generator vectors of E/3/:

$$P_{\alpha|}^{\ a} = \delta^a_\alpha,\ M_{\alpha\beta|}^{\ \ a} = x_\beta\, \delta^a_\alpha - x_\alpha\, \delta^a_\beta ,$$
$$x_\alpha = \delta_{\alpha\rho}\, x^\rho,\ (a = 1,2,\dots,n). \qquad /4.10/$$

The corresponding solution /4.3/ of condition /2.5/ for the metric tensor of V_n is

$$g_{\alpha\beta} = t\, \delta_{\alpha\beta}\ ,\ t_{,\alpha} = 0,$$
$$\left.\begin{array}{l} g_{\alpha a} = 0, \\ g_{ab,\alpha} = 0, \end{array}\right\} (a,b = 4,5,\dots,n), \qquad /4.11/$$

where t and g_{ab} are functions of x^a $(a = 4,5,\dots,n)$ only such that t and $\det(g_{ab})(a,b = 4,5,\dots,n)$ do not vanish identically. For this solution condition /3.1/ is equivalent to $t \neq 0$.

In the second case we have the solution /4.8/ of condition /2.4/ for the generator vectors of E/3/:

$$P_{\alpha|}^{\ a} = \delta^a_\alpha\ (a = 1,2,\dots,n),$$
$$M_{\alpha\beta|}^{\ \ \gamma} = x_\beta\, \delta^\gamma_\alpha - x_\alpha\, \delta^\gamma_\beta\ ,\ x_\alpha \equiv \delta_{\alpha\rho}\, x^\rho,$$
$$M_{\alpha\beta|}^{\ \ A} = N_{\alpha|,}^{\ \ A}\, N_{\beta|} - N_{\beta|,}^{\ \ A}\, N_{\alpha|}, \qquad /4.12/$$
$$M_{\alpha\beta|}^{\ \ a} = 0\ (a = 6,7,\dots,n),$$

where $N_{\alpha|}$ is given by /3.9/. The corresponding solution /4.4/ and /4.9/ of condition /2.5/ for the metric tensor of V_n is

$$g_{\alpha\beta} = (h - f)\delta_{\alpha\beta} + fN_{\alpha|}N_{\beta|}\ ,$$
$$g_{\alpha A} = dN_{\alpha|,A} + \epsilon_A^{\ R}\, N_{\alpha|,R}\ ,$$
$$\left.\begin{array}{l} g_{\alpha a} = u_a\, N_{\alpha|}, \\ g_{AB} = q\, \gamma_{AB}, \\ g_{Aa} = 0, \\ g_{ab,\alpha} = g_{ab,A} = 0 , \end{array}\right\} (a,b = 6,7,\dots,n), \qquad /4.13/$$

where h,f,d,e,u_a $(a = 6,7,\dots,n)$, q and $g_{ab}(a,b = 6,7,\dots,n)$ depend on x^a $(a = 6,7,\dots,n)$ only, γ_{AB} is given by /3.12/ and ϵ_{AB} is the completely antisymmetric tensor in the indices 4, 5 corresponding to γ_{AB}. For this solution condition /3.1/ is equivalent to $h \neq 0, h - f \neq 0$.

We investigate now the geodesics of V_1 which according to condition /2.1/ describe the motion of the system and field equations resulting from condition /2.2/.

§/5/: THE FIELD EQUATIONS AND GEODESICS

Both of the metric tensors /4.11/ and /4.13/ have the property /4.1/:

$$g_{ab,\alpha} = 0\ (a,b, = 1,2\dots,n), \qquad /5.1/$$

and meat the condition /3.1/ which using $P_{\alpha|}^{\ a}$ given by either the first of /4.10/ or the first of /4.12/ can be written in the form

$$\det(g_{\alpha\beta}) \neq 0. \qquad /5.2/$$

Let $\gamma^{\alpha\beta}$ denote the inverse of $g_{\alpha\beta}$:

$$\gamma^{\alpha\rho} g_{\rho\beta} = \delta^\alpha_\beta. \qquad /5.3/$$

We also introduce the quantities $\psi^\alpha_{\ a}$ and $\bar{g}_{ab}$:

$$\left.\begin{array}{l} \psi^\alpha_{\ a} = \gamma^{\alpha\rho} g_{\rho a}, \\ \bar{g}_{ab} = g_{ab} - \psi^\rho_{\ a}\, g_{\rho b}, \end{array}\right\} (a,b = 4,5,\dots,n) \qquad /5.4/$$

It can easily be shown that in consequence of /5.2/ $\det(\bar{g}_{ab})(a,b = 4,5,\dots,n)$ does not vanish identically.

