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TEST OF GAUGE INVARIANCE AND UNITARITY OF THE QUANTIZED EINSTEIN THEORY OF GRAVITY*

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ABSTRACT

Explicit calculations at the 1-loop level verify that the usual quantized Einstein theory of gravity is indeed gauge independent and unitary for all values of the gauge parameter α. This lends non-trivial support to a general formal proof.

The quantized Einstein theory of gravity has been formulated as a non-Abelian gauge theory, based on the path-integral method.[1,2,3] The classical Einstein Lagrangian possesses the essential property of gauge invariance. Feynman rules have been derived for general gauges,[3] and a general, though formal proof exists that the gravitational field quantized in this manner is both gauge invariant and unitary.[3] That is, physical amplitudes are independent of the gauge parameter α and are unitary for all values of α, just as in quantum electrodynamics.

In other non-Abelian gauge theories, explict calculations have been carried out to substantiate the general proof of gauge invariance and unitarity of the S-matrix elements.[4] The general proof is carried out by using the Ward-Takahashi identities and the path integral and hence it is a formal proof. Thus, it is imperative to perform an explicit calculation to test the claims of the general proof. Such a test has been done before in the Yang-Mills case.[5] Here we have calculated the imaginary amplitude of the 1-loop self-energy of the physical graviton for arbitrary values of α. The results show that this amplitude is both gauge independent and unitary, as advertised.[3]

As an illustration of the form of such a calculation, consider the massless Yang-Mills theory, where the effective

Lagrangian for the 4-vector field $\vec{f}_\mu(x)$ is[6]

$$L_{YM} = -\frac{1}{4}(\partial_\mu \vec{f}_\nu - \partial_\nu \vec{f}_\mu + g\vec{f}_\mu \times \vec{f}_\nu)^2 - \frac{\xi}{2}(\partial_\mu \vec{f}^\mu)^2$$

$$- \vec{\bar{D}} \cdot [\Box \vec{D} + g(\partial_\mu \vec{f}^\mu + \vec{f}^\mu \partial_\mu) \times \vec{D}] , \qquad (1)$$

and $\bar{D}^a(x)$ and $D^a(x)$ are the usual complex fictitious scalar-fermions. At the 1-loop level, unitarity and gauge independence require that the imaginary amplitudes of the self-energy must satisfy

$$\mathrm{Im}[A(F^a_\alpha \to f^b_\mu f^c_\nu \to F^d_\beta) + B(F^a_\alpha \to \bar{D}^b D^c \to F^d_\beta)]$$

$$= \mathrm{Im}\ C(F^a_\alpha \to F^b_\mu F^c_\nu \to F^d_\beta) \qquad (2)$$

for arbitrary values of the gauge parameter ξ, where F^a_α denotes the two physical components of the massless 4-vector field f^a_μ. One can show that (for arbitrary ξ)

$$\mathrm{Im}\ A(F^a_\alpha(p) \to f^b_\mu(k) f^c_\nu(q) \to F^d_\beta(p))$$

$$= \mathrm{Im}(-g^2 \delta_{ad})\{[10Y + Z(4p^2 + k^2 + q^2)]/k^2q^2$$

$$- (1-1/\xi)[Y(2p^2 - q^2) + Z(p^2 - q^2)^2]/k^4q^2$$

$$- (1-1/\xi)[Y(2p^2 - k^2) + Z(p^2 - k^2)^2]/k^2q^4$$

$$+ (1-1/\xi)^2 Yp^4/(k^4q^4)\}$$

$$= 4\pi^2 g^2 \delta_{ad}(10Y)\delta(k^2)\delta(q^2) \qquad , \ p^2 = 0 \qquad (3)$$

$$\text{Im } B[F^a_\alpha(p)\to\bar{D}^b(k)D^c(q)\to F^d_\beta(p)]$$

$$= -4\pi^2 g^2 \delta_{ad}(2Y)\delta(k^2)\delta(q^2), \qquad (4)$$

and

$$\text{Im } C[F^a_\alpha(p)\to F^b_\mu(k)F^c_\nu(q)\to F^d_\beta(p)]$$

$$= 4\pi^2 g^2 \delta_{ad}(8Y)\delta(k^2)\delta(q^2), \qquad (5)$$

where $Y = e_\alpha q^\alpha e'_\beta q^\beta$, $Z = e^\beta e'_\beta$; with $e_\alpha(p)$ and $e'_\beta(p)$ the polarization vectors for $F^a_\alpha(p)$ and $F^d_\beta(p)$ respectively. In this paper we ignore the phase space integration for simplicity. One sees that the relation (2) is indeed satisfied for arbitrary ξ. Note that (5) is obtained by using the propagator for the physical (transverse) field F^b_μ. The results (3) - (5) have been derived previously for the simple case $\xi = 1$ to check unitarity.[5] If one uses the fact that $\vec{p}\parallel\vec{q}\parallel\vec{k}$, with $p^2 = q^2 = k^2 = 0$ for massless particles, it is clear that $e\cdot q = e'\cdot q = 0$, and therefore that $Y = 0$ since both q_μ and k_μ are proportional to p_μ. We emphasize that the relation (2) is satisfied not just trivially as a result of $Y = 0$, a consequence of the simple 1-loop calculation with massless particles, but because of the structural relations of the terms A, B, and C due to gauge invariance.

