ORO [redacted] 3992-247

Relativistic Rotators-
A Quantum Mechanical de Sitter Bundle

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Talk presented at the International Symposium on Mathematical Physics, January 5-8, 1976, Mexico City

* Work Supported in part by ERDA Contract NO. (40-1)3992.

Abstract

If de Sitter fibre bundle over space time is the classical picture of hadrons then for a quantum mechanical description one has to generalise the concept of a principal fibre bundle to a bundle that contains the representation of the group of motion. This idea is related to the relativistic rotator model, and the radius of the de Sitter fibre is determined from the experimental hadron spectrum.

The idea that matter consists of ever smaller objects has so deeply penetrated the minds of physicists that they seem almost incapable to liberate themselves of it. The molecule is considered as consisting of electrons and nuclei, the nucleus is made up of neutrons and protons and the proton must therefore be built from still more elementary constituents. That this way of understanding by dissecting may not even be in conformity with the fundamental ideas of quantum mechanics - and is already quite impractical for molecules, which are much better described as e.g., vibrators or rotators or combinations of both - was no deterrent from working with quark and similar kind of models. The reason for this, probably, was that it is so hard to conceive a geometrical picture of an extended yet "elementary" object. Now that the idea of the hadron a a fibre bundle over space time has been introduced in the preceding talk by W. Drechsler[1)], this obstacle may be overcome.

In this talk, I shall describe a model which is a relativistic generalization of the rotator model as realized e.g., by diatomic molecules. It is based upon work that I did some time ago, but for which at that time I did not have a geometrical understanding. Now, that I have learned from W. Drechsler about de Sitter fibre bundles this old rotator model appears in a new light.

To start, we take the de Sitter fiber bundle over (flat) space time as the classical - that is to mean non-quantum-picture of hadrons. In quantum mechanics the mathematical image of a physical system is an algebra of operators in a linear space. Thus, believing that this basic axiom of quantum mechanics holds for hadron physics, we have to find an algebra of operators for the de Sitter fibre over space time. I do not know how to do this and I do not even know how to give a concrete formulation of the problem. I shall, therefore, state in the first part some vague ideas and try to connect them to the basic relation for the relativistic rotator model, which, I believe, will be one of the basic ingredients of the algebra for the de Sitter fiber. In the second part I shall then describe results of the relativistic rotator model, which will allow us to compute the size of the de Sitter fibre from experimental data.

Fibre bundles have been discussed in the talk of A. Taub[2] and I shall only briefly and vaguely recall some of these notions here in order to show where I think one has to go beyond the classical concepts of differential geometry:
A bundle is a triplet $[E, V_4, \pi]$ where E, V_4 are topological spaces and π is a projection
$\pi: E \to V_4$
As V_4 we will always consider the space time manifold.
If $\mathcal{O} \subset V_4$ is an element of an open covering then $(\pi^{-1}(\mathcal{O}), \mathcal{O}, \pi)$ is called a sub-bundle.
If the inverse images $\pi^{-1}(x)$ of $x \in V_4$ are all (topologically) isomorphic then $[E, V_4, \pi]$ is called a fibre bundle and the fibre F is the topological space which is isomorphic to the inverse images $\pi^{-1}(x)$.
E.g., the de Sitter bundle of Drechsler $T^R(V_4)$ is a fibre bundle with fibre V_4' = (4+1) de Sitter space.

The physical observables are usually connected not with the underlying space time but with the group of motion in the space time. In particular they are generators or representatives of these generators by linear operators in the space of physical states. The group of motion in the de Sitter space V_4' is SO(4,1), i.e. $(V_4', SO(4,1), \iota)$, where ι_g, $g \in SO(4,1)$, acts on V_4' from the left, is a transformation group.

A mathematical structure $\{E, V_4, F, G, \pi, \iota\}$ is a fibre burdle with structure goup G and fiber F if $\{E, V_4, \pi\}$ is a fibre bundle with fiber F and $\{F, G, \iota\}$ is a transformation group.