First we investigate the consequences of assumption /2.1/. From the Lagrangian $L = \frac{1}{2} g_{rs}\, \dot{x}^r \dot{x}^s$ $(r,s = 1,2,\dots,n)$ leading to /2.1/ we calculate the Routhian R

$$R \equiv \pi_\rho\, \dot{x}^\rho - L, \qquad /5.5/$$

where

$$\pi_\alpha \equiv \frac{\partial L}{\partial \dot{x}^\alpha} = g_{\alpha\rho}\dot{x}^\rho + g_{\alpha r}\, \dot{x}^r\ (r = 4,5,\dots,n). \qquad /5.6/$$

Note that in consequence of /5.1/ and the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\alpha} = 0$$

the momenta π_α are constants of the motion. Due to /5.2/ $\dot{x}^\alpha$ can be expressed from /5.6/ in terms of π_α and $\dot{x}^a$ $(a = 4,5,\dots,n)$; if these are substituted in /5.5/ and use is made of /5.3/ and /5.4/ we get

$$R = \frac{1}{2}\pi_\rho\pi_\sigma\gamma^{\rho\sigma} - \frac{1}{2}\bar{g}_{rs}\dot{x}^r\dot{x}^s - \pi_\rho\psi^\rho{}_r\dot{x}^r (r,s = 4,5,\dots,n).$$

R can be used as a Hamiltonian for x^α and π_α and as a Lagrangian for x^a and $\dot{x}^a$ $(a = 4,5,\dots,n)$ [5], thus we have

$$\pi_\alpha = \text{const.},$$

$$\ddot{x}^a + \bar{\Gamma}^a_{rs}\dot{x}^r\dot{x}^s = \bar{g}^{ar}\pi_\rho(\psi^\rho{}_{r,s} - \psi^\rho{}_{s,r})\dot{x}^s - \frac{1}{2}\bar{g}^{ar}\pi_\rho\pi_\sigma\gamma^{\rho\sigma}{}_{,r}$$

$$(a,r,s = 4,5,\dots,n), \qquad /5.7/$$

where $\bar{g}^{ab}(a,b = 4,5,\dots,n)$ is the inverse of $\bar{g}_{ab}$ of the second of /5.4/, and $\bar{\Gamma}^a_{bc}$ $(a,b,c = 4,5,\dots,n)$ are the Christoffel symbols corresponding to $\bar{g}_{ab}$.

The second of /5.7/ shows that to the free motion of the system described by a geodesic of V_n there corresponds a non-geodesic motion in a manifold V_{n-3} of the coordinates x^a $(a = 4,5,\dots,n)$. The metric tensor of this V_{n-3} is $\bar{g}_{ab}$ given by the second of /5.4/. The first term on the right hand side of the second of /5.7/ has a structure typical for the Lorentz force; the quantities $\psi^\alpha{}_{a,b} - \psi^\alpha{}_{b,a}$ play the role of the field strength. The second term is a force arising from the potentials $\gamma^{\alpha\beta}$. In both terms π_α can be considered as a generalized charge.

Second we investigate the consequences of assumption /2.2/. A tedious but straightforward calculation using /5.4/ shows that /2.2/ yields the equations

$$\bar{R}_{ab} = -\frac{1}{2}g_{\rho\sigma}\psi^\rho{}_{ra}\psi^{\rho r}{}_b - \frac{1}{4}\gamma^{\rho\sigma}{}_{,a}g_{\rho\sigma,b} - \frac{1}{2}\gamma^{\rho\sigma}g_{\rho\sigma;ab},$$

$$(\sqrt{\gamma}\,\gamma^{\alpha\rho}g_{\rho\beta,r})^{;r} + (\sqrt{\gamma}\,{}_{\beta\rho}\psi^{\alpha t}\psi^\rho{}_{rt})^{;r} = 0$$

$$(\sqrt{\gamma}g_{\alpha\rho}\psi^{\rho r}{}_a)_{;r} = 0 \qquad /5.8/$$

$$\psi^\rho{}_{ab} = \psi^\rho{}_{a,b} - \psi^\rho{}_{b,a},$$

$$(a,b,r,t = 4,5,\dots,n).$$

where $\bar{R}_{ab}$ is the Ricci tensor for $\bar{g}_{ab}$, $\sqrt{\gamma} = \det(g_{\alpha\beta})$, for the raising and lowering of small Latin indices and for covariant derivations $\bar{g}_{ab}$ $(a,b = 4,5,\dots,n)$ is used.

The first of /5.8/ is Einstein's equation for V_{n-3}. The energy-momentum tensor is determined by $\psi^\alpha{}_{ab}$ and $\gamma^{\alpha\beta}$ in a way which is a straightforward generalization of the well known form of General Relativity for the case when one has several field strengths and potentials.