The Feynman rules for the Einstein theory are somewhat more complicated than those for the Yang-Mills theory. For example, the normal 3-vertex in quantized Einstein's theory involves at least 171 separate terms[2] and the propagator involves 7 terms (cf. equation (5) below); while the massless Yang-Mills 3-vertex and propagator involve only 6 and 2 terms, respectively. However, the high degree of symmetry of the gravity theory allows the reduction of the normal 3-vertex to only 45 terms for performing calculations.

To be specific, let us consider the class of linearized harmonic gauges described by[3]

$$\psi_\mu \equiv \eta^{\sigma\lambda}(\partial_\sigma g_{\lambda\mu} - \tfrac{1}{2}\partial_\mu g_{\lambda\sigma}) = 0, \tag{6}$$

where $\eta^{\sigma\lambda}$ is the Minkowski tensor and the $g_{\mu\nu}$ are treated as independent field variables. The generating functional characterized by the parameter α is[3]

$$Z_\alpha = \int d[g_{\rho\lambda}]\exp\{i\int d^4x(L_0 - \tfrac{1}{2\alpha}\psi_\mu\eta^{\mu\nu}\psi_\nu + g_{\mu\nu}J^{\mu\nu} + \mathrm{Tr}\ln Q\overleftarrow{\Box}^{-1}\}, \tag{7}$$

$$Q_{\mu\nu} = g_{\mu\nu}\Box + (\partial_\rho g_{\mu\lambda} - \tfrac{1}{2}\partial_\mu g_{\rho\lambda}) \times (\eta^{\sigma\rho}\partial_\sigma\eta^\lambda_\nu + \eta^{\sigma\lambda}\partial_\sigma\eta^\rho_\nu - \eta^{\rho\lambda}\partial_\nu), \tag{8}$$

where L_0 is the usual Einstein Lagrangian density and we have set $16\pi^2 G_0 = 1$ (G_0 being the gravitational constant). The

Feynman rules consequent upon (2) in powers of $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ have been derived.[3,2] For our purpose it suffices to have the expressions for the gravity 3-vertex[2] and propagator[3]

$$
\begin{aligned}
\Gamma(p,\mu\nu;k,\sigma\tau;q,\rho\lambda) = \mathrm{Sym}[& -\tfrac{1}{4}P_3(p\cdot k\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}) \\
& - \tfrac{1}{4}P_6(p^\sigma p^\tau \eta^{\mu\nu}\eta^{\rho\lambda}) + \tfrac{1}{4}P_3(p\cdot k\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}) \\
& + \tfrac{1}{2}P_6(p\cdot k\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}) + P_3(p^\sigma p^\lambda \eta^{\mu\nu}\eta^{\tau\rho}) \\
& - \tfrac{1}{2}P_3(p^\tau k^\mu \eta^{\nu\sigma}\eta^{\rho\lambda}) + \tfrac{1}{2}P_3(p^\rho k^\lambda \eta^{\mu\sigma}\eta^{\nu\tau}) \\
& + \tfrac{1}{2}P_6(p^\rho p^\lambda \eta^{\mu\sigma}\eta^{\nu\tau}) + P_6(p^\sigma k^\lambda \eta^{\tau\mu}\eta^{\nu\rho}) \\
& + P_3(p^\sigma k^\mu \eta^{\tau\rho}\eta^{\lambda\nu}) - P_3(p\cdot k\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})] , \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
D_{\rho\lambda\rho'\lambda'}(q) = -i[& \eta_{\rho\lambda}\eta_{\rho'\lambda'} - \eta_{\rho\rho'}\eta_{\lambda\lambda'} - \eta_{\rho\lambda'}\eta_{\lambda\rho'} \\
& + ((1+\alpha)/q^2)(\eta_{\rho\rho'}q_\lambda q_{\lambda'} + \eta_{\rho\lambda'}q_\lambda q_{\rho'} + \eta_{\lambda\rho'}q_\rho q_{\lambda'} \\
& + \eta_{\lambda\lambda'}q_\rho q_{\rho'})]/q^2 \quad ; \qquad (10)
\end{aligned}
$$