The de Sitter bundle $T_R(V_4)$ is a fibre bundle with structure group G = SO(4,1).

A principal fibre bundle, $\{E, V_4, G\}$ is a fibre bundle with structure group for which the typical fiber F = group G.

A representation of a group G is a continuous homomorphism $G \ni g \to U(g) \in \mathcal{U}(g) \subset A(\Phi)$ into the algebra of (unitary) operators of a linear space Φ.

If $\mathcal{U}_\alpha(G) \subset A(\Phi)$ denotes a particular representation (of type α) then one may attempt to construct a "representation fiber bundle" $(E, V_4, \mathcal{U}_\alpha(G))$ by replacing G in the principle fibre bundle by $\mathcal{U}_\alpha(G)$ (with a suitably chosen topology).

In quantum physics the physical observables are the elements of the Lie algebra and enveloping algebra of $\mathcal{U}(G)$, (which are the representatives of the Lie algebra of G).

A sub-bundle $(\pi^{-1}(\mathcal{O}), \mathcal{O} \subset V_4, \mathcal{U}_\alpha(SO(4,1)))$ of a - yet to be properly defined - "representation fibre bundle" is suggested as the mathematical image of a one hadron system.
If the classical geometrical picture of a one-hadron system

is the de Sitter fibre V_4', this suggestion constitutes the obvious transition from classical to quantum physics.

The requirement that the $U_a(G)$ are all equivalent, i.e. that a does not depend upon the θ (or x) is too restrictive for the consideration of multi-hadron systems. If one relaxes this requirement one will obtain a structure $\{E, V_4, u(G), \neg(a)\}$ which is a bundle but not a fibre bundle. Such a - yet to be properly defined - representation bundle (with V_4=Minkowski space and $G = SO(4,1)$) may be needed for the description of hadrons.

The following considerations, taken together with Drechsler's presentation, may give support to this idea. And perhaps it may stimulate the mathematicians to make the above vague considerations precise by creating the mathematics for it.

I shall now report, without giving the detailed calculation here, on a model which may be naturally viewed in the frame of an SO(4,1) - representation sub-bundle. However, I should like to stress, that though this frame leads to a nice geometric interpretation of the one-hadron system its use is not necessary and the original calculations have in fact been done without the knowledge of this geometric interpretation.

We start with Drechsler's[2)] figure for the de Sitter fibre bundle and take for V_4 the Minkowski space. Then the group of motion in V_4 is the Poincaré group E(3,1) with the generators $\Lambda_{\mu\nu}$ $\mu,\nu = 0,1,2,3$ for the homogeneous Lorentz transformations, i.e. for the pseudorotations around any reference point e.g., the point x and the generators p_μ for

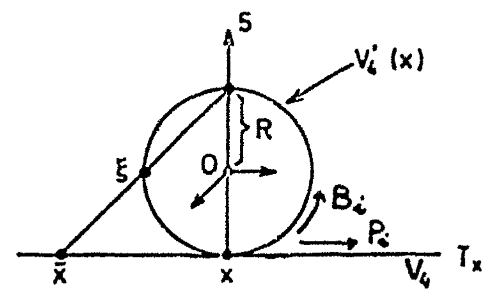


Fig. 1

the space time translations. The transformation group in the de Sitter fiber is SO(4,1) with the generators L_{ab} a,b = 0,1,2,3,5.

In the representation

$$E(3,1) \rightarrow U(E(3,1))$$

$$SO(3,1) \rightarrow U(SO(3,1))$$

the generaters are represented by Hermitian operators

$$P_\mu \rightarrow P_\mu$$
$$L_{ab} \rightarrow L_{ab}$$

which obey the well known commutation relation listed below.