The second of /5.8/ is a scalar equation if the coordinates x^a $(a = 4,5,\dots,n)$ are subject to transformations of the form

$$x'^a = f^a(x^4, x^5,\dots,x^n) \qquad /5.9/$$

$$(a = 4,5,\dots,n),$$

and thus is an analogue of the Klein-Gordon equation for the potential.

The third and fourth of /5.8/ are vector and tensor equations respectively for the transformations /5.9/; these are the generalizations of Maxwell's equations.

The transformations /3.5/ with $F^a = x^a$ can be proved to have the effect

$$g'_{\alpha\beta} = g_{\alpha\beta}$$

$$\bar{g}'_{ab} = \bar{g}_{ab} \qquad /5.10/$$

$$\psi'^\alpha{}_a = \psi^\alpha{}_a - F^\alpha{}_{,a}$$

Using /5.10/ it can be shown easily that these transformations do not change the forms of equations /5.7/ and /5.8/. On account of the third of /5.10/ these may be called gauge transformations.

Summarizing, the geodesic motion in V_n if viewed from the manifold V_{n-3} with the metric tensor $\bar{g}_{ab}$ becomes a motion under the influence of forces of the Lorentz and the potential type. The charges are constants of the motion characteristic for the geodesics of V_n. The vacuum equations of V_n split into Einstein's equation for V_{n-3} and generalized Klein-Gordon and Maxwell equations. These behave in the usual way for the transformations of the coordinates x^a $(a = 4,5,\dots,n)$. There exist analogues of the gauge transformations too.

We proceed now to investigate the structure of the representations /4.10/ and /4.12/ of the group E/3/ and to show how position coordinates and pointlike particles enter into our model.

§/6/: THE REPRESENTATIONS OF E/3/

First we consider /4.10/. From the form of the generator vectors it is seen that in this case the point transformations /2.3/ affect the three coordinates x^α i.e. x^1, x^2 and x^3 only. Evidently the number of dimensions of the corresponding V_n is at least three: $n \geq 3$. The transformations of the group E/3/ act transitively on the manifold of x^1, x^2 and x^3, we indicate the number of dimensions of this manifold of transitivity by a subscript after the symbol of the group so we write $E/3/_3$ instead of E/3/ in the case /4.10/. Moreover this representation is the same as that of E/3/ on the manifold of the position coordinates of a pointlike particle in ordinary three dimensional space.

On these grounds we interpret x^α i.e. x^1, x^2 and x^3 as the three position coordinates of a pointlike particle in three dimensional space. The coordinates x^a $(a = 4,5,\dots,n)$ of V_n can be considered to correspond to internal degrees of freedom from the point of view of x^α. Thus we call the solutions of the type /4.10/, /4.11/ "one particle systems".

Next we consider /4.12/. It is seen from the form of the generator vectors that in this case the point transformations /2.3/ affect the five coordinates x^α and x^A i.e. x^1, x^2, x^3 and x^4, x^5 only, thus the group E/3/ generated by the vectors /4.12/ is an $E/3/_5$ in our notation. Evidently the number of dimensions of the corresponding V_n is at least five: $n \geq 5$. The generators $M_{\alpha\beta|}{}^a$ can be written now as a sum of two vectors

$$M_{\alpha\beta|}{}^a = {}_cM_{\alpha\beta|}{}^a + {}_rM_{\alpha\beta|}{}^a \quad (a = 1,2,\dots,n), \qquad /6.1/$$

where

$${}_cM_{\alpha\beta|}{}^a = x_\beta \delta^a_\alpha - x_\alpha \delta^a_\beta, \quad x_\alpha = \delta_{\alpha\rho} x^\rho \qquad /6.2/$$
$$(a = 1,2,\dots,n)$$

and

$${}_rM_{\alpha\beta|}{}^\gamma = 0,$$
$${}_rM_{\alpha\beta|}{}^A = N_{\alpha|,}{}^A N_{\beta|} - N_{\beta|,}{}^A N_{\alpha|} \, , \qquad /6.3/$$
$${}_rM_{\alpha\beta|}{}^a = 0 \quad (a = 6,7,\dots,n) .$$

From /6.2/ and the first of /4.12/ it is seen that the vectors $P_{\alpha|}{}^a$ and ${}_cM_{\alpha\beta|}{}^a$ have the same form as the generators in /4.10/ have, thus their Lie brackets satisfy relations of the type /2.4/ and are the generators of a group E/3/. As the infinitesimal point transformations

$$\left.\begin{aligned} x'^a &= x^a + P_{\alpha|}{}^a \varepsilon^\alpha, \\ x'^a &= x^a + {}_cM_{\alpha\beta|}{}^a \varepsilon^{\alpha\beta}, \end{aligned}\right\} \quad (a = 1,2,\dots,n) , \qquad /6.4/$$

affect the coordinates x^1, x^2, and x^3 only, this group is an $E/3/_3$ in our notation. According to the reasoning used in the previous discussion of the generators /4.10/ we interpret the coordinates x^α i.e. x^1, x^2 and x^3 as the three position coordinates of a particle in three dimensional space.