as well as the fictitious vector-fermion vertex and propagator

$$
\begin{aligned}
V(p,\sigma\tau;q,\rho;k,\lambda) = i(& k\cdot q\eta_{\rho\sigma}\eta_{\tau\lambda} + q_\lambda k_\tau \eta_{\rho\sigma} \\
& - q_\tau k_\lambda \eta_{\rho\sigma} + p_\rho k_\sigma \eta_{\tau\lambda} - \tfrac{1}{2}p_\rho k_\lambda \eta_{\sigma\tau}) , \qquad (11)
\end{aligned}
$$

and $$G_{\mu\nu}(p) = -i\eta_{\mu\nu}/p^2 \tag{12}$$

where all momenta are incoming to a vertex, and $p_\lambda + q_\lambda + k_\lambda = 0$. The notation "Sym" in (9) denotes that a symmetrization is to be performed on each pair $\mu\nu$, $\sigma\tau$, etc. The symbol P indicates that one must sum over all distinct permutations of the momentum - index triplets, with the subscript giving the number of permutations required in each case.[2]

We now consider the 1-loop self-energy of the physical graviton $\hat{h}_{\mu\nu}$ with 4-momentum p_μ and symmetric polarization tensor $\varepsilon_{\mu\nu}(p)$ satisfying $p^\mu\varepsilon_{\mu\nu} = 0$ and $\varepsilon^\mu_\mu = 0$. The absorptive amplitude for the process $\hat{h}_{\mu\nu}(p) \to h_{\sigma\tau}(k)h_{\rho\lambda}(q) \to \hat{h}_{\mu'\nu'}(p)$ can be obtained from the amplitude

$$A = \frac{1}{2}\varepsilon_{\mu\nu}(p)\Gamma(p,\mu\nu;k,\sigma\tau;q,\rho\lambda)D_{\rho\lambda\rho'\lambda'}(q) \tag{13}$$

$$\times\ \varepsilon'_{\mu'\nu'}(p)\Gamma(-p,\mu'\nu';-k,\sigma'\tau';-q,\rho'\lambda')D_{\sigma\tau\sigma'\tau'}(k),$$

where $\varepsilon'_{\mu'\nu'}(p)$ is the polarization tensor of $\hat{h}_{\mu'\nu'}(p)$, and $p^2 = k^2 = q^2 = 0$. The amplitude A formally involves gauge-dependent terms, i.e., terms with the factors $(1+\alpha)$ and $(1+\alpha)^2$. Since the fictitious vertex and propagator have no α-dependence, these α-dependent terms in Im A must vanish if the theory is gauge invariant. The result of our calculation shows the α-dependent parts of Im A do vanish, and the absorptive part of the graviton loop to have the form (cf. appendix)

$$\text{Im } A = \frac{1}{2}\varepsilon_{\mu\nu}(p)k^{\mu}k^{\nu}\varepsilon'_{\alpha\beta}(p)k^{\alpha}k^{\beta}(4\pi^2)\delta(k^2)\delta(q^2). \qquad (14)$$

In addition, 1-loop unitarity requires

$$\text{Im}[A(\hat{h}_{\mu\nu}(p)\rightarrow h_{\sigma\tau}(k)h_{\rho\lambda}(q)\rightarrow \hat{h}_{\mu'\nu'}(p)$$

$$+ B(\hat{h}_{\mu\nu}(p)\rightarrow \overline{D}_{\rho}(k)D_{\lambda}(q)\rightarrow \hat{h}_{\mu'\nu'}(p)]$$

$$= \text{Im } C(\hat{h}_{\mu\nu}(p)\rightarrow \hat{h}_{\sigma\tau}(k)\hat{h}_{\rho\lambda}(q)\rightarrow \hat{h}_{\mu'\nu'}(p)) \qquad (15)$$

where $\overline{D}_{\rho}(x)$, $D_{\rho}(x)$ are the familiar complex fictitious vector-fermions, and $\hat{h}_{\mu\nu}$ denotes the two physical components of the gravitational field $h_{\mu\nu}$. The first term is given in equation (14). The polarization sum to be performed in (15) is simplified in the frame $p \equiv (p,0,0,p)$. The remaining terms are

$$\text{Im } B(\hat{h}_{\mu\nu}\rightarrow \overline{D}_{\rho}D_{\lambda}\rightarrow \hat{h}_{\mu'\nu'}) = 0 \qquad (16)$$

$$\text{Im } C(\hat{h}_{\mu\nu}\rightarrow \hat{h}_{\sigma\tau}\hat{h}_{\rho\lambda}\rightarrow \hat{h}_{\mu'\nu'})$$

$$= \frac{1}{2}\varepsilon_{\mu\nu}(p)k^{\mu}k^{\nu}\varepsilon'_{\alpha\beta}(p)k^{\alpha}k^{\beta}(4\pi^2)\delta(k^2)\delta(q^2)$$

$$= \text{Im } A(\hat{h}_{\mu\nu}\rightarrow h_{\sigma\tau}h_{\rho\lambda}\rightarrow \hat{h}_{\mu'\nu'}) \qquad (17)$$

and thus the required 1-loop unitarity relation is satisfied, with no contribution from the ghost fields. This is to be contrasted with the manner in which 1-loop unitarity is satisfied in the massless Yang-Mills theory, where the fictitious loop gives an imaginary contribution as shown in (4).