The quantum mechanical observables that are represented by P_μ and $L_{\mu\nu}$ are well known: if the distance in V_4 (parameters of the translation goup) is measured in cm. (units with c = 1 $\hbar$ = 1, 1MeV $\hat{=} \frac{1}{1.973} \cdot 10^{11}$ cm^{-1}), then P_μ is the energy momentum measured in cm^{-1}, if the parameters of the Lorentz group are measured in rad., then L_{ij} is the angular momentum measured in units of 1. The $L_{5\mu}$ represent the generators of rotation in the 5-μ surface, i.e. the operators $B_\mu = \frac{1}{R} L_{5\mu}$ represent the generators of translation along the de Sitter fibre, as shown in figure 1. This new observable B_μ has the same dimension as P_μ, i.e. cm^{-1} if R is measured in cm.

Between the P_μ, B_μ, and $L_{\mu\nu}$ exists a remarkable relation[3], whose geometrical interpretation I am not aware of:

$$(1) \quad B_\mu = P_\mu + \frac{1}{2}\lambda\,(P_\mu P^\mu)^{-1/2}\,(P^\rho L_{\rho\mu} + L_{\rho\mu} P^\rho)\,, \quad \lambda = \frac{1}{R}$$

More precisely, if P_μ $L_{\mu\nu}$ fulfill the commutation relations of the Poincaré group:

$$[P_\mu, P_\nu] = 0$$
$$(2) \quad [L_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu),$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = -i(g_{\mu\rho}L_{\nu\sigma} + g_{\nu\sigma}L_{\mu\rho} - g_{\mu\sigma}L_{\nu\rho} - g_{\nu\rho}L_{\mu\sigma}),$$
$$(g_{ii} = -1,\ i = 1,2,3,\ g_{00} = 1)$$

then B_μ and $L_{\mu\nu}$ fulfill the commutation relations of the (4+1) de Sitter group, i.e.

$$[B_\mu, B_\nu] = i\lambda^2 L_{\mu\nu}\,,$$
$$(3) \quad [L_{\mu\nu}, B_\rho] = i(g_{\nu\rho}B_\mu - g_{\mu\rho}B_\nu).$$

The relation (1) introduces the elementary length R as the radius of the de Sitter space V'_4 of which the $U(SO(4,1))$ generated by B_μ and $L_{\mu\nu}$ are the representatives of the group of motion (i.e. $U(SO(4,1))$ is the quantum mechanical group of motions).

This relation (1) establishes a connection between representations of the Poincaré group and a representation of the de Sitter group.[4] The irreducible representations of the Poincaré group that describe a hadron are characterized by mass m and spin s. The representations of the de Sitter group are also characterized by two numbers;[4] a positive real number

α^2 = eigenvalue of the second order Casimir operator

$$Q = \frac{1}{\lambda^2} B^\mu B_\mu - \frac{1}{2} L_{\mu\nu} L^{\mu\nu}$$

and an integer or half integer σ.

These numbers m, s, α^2, σ are not unrelated but as a consequence of (1) one can derive

$$(4) \qquad m^2 = \lambda^2(\alpha^2 - 9/4) + \lambda^2 s(s+1) \qquad s = \sigma$$

If the hadron is, as perhaps suggested by Drechsler's talk, a de Sitter fibre of type α, i.e. described by U_α (SO(4 1)), then (4) gives the mass formula for hadrons that belong to the same de Sitter fibre of the same representation type α. The de Sitter fibre of type α can be in a discrete set of states of rotational excitation with $s = 0,1,2,3 \ldots$ (or $s = 1/2, 3/2, 5/2 \ldots$), and the mass in these states is given by (4). (4) relates the mass-spectrum to the spin spectrum in the same way as the energy spectrum is related to the angular momentum spectrum for the non-relativistic rotator (diatomic molecule).