The Lie brackets of ${}_rM_{\alpha\beta|}{}^a$ and $P_{\alpha|}{}^a$ or ${}_cM_{\alpha\beta|}{}^a$ vanish whereas the Lie brackets of two of the vectors ${}_rM_{\alpha\beta|}{}^a$ are the same as those of the generators of the rotation group SO(3) As it can be seen from /6.3/ the infinitesimal point transformations

$$x'^a = x^a + {}_rM_{\alpha\beta|}{}^a \varepsilon^{\alpha\beta} \quad (a = 1,2,\dots,n) \qquad /6.5/$$

affect the coordinates x^A i.e. x^4 and x^5 only, thus the vectors ${}_rM_{\alpha\beta|}{}^a$ are the generators of a group SO(3) which in our notation is a $SO(3)_2$. Consequently in this case the group $E/3/_5$ generated by the vectors given by /4.12/ is a subgroup of the direct product of the $E/3/_3$ generated by $P_{\alpha|}{}^a$ and ${}_cM_{\alpha\beta|}{}^a$ and of the $SO(3)_2$ generated by ${}_rM_{\alpha\beta|}{}^a$:

$$E(3)_5 \subset E(3)_3 \otimes SO(3)_2 \, .$$

There exists however a different decomposition too for the generators of /4.12/; this clarifies the meaning of the coordinates x^A. There exists namely a coordinate system such that the generator vectors of /4.12/ assume the forms

$$\left.\begin{aligned} P'_{\alpha|}{}^a &= {}_1P_{\alpha|}{}^a + {}_2P_{\alpha|}{}^a \, , \\ M'_{\alpha\beta|}{}^a &= {}_1M_{\alpha\beta|}{}^a + {}_2M_{\alpha\beta|}{}^a , \end{aligned}\right\} \quad (a = 1,2,\dots,n) , \qquad /6.6/$$

where

$$_1P_{\alpha}^{\ a} = \delta_{\alpha}^{a},$$

$$_2P_{\alpha}^{\ a} = \delta_{\alpha+3}^{a},$$

$$_1M_{\alpha\beta}^{\ \ a} = x'_{\beta}\,\delta_{\alpha}^{a} - x'_{\alpha}\,\delta_{\beta}^{a},\quad x'_{\alpha} = \delta_{\alpha\rho}x'^{\rho},$$

$$_2M_{\alpha\beta}^{\ \ a} = x'_{\beta+3}\,\delta_{\alpha+3}^{a} - x'_{\alpha+3}\,\delta_{\beta+3}^{a},\quad x'_{\alpha+3} = \delta_{\alpha+3,\rho+3}\,x'^{\rho+3}, \qquad /6.7/$$

$$(a = 1,2,\dots,n).$$

To prove this statement we determine the transformation of coordinates carrying the forms of the generators given by /4.12/ over into the forms given by /6.6/ and /6.7/. Applying again the method of expanding any triplet of functions in terms of N_α and $N_{\alpha,A}$ and using /3.10/-/3.14/ we find for this transformation

$$x'^{\alpha} = x^{\alpha} + EN^{\alpha} ,$$

$$x'^{\alpha+3} = x^{\alpha} + HN^{\alpha}, \qquad /6.8/$$

$$x'^{a} = f^{a}\ (a = 7,8,\dots,n),$$

where the functions E, H, f^a $(a = 7,8,\dots,n)$ depend on x^a $(a = 6,7,\dots,n)$ only, and the Jacobian of /6.8/ does not vanish if and only if the rank of the matrix

$$\begin{pmatrix} (E-H)_{,6} & \dots & (E-H)_{,n} \\ f^7_{\ ,6} & \dots & f^7_{\ ,n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f^n_{\ ,6} & \dots & f^n_{\ ,n} \end{pmatrix} \qquad /6.9/$$

is $n-5$. Thus it is possible to transform the form of the generators /4.12/ into the form given by /6.6/ and /6.7/ if and only if $n \geq 6$ for V_n. However according to /4.13/ the metric tensor of a V_5 corresponding to /4.12/ is

$$g_{\alpha\beta} = (h-f)\delta_{\alpha\beta} + fN_{\alpha}N_{\beta}$$

$$g_{\alpha A} = dN_{\alpha,A} + e\epsilon_{A}^{\ R}N_{\alpha,R}$$

$$g_{AB} = q\gamma_{AB}$$

where h, f, d, e and q must be constants for this V_5. An elementary but lengthy calculation shows that in this case the Ricci tensor $R_{ab}(a,b = 1,2,\dots,5)$ does not vanish in contradiction to our condition /2.2/. Thus for a V_n characterized by /4.12/ and /4.13/ we have $n \geq 6$, and one is always able to perform the transformation /6.8/.

