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INTEGRALS OF PRODUCTS OF SPHERICAL FUNCTIONS

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ABSTRACT

In different parts of mathematical physics the integral formulas of the products of spherical functions are used. In quantum mechanics and in transport theory we have to do with the integrals

$$\int\limits_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega}) Y_l^k(\vec{\Omega}) Y_n^m(\vec{\Omega}),\quad \int\limits_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^m(\mu),\quad \int\limits_{-1}^{1} d\mu\, P_s(\mu) P_l(\mu) P_n(\mu),$$

where $Y_\alpha^\beta(\vec{\Omega})$ are spherical harmonics, $P_\alpha^\beta(\mu)$ are associated Legendre functions, and $P_\alpha(\mu)$ are Legendre polynomials. In this paper the general procedure of the calculation of integrals of the products of any combination of spherical functions is given. We refer to this report in our paper discussing the boundary conditions for the cylindrical geometry in neutron transport theory for both the outer and inner cylindrical boundary, which is to be published.

Note: The trivial case of $s = l = n = 0$ is an exception as $Y_0^0(\vec{\Omega}) = 1$, $P_0^0(\mu) = 1$, $P_0(\mu) = 1$. Hence the first integral has the value 4π, and the second and the third ones have the value 2.

TABLE OF CONTENTS

1. Introduction

We show here a practical problem where the integral $\int_{-1}^{1} d\mu P_p^t(\mu) P_\ell^k(\mu) P_n^m(\mu)$ is to be used. In neutron transport theory the problem of penetration of neutrons through a cylindrical vacuum channel is to be solved. We suppose a long channel and cylindrically symmetrical neutron flux. The boundary condition is $\varphi(R,\vartheta,\psi) = \varphi(R,\vartheta,\pi-\psi)$ for $0 \le \vartheta \le \pi$, $-\frac{\pi}{2} \le \psi \le \frac{\pi}{2}$.

To get this condition in P_N -approximation the upper equation is to be multiplied by the spherical harmonic $\vec{\Omega}\vec{n} P_{2\ell}^k(\vec{\Omega})$ and integrated over $\vec{\Omega}\vec{n} < 0$. These operations give the well known optimal Wladimirow boundary condition

$$\int d\vec{\Omega}\, \vec{\Omega}\vec{n}\, \varphi(R,\vartheta,\psi) P_{2\ell}^k(\vec{\Omega}) = \int d\vec{\Omega}\, \vec{\Omega}\vec{n}\, \varphi(R,\vartheta,\pi-\psi) P_{2\ell}^k(\vec{\Omega}) \ .$$

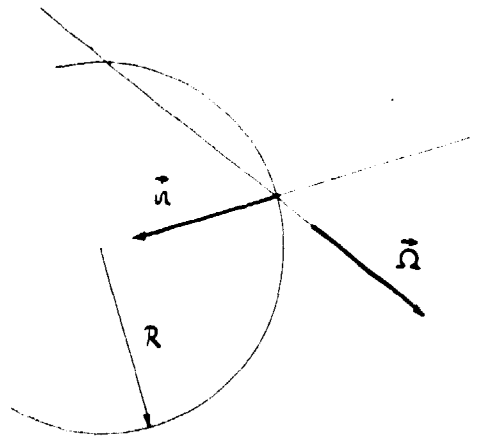
Using the Fourier series

$$\varphi(R,\vartheta,\psi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{2n+1}{4\pi} \varphi_{nm}(R) P_n^m(\vec{\Omega}) ,$$

$$\varphi(R,\vartheta,\pi-\psi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{2n+1}{4\pi} \varphi_{n,-m}(R) P_n^m(\vec{\Omega}) ,$$

and $d\vec{\Omega} = \sin\vartheta\, d\vartheta\, d\psi$, $\vec{\Omega}\vec{n} = -\sin\vartheta \cos\psi$, $\sin\vartheta = P_1^1(\cos\vartheta)$ we get a system of equations for Fourier moments $\varphi_{nm}(R)$ with coefficients

$$C_{nm\ell k} \sim \int_0^{\pi} \sin\vartheta\, d\vartheta\, P_1^1(\cos\vartheta) P_{2\ell}^k(\cos\vartheta) P_n^m(\cos\vartheta) = \int_{-1}^{1} d\mu\, P_1^1(\mu) P_{2\ell}^k(\mu) P_n^m(\mu) \ .$$