The derivation of the spin spectrum, i.e. the possible values of s for one particular representation type α, requires, of course, additional assumptions[5], which I do not want to discuss here. Also, I only want to mention that this model is the simplest possible model of a relativistic rotator and that the particle spectrum is more reminiscent of the symmetric top spectrum than of the simple rotator spectrum; such generalizations[6] can be obtained if one does not take a constant radius for the de Sitter fiber but assumes that R is a quantum mechanical observable with a discrete spectrum.

Finally, I wish to determine the value of λ or $R = \frac{1}{\lambda}$ from the experimental data. In order to do that I have plotted in fig. 2 and 3 m^2 versus $s(s+1)$ for a collection of hadrons. We observe that hadrons of the same type, i.e. the same internal quantum numbers, characterized by the same value α, lie on the same straight line and all lines have the same slope. This is exactly as required by (4), and the value of λ is obtained from the slope to be $\lambda^2 = 0.285$ GeV^2 or $R = 0.37 \cdot 10^{-13}$ cm. An elementary length of this magnitude is quite reasonable for the size of an elementary particle.

Summarizing, the idea that hadrons are de Sitter fiber bundles, introduces in a natural way an elementary length, the radius of these fibres. From the experimental hadron spectrum one can calculate this elementary length to be around 0.37 fermi.

Fig. 2 : Mass squared of the meson resonances plotted versus s(s+1) with s the spin of the resonance.