From /6.7/ it is seen that the infinitesimal point transformations

$$\bar{x}'^{a} = x'^{a} + {_1P_{\alpha}}^{a}\epsilon^{\alpha} ,$$

$$\bar{x}'^{a} = x'^{a} + {_1M_{\alpha\beta}}^{a}\,\epsilon^{\alpha\beta},$$

and

$$\bar{x}'^{a} = x'^{a} + {_2P_{\alpha}}^{a}\,\epsilon^{\alpha},$$

$$\bar{x}'^{a} = x'^{a} + {_2M_{\alpha\beta}}^{a}\epsilon^{\alpha\beta} ,$$

affect the coordinates x'^{α} and $x'^{\alpha+3}$ i.e. x'^1, x'^2, x'^3 and x'^4, x'^5, x'^6 respectively. Thus V_n admits a direct product of an $E/3/_3$ generated by the vectors ${_1P_{\alpha}}^{a}$ and ${_1M_{\alpha\beta}}^{a}$ and of another $E/3/_3$ generated by the vectors ${_2P_{\alpha}}^{a}$ and ${_2M_{\alpha\beta}}^{a}$: the original $E/3/_5$ generated by the vectors /4.12/ is a subgroup of this direct product:

$$E(3)_5 \subset E(3)_3 \otimes E(3)_3$$

Hence we may interpret the two triplets of coordinates x'^{α} and $x'^{\alpha+3}$ given by /6.8/ as the position coordinates of two pointlike particles is three dimensional space.

According to /6.8/ the usual relative coordinates of these two particles are

$$x'^{\alpha} - x'^{\alpha+3} = (E - H)N^{\alpha} \qquad /6.10/$$

which in consequence of /3.9/ suggests to interpret x^4 and x^5 as the polar and azimuthal angles of the relative position vector. According to /3.10/ the square of the distance of the two particles is

$$r^2 = \sum_{\alpha=1}^{3} (x'^{\alpha} - x'^{\alpha+3})^2 = (E - H)^2. \qquad /6.11/$$

On these grounds we call the solutions of the type /4.12/, /4.13/ "two particle systems".

We stress that there is an infinity of ways for choosing the functions E,H and hence E-H, thus the "two particle decomposition" given by /6.6/, /6.7/ and /6.8/ is not unique. In the next paragraph we consider in detail a V_6 which in accordance with our interpretations is supposed to describe the motion of two pointlike particles, there we try to give a reasonable criterion for choosing E and H.

§/7/: THE KEPLERIAN MOTION

We consider now a V_6 wich according to the discussion following /6.9/ has the lowest possible number of dimensions in the case of a solution of type /4.13/. According to our assumptions and the interpretations given in the preceeding paragraph the geodesics of this V_6 corresponding to a metric tensor /4.13/ with $n = 6$ subject to the vacuum equation $R_{ab} = 0$ $(a,b=1,2,\dots,6)$ /condition 2,2/ are supposed to describe the motion of two pointlike particles which can interact with each other but are isolated from the rest of the universe. There are only the six degrees of freedom corresponding to the six position coordinates appearing in /6.8/, thus internal coordinates are absent.

We strive to deduce the gravitational interaction and the resulting motion of the two particles. We suppose spherical symmetry in the relative coordinates /6.10/. From /6.3/, /3.9/, the discussion following /6.5/ and the interpretation of x^4 and x^5 based on /6.10/ it can easily be seen that the point transformations /6.5/ generate infinitesimal rotations of the relative position vector with components given by /6.10/. Thus we shall consider the case when in addition to $P_{\alpha}{}^{a}$ and $M_{\alpha\beta I}{}^{a}$ given by /4.12/ the vectors ${}_{r}M_{\alpha\beta}{}^{a}$ given by /6.3/ also satisfy Killing's equations:

$${}_{r}M_{\alpha\beta I\, a;b} + {}_{r}M_{\alpha\beta I\, b;a} = 0 \quad (a,b = 1,2,\dots,n) \qquad /7.1/$$

It can be shown easily that the condition /7.1/ gives for the quantities appearing in /4.13/

$$f = d = e = u_a = 0 \quad (a = 6,7,\dots,n).$$

Thus for our V_6 we have the metric tensor

$$g_{\alpha\beta} = h\delta_{\alpha\beta},\ g_{AB} = q\gamma_{AB},\ g_{66} = p,$$
$$g_{\alpha A} = g_{\alpha 6} = g_{A6} = 0, \qquad /7.2/$$

where h, q and p depend on x^6 only, and do not vanish identically.