The 1-loop self-energy of the graviton has been calculated before for the harmonic (and therefore, different) gauge condition $\psi'_{\nu} \equiv \partial^{\mu}\{\sqrt{-g}\, g_{\mu\nu}\}$ with the particular value of $\alpha=-1$ for the gauge parameter.[7] However, the unitarity relation (15) has not been checked.

The plan of this calculation has been to test the claim that the Feynman rules of the Einstein field lead to unitary amplitudes for arbitrary values of the gauge parameter α. The results, which support the formal proof, are greater cause for confidence than just the formal proof. Unfortunately, the complexity of gravitational calculations inhibits perturbative tests of a higher order.

Appendix

As to be expected, the calculation is quite lengthy although the results are simple. In the amplitude A given by equation (8) each of the propagators $D_{\rho\lambda\rho'\lambda'}(q)$ and $D_{\sigma\tau\sigma'\tau'}(k)$ consists of two parts, one being α-dependent. If one multiplies out these terms the amplitude falls into four pieces

$$A = L\bar{L} + L\bar{R} + \bar{L}R + R\bar{R} \tag{A1}$$

where $L\bar{L}$ has no α-dependence, $L\bar{R}$ and $\bar{L}R$ are the two cross terms linear in $(\alpha+1)$, and $R\bar{R}$ is quadratic in $(\alpha+1)$. These have the values:

$$L\bar{L} = \frac{1}{2}\varepsilon_{\mu\nu}(p)k^{\mu}k^{\nu}\varepsilon'_{\mu'\nu'}k^{\mu'}k^{\nu'}/k^2q^2, \tag{A2}$$

$$L\bar{R} = -\frac{(\alpha+1)}{2}p^2\varepsilon_{\mu\nu}(p)k^{\mu}k^{\nu}\varepsilon'_{\mu'\nu'}k^{\mu'}k^{\nu'}/k^2q^4, \tag{A3}$$

$$\bar{L}R = -\frac{(\alpha+1)}{2}p^2\varepsilon_{\mu\nu}(p)k^{\mu}k^{\nu}\varepsilon'_{\mu'\nu'}k^{\mu'}k^{\nu'}/k^4q^2, \tag{A4}$$

and $R\bar{R} = 0$ when $p^2 = 0$. Since the calculation of Im A involves the use of the mass-shell conditions $p^2 = 0$, and $k^2 \to 0$, $q^2 \to 0$ the only imaginary part of (A1) which survives is Im $L\bar{L}$. The imaginary part of the fictitious loop is

$$\text{Im } B(\hat{h}_{\mu\nu} \to \bar{D}_{\rho}D_{\lambda} \to \hat{h}_{\mu'\nu'})$$

$$= \text{Im}\{[-(k\cdot q)^2\varepsilon_{\mu\nu}\varepsilon'^{\mu\nu} + 2p^2\varepsilon_{\mu\nu}k^{\mu}\varepsilon'^{\nu\sigma}k_{\sigma}]/k^2q^2\}$$

$$= 0 \text{ since } p^2=0,\ p + q + k = 0, \text{ and } k^2,\ q^2 \to 0. \tag{A5}$$

The process C corresponds to the physical (transverse) graviton loop, so we have

$$\text{Im } C(\hat{h}_{\mu\nu} \to \hat{h}_{\sigma\tau}\hat{h}_{\rho\lambda} \to \hat{h}_{\mu'\nu'}) = \frac{1}{2} \sum_{\text{pol.}} |\hat{h}_{\mu\nu} \to \hat{h}_{\sigma\tau}\hat{h}_{\rho\lambda}|^2$$

$$= \frac{1}{2} \sum_{(2),(3)} |\varepsilon^{\mu\nu}_{(1)}\Gamma(p,\mu\nu;k,\sigma\tau;q,\rho\lambda)\varepsilon^{\sigma\tau}_{(2)}\varepsilon^{\rho\lambda}_{(3)}|^2 \qquad \text{(A6)}$$

If one works in the frame defined by $p \equiv (p,0,0,p)$ the completeness relations of the polarization tensors take the simple form[5]

$$\sum_{i=1}^{2} \varepsilon^{\mu\nu}_{(a)}(p_a,i)\varepsilon^{\alpha\beta}_{(a)}(p_a,i) =$$

$$= \begin{cases} \eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta} & ;\mu,\nu,\alpha,\beta = 1,2 \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(A7)}$$

Applying this to (A6) one obtains Im C = Im A.

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