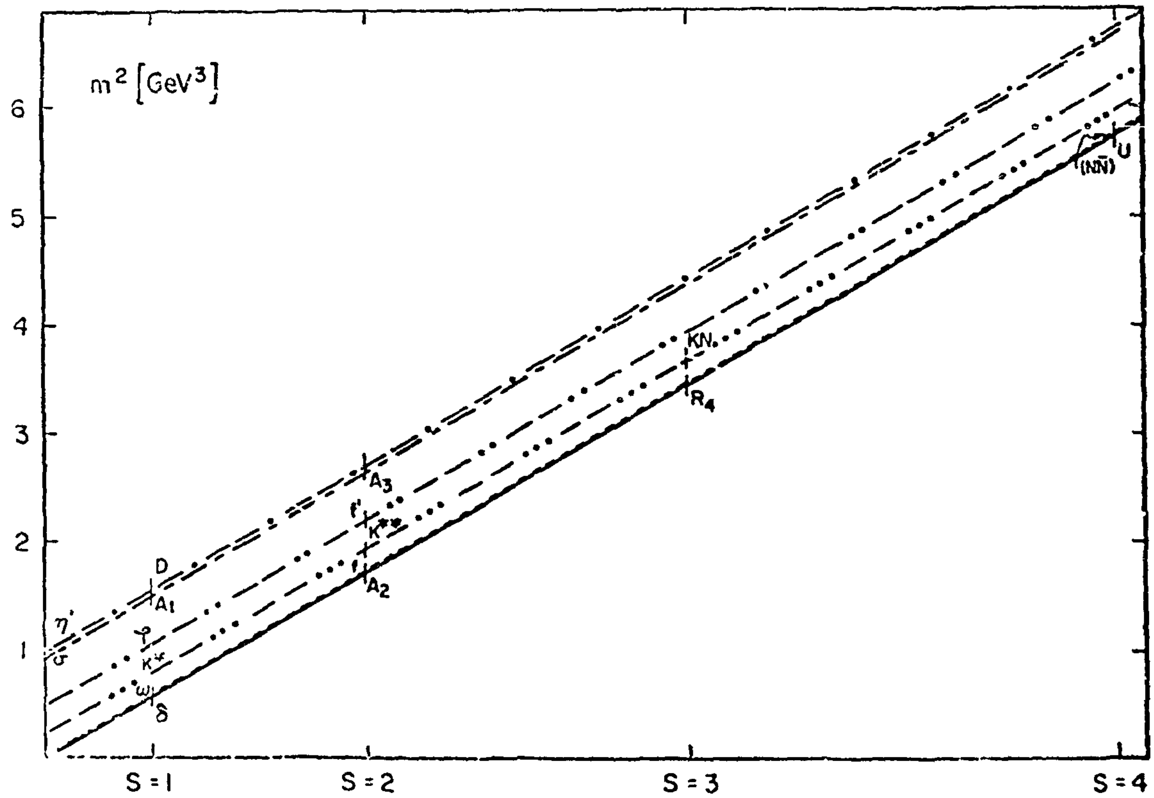
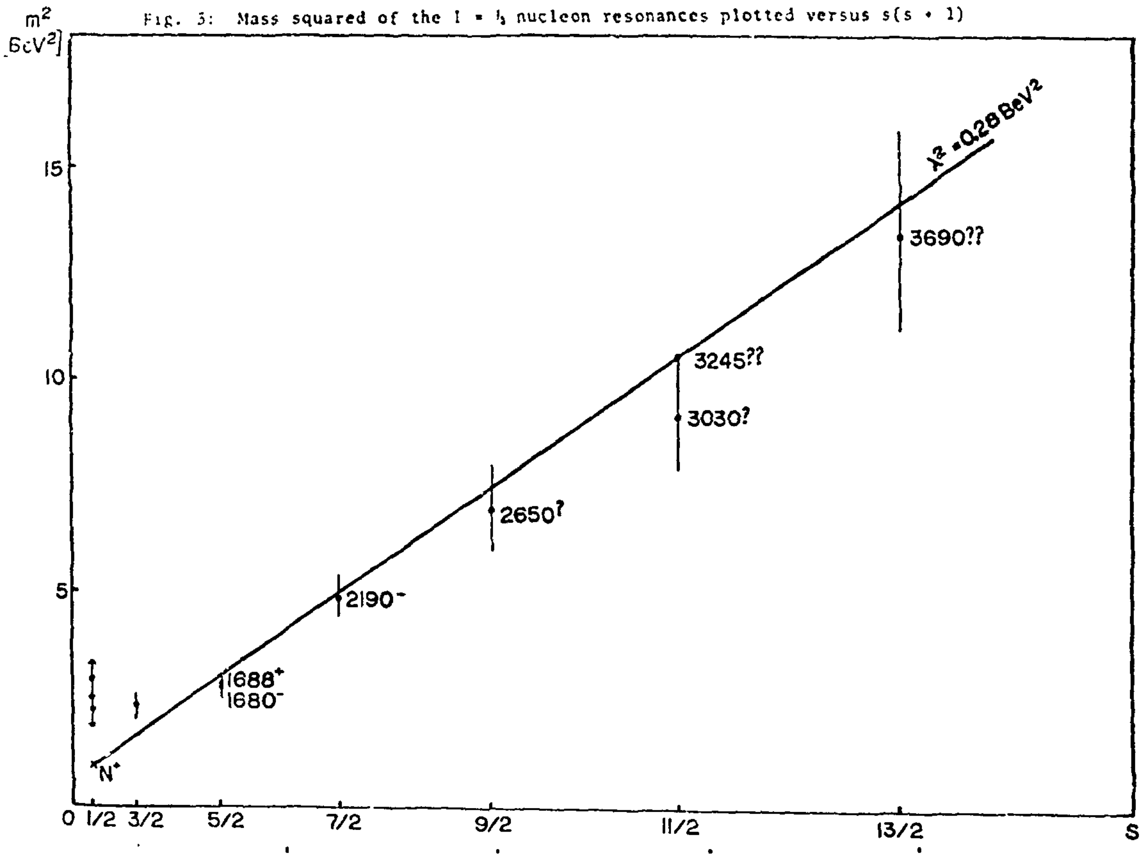


Fig. 3: Mass squared of the I = ½ nucleon resonances plotted versus s(s + 1)



References and Footnotes

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2. A. Taub, "Fibre Bundles Over Space Time". These proceedings
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