According to assumption /2.2/ $R_{ab} = 0$ $(a,b = 1,2,\dots,6)$ for the metric tensor /7.2/. This yields after a lengthy calculation

$$h''h + \frac{1}{2}(h')^2 + \frac{q'}{q}h'h - \frac{1}{2}\frac{p'}{p}h'h = 0,$$
$$\frac{2p}{q} - \frac{q''}{q} + \frac{1}{2}\frac{p'}{p}\frac{q'}{q} - \frac{3}{2}\frac{h'}{h}\cdot\frac{q'}{q} = 0, \qquad /7.3/$$
$$\frac{3}{2}\left(\frac{h'}{h}\right)^2 - \frac{q''}{q} + \frac{1}{2}\frac{p'}{p}\cdot\frac{q'}{q} + \frac{1}{2}\left(\frac{q'}{q}\right)^2 + \frac{3}{2}\frac{h'}{h}\cdot\frac{q'}{q} = 0,$$

where the prime denotes derivation with respect to x^6. From the second of /7.3/ it is seen that q' must not vanish identically otherwise p would vanish too. It can be shown that function $F \equiv \int \frac{2p}{q'}dx^6$ is invariant under the transformations $x'^6 = f(x^6)$, $f' \neq 0$. By choosing $x^6 = e^F$ we find for the solution of /7.3/

$$h = B\left|\frac{L}{K}\right|^{\frac{1}{\sqrt{6}}},\quad q = C\left|\frac{K}{L}\right|^{\sqrt{\frac{3}{2}}}\cdot KL,\quad p = C\left|\frac{K}{L}\right|^{\sqrt{\frac{3}{2}}} \qquad /7.4/$$

where

$$K \equiv (x^6 + 3\mu + \sqrt{6}\mu),\quad L \equiv (x^6 + 3\mu - \sqrt{6}\mu).$$

B,C and μ are arbitrary constants with the restrictions $B \neq 0$, $C \neq 0$. In the case $\mu = 0$ we have

$$h = B, \qquad q = C(x^6)^2, \quad p = C, \qquad /7.5/$$

which if substituted in /7.2/ can be shown to give a flat V_6 . Hence in the limit $x^6 \to \infty$ we have in /7.4/

$$h\to B, \quad q\to C(x^6)^2, \quad p\to C, \qquad /7.6/$$

thus the metric /7.2/ becomes flat for $x^6\to\infty$.

According to /6.8/ the functions E,H and hence E - H depend only on x^6 in our case and $E' - H' \neq 0$, otherwise the Jacobian of /6.8/ would vanish in accordance with the discussion following /6.8/. At the end of §/6/ it was stressed that there is an infinity of ways for choosing E,H and thus the distance of the two particles r = E - H given by /6.11/. We give a condition now which serves to determine the form of the functions E and H. We write down the Lagrangian $L = \frac{1}{2} g_{rs} \dot{x}^r \dot{x}^s$ (r,s = 1,2,...,6) in the coordinate system of the "two particle" decomposition given by /6.6/, /6.7/ and /6.8/:

$$L = \frac{1}{2} (g'_{\rho\sigma} \dot{x}'^{\rho} \dot{x}'^{\sigma} + 2 g'_{\rho,\sigma+3} \dot{x}'^{\rho} \dot{x}'^{\sigma+3} + g'_{\rho+3,\sigma+3} \dot{x}'^{\rho+3} \dot{x}'^{\sigma+3}) . \qquad /7.7/$$

We require that the mixed term $g'_{\rho,\sigma+3} \dot{x}'^{\rho} \dot{x}'^{\sigma+3}$ of /7.7/ vanish in the case of the flat space /7.5/. We find using /6.8/, /7.2/, /7.5/ and /3,10/-/3.13/ that the necessary and sufficient conditions for this are

$$EH = \frac{C}{B}(x^6)^2 , \quad E'H' = - \frac{C}{B} , \qquad /7.8/$$

where the prime denotes derivation with respect to x^6. The solution of /7.8/ is

$$E = m_2 x^6 , \quad H = -m_1 x^6 ,$$

where the constants m_1, m_2 satisfy

$$m_1 m_2 = \frac{C}{B} , \quad m_1 + m_2 \neq 0 , \qquad /7.9/$$

the second being the consequence of $E' - H' \neq 0$.

If we introduce the constants M_1, M_2, ν:

$$M_1 = B \frac{m_1}{m_1+m_2} , \quad M_2 = B \frac{m_2}{m_1+m_2} , \quad \nu = \mu(m_1+m_2) ,$$

perform a coordinate transformation

$$\bar{x}^a = x^a \quad (a = 1,2,\dots,5) ,$$
$$\bar{x}^6 = r \equiv E - H = (m_1+m_2)x^6$$

and use /7.2/, /7.4/ and /7.9/ we get for the transform of the metric tensor

$$g_{\alpha\beta} = (M_1+M_2) \left|\frac{\bar{L}}{\bar{K}}\right|^{\frac{1}{\sqrt{6}}} \delta_{\alpha\beta} ,$$

$$g_{AB} = \frac{M_1 M_2}{M_1+M_2} \left|\frac{\bar{K}}{\bar{L}}\right|^{\frac{\sqrt{3}}{2}} \bar{K}\bar{L}\, \gamma_{AB} ,$$

$$g_{\bar{6}\bar{6}} = \frac{M_1 M_2}{M_1+M_2} \left|\frac{\bar{K}}{\bar{L}}\right|^{\frac{\sqrt{3}}{2}} , \qquad /7.10/$$

$$g_{\alpha A} = g_{\alpha\bar{6}} = g_{A\bar{6}} = 0,$$

where

$$\bar{K} \equiv (r + 3\nu + \sqrt{6}\nu), \quad \bar{L} \equiv (r + 3\nu - \sqrt{6}\nu), \quad \bar{x}^6 \equiv r .$$

In the limit $r\to\infty$ we have

$$g_{\alpha\beta} \to (M_1+M_2)\, \delta_{\alpha\beta} , \quad g_{AB} = \frac{M_1 M_2}{M_1+M_2}\, r^2\, \gamma_{AB} ,$$

$$g_{\bar{6}\bar{6}} \to \frac{M_1 M_2}{M_1+M_2} , \quad g_{\alpha A} = g_{\alpha\bar{6}} = g_{A\bar{6}} = 0, \qquad /7.11/$$

For the transformation /6.8/ of the two particle decomposition we have

$$x'^{\alpha} = x^{\alpha} + \frac{M_2}{M_1+M_2}\, r\, N^{\alpha} ,$$

$$x'^{\alpha+3} = x^{\alpha} - \frac{M_1}{M_1+M_2}\, r\, N^{\alpha} . \qquad /7.12/$$

If we calculate now the Lagrangian /7.7/ in the limit /7.11/ using /7.12/ and /3.10/-/3.13/ we get

$$\lim_{r\to\infty} L = \frac{1}{2} M_1 \sum_{\rho=1}^{3} (\dot{x}'^{\rho})^2 + \frac{1}{2} M_2 \sum_{\rho=1}^{3} (\dot{x}'^{\rho+3})^2 ,$$

showing that if the relative distance r tends to ∞ we retain the Lagrangian of two freely moving pointlike particles. From /7.12/ it is seen that x'^{α} and $x'^{\alpha+3}$ i.e. x'^1, x'^2, x'^3 and x'^4, x'^5, x'^6 are the position coordinates of two particles with masses M_1 and M_2 respectively, expressed in terms of the coordinates of the center of mass and the relative position vector.

Now we calculate the motion of the two particles for large r. From assumption /2.5/ it follows that there corresponds a constant $P_{\alpha|}$ of motion to each vector $P_{\alpha|}{}^{a}$ [2]:

$$P_{\alpha|} = P_{\alpha|}{}^{r}\,\dot{x}^{s}\,g_{rs} \quad (r,s = 1,2,\dots,6),$$

which according to /7.10/ and /4.12/ reads

$$P_{\alpha|} = (M_1+M_2)\left|\frac{\bar{L}}{\bar{K}}\right|^{\frac{1}{\sqrt{6}}}\dot{x}^{\alpha} \qquad /7.13/$$

Also in view of the additional requirement of spherical symmetry /7.1/ there corresponds a constant of motion to each vector ${}_{r}M_{\alpha\beta|}{}^{a}$ given by /6.3/. However it can allways be achieved by applying a suitable rotation depending only on these constants to the functions ${}_{r}M_{\alpha\beta|}{}^{a}$ and N_{α} /see 3.9/ considered as components of antisymmetric tensors and a vector respectively in three dimensional space i.e. in the indices α and β such that the set of these constants of motion be the following:

$$\left.\begin{aligned} {}_{r}M_{12|}{}^{t}\,\dot{x}^{q}\,g_{tq} &= -J\,,\\ {}_{r}M_{23|}{}^{t}\,\dot{x}^{q}\,g_{tq} &= 0\,,\\ {}_{r}M_{31|}{}^{t}\,\dot{x}^{q}\,g_{tq} &= 0\,. \end{aligned}\right\} \quad (t,q = 1,2,\dots,6) \qquad /7.14/$$

Making use of /6.3/, /3.9/./3.13/ and /7.10/ we get from /7.14/

$$x^4 = 0, \quad \dot{x}^5 = \frac{J}{\frac{M_1M_2}{M_1+M_2}} \cdot \frac{1}{\left|\frac{\bar{K}}{\bar{L}}\right|^{\frac{\sqrt{3}}{2}} \cdot \bar{K}\bar{L}} \,. \qquad /7.15/$$

Remembering the interpretation of x^4 and x^5 following /6.10/, from the first of /7.15/ it is seen that the relative motion takes place in the equatorial plane characterized by zero polar angle. The second of /7.15/ is Kepler's

second law; taking into account /7.10/ in the limit $r \to \infty$ we get

$$\dot{x}^5 = \frac{J}{\frac{M_1M_2}{M_1+M_2}} \cdot \frac{1}{r^2} \qquad /7.16/$$

For the determination of the relative orbit we note that the Lagrangian $\frac{1}{2}\,g_{rs}\,\dot{x}^r\dot{x}^s$ $(r,s = 1,2,\dots,n)$ is a constant e of the motion [2]. Using /7.10/, /7.13/, /7.15/ and /3.12/ we obtain

$$\begin{aligned} 2e &= g_{rs}\,\dot{x}^r\dot{x}^s = \\ &= \frac{P^2}{M_1+M_2}\left|\frac{\bar{K}}{\bar{L}}\right|^{\frac{1}{\sqrt{6}}} + \frac{J^2}{\frac{M_1M_2}{M_1+M_2}} \cdot \left|\frac{\bar{L}}{\bar{K}}\right|^{\frac{\sqrt{3}}{2}} \cdot \frac{1}{\bar{K}\bar{L}} + \dot{r}^2\,\frac{M_1M_2}{M_1+M_2}\left|\frac{\bar{K}}{\bar{L}}\right|^{\frac{\sqrt{3}}{2}}, \end{aligned} \qquad /7.17/$$

$$P^2 \equiv \sum_{\alpha=1}^{3}(P_{\alpha})^2 \,.$$

If we introduce $\phi \equiv x^5$ express $\dot{r}$ by means of $\frac{dr}{d\varphi}$ and ϕ and use /7.15/ then /7.17/ yields

$$\begin{aligned}\varphi &= \\ &= \int\left[\frac{2eM_1M_2}{M_1+M_2}\left|\frac{\bar{K}}{\bar{L}}\right|^{\frac{\sqrt{3}}{2}}\frac{(\bar{K}\,\bar{L})^2}{r^4} - \frac{P^2M_1M_2}{M_1+M_2}\left|\frac{\bar{K}}{\bar{L}}\right|^{2\sqrt{\frac{2}{3}}} \cdot \frac{(\bar{K}\,\bar{L})^2}{r^4} - \frac{J^2(\bar{K}\,\bar{L})}{r^4}\right]^{-\frac{1}{2}}\frac{J}{r^2}\,dr. \end{aligned} \qquad /7.18/$$

If we consider the motion in the case of large r and expand the function under the square root in /7.18/ in a power series of $\frac{1}{r}$ up to the second order we get

$$\varphi = \int\left[A + \frac{B}{r} - \frac{J^2}{r^2}(1 - C)\right]^{-\frac{1}{2}}\frac{J}{r^2}\,dr, \qquad /7.19/$$

where

$$\begin{aligned} A &= \frac{2M_1M_2}{M_1+M_2}\left[e - \frac{P^2}{2(M_1+M_2)}\right],\\ B &= \nu\,\frac{40M_1M_2}{M_1+M_2}\left[\frac{9e}{10} - \frac{P^2}{2(M_1+M_2)}\right],\\ C &= \nu^2\,\frac{29\,2M_1M_2}{M_1+M_2}\left[\frac{57e}{73} - \frac{P^2}{2(M_1+M_2)}\right]. \end{aligned} \qquad /7.20/$$

From /7.19/ it is seen that the orbit of the relative motion is a conic section and if the orbit is an ellipse the shift of the perihelion per one revolution in the case of small ν is $2\pi C$.

Our model does not contain any numerical input parameters· the constants appearing in /7.18/ and /7.19/ are built up of constants of integration of either the field equations or of the equations of motion. Thus we are able to predict only the form of the orbit.

If we compare the constant A given by the first of /7.20/ with the corresponding one of the Newtonia theory and identify $\frac{P^2}{2(M_1+M_2)}$ with the kinetic energy of the center of mass motion we find that e is the total energy of the two body system, an interpretation also suggested by the asymptotic Lagrangian following the formula /7.12/. However this comparison is unrealistic and can serve only as a first orientation in interpreting the constants of our model as the classical theory predicts no perihelion motion.

Thus we have completed the discussion of our model based on the extended principle of covariance. We have shown the existence of gravitational interaction which is a consequence of the geometry of V_6. Also we have deduced the perihelion motion.

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Felelős kiadó: Pintér György, a KFKI
Részecske és Magfizikai Tudományos Tanácsának
szekcióelnöke
Szakmai lektor: Perjés Zoltán
Nyelvi lektor: Lukács Béla
Példányszám: 310 Törzsszám: 75-312
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