2. List of used spherical functions

Generalized spherical function (according to /1/, p. 123 etc.) for $n = 0\ 1\ 2\ \dots,\ m = -n\ \dots\ n,\ m' = -n\ \dots\ n$:

$$P_n^{mm'}(\mu) = \frac{(-1)^{n-m'}\, i^{m'-m}}{2^n}\sqrt{\frac{(n+m)!}{(n-m')!(n+m')!(n-m)!}}\,(1+\mu)^{\frac{-m'-m}{2}}(1-\mu)^{\frac{m'-m}{2}}\frac{d^{n-m}}{d\mu^{n-m}}\left[(1+\mu)^{n+m'}(1-\mu)^{n-m'}\right],$$

$$\mu = \cos\vartheta \in \langle -1, 1\rangle,\ \vartheta \in \langle 0, \pi\rangle; \tag{2.1}$$

associated Legendre functions for $n = 0\ 1\ 2\ \dots,\ m = 0\ 1\ 2\ \dots\ n$:

$$P_n^m(\cos\vartheta) = \sin^m\vartheta\,\frac{d^m P_n(\cos\vartheta)}{d\cos\vartheta^m} = \sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]}\frac{(-1)^\alpha(2n-2\alpha)!}{2^n\alpha!(n-\alpha)!(n-m-2\alpha)!}\sin^m\vartheta\cos^{n-m-2\alpha}\vartheta; \tag{2.2}$$

Legendre polynomials for $n = 0\ 1\ 2\ \dots$:

$$P_n(\cos\vartheta) = \frac{1}{2^n n!}\frac{d^n(\cos^2\vartheta-1)^n}{d\cos\vartheta^n} = \sum_{\alpha=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^\alpha(2n-2\alpha)!}{2^n\alpha!(n-\alpha)!(n-2\alpha)!}\cos^{n-2\alpha}\vartheta = P_n^0(\cos\vartheta); \tag{2.3}$$

spherical harmonics for $n = 0\ 1\ 2\ \dots,\ m = -n\ \dots\ n$:

$$Y_n^m(\vec{\Omega}) = Y_n^m(\vartheta,\psi) = e^{im\psi}P_n^m(\cos\vartheta), \tag{2.4}$$

$$Y_n^{m*}(\vec{\Omega}) = Y_n^{m*}(\vartheta,\psi) = e^{-im\psi}P_n^m(\cos\vartheta), \tag{2.5}$$

$$P_n^m(\vec{\Omega}) = (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,Y_n^m(\vec{\Omega}), \tag{2.6}$$

$$P_n^{m*}(\vec{\Omega}) = (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,Y_n^{m*}(\vec{\Omega}), \tag{2.7}$$

$$\vec{\Omega} = [\sin\vartheta\cos\psi,\ \sin\vartheta\sin\psi,\ \cos\vartheta],\quad 0 \le \vartheta \le \pi,\quad 0 \le \psi < 2\pi. \tag{2.8}$$

According to /1/, p. 133, the following formulae are true :

$$P_n^m(\mu) = i^m\sqrt{\frac{(n+m)!}{(n-m)!}}\,P_n^{m0}(\mu) = i^m\sqrt{\frac{(n+m)!}{(n-m)!}}\,P_n^{-m,0}(\mu),\quad m \ge 0. \tag{2.9}$$

Hence the function (2.2) may be generalized for $m<0$ as

$$P_n^{-m}(\cos\vartheta) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\vartheta). \tag{2.10}$$

From (2.9) may be written

$$P_n(\mu) = P_n^{00}(\mu). \tag{2.11}$$

Further the following formulae are true :

$$P_n^{m*}(\vec{\Omega}) = (-1)^m P_n^{-m}(\vec{\Omega}), \tag{2.12}$$

$$P_0^0(\vec{\Omega}) = P_0^{0*}(\vec{\Omega}) = P_0^0(\mu) = P_0(\mu) = 1. \tag{2.13}$$

Note: In accordance with (2.2), (2.4), (2.5) $P_n^m(\cos\vartheta) = 0$, $Y_n^m(\vec{\Omega}) = 0$, $Y_n^{m*}(\vec{\Omega}) = 0$ for $|m| > n$.

3. Clebsh-Gordan coefficients and Wigner's 3j-symbols

In this chapter the following integral formula is calculated:

$$\int_{-1}^{1} d\mu\, P_s^{tt'}(\mu) P_\ell^{\kappa\kappa'}(\mu) \overline{P_u^{\kappa+t,\kappa'+t'}(\mu)} = \frac{2}{2u+1} \overline{C}(s,\ell,u;t,\kappa,\kappa+t)\, C(s,\ell,u;t',\kappa',\kappa'+t'). \tag{3.1}$$

Here $\overline{P_u^{\kappa+t,\kappa'+t'}}(\mu)$ is the complex conjugate of $P_u^{\kappa+t,\kappa'+t'}(\mu)$ and C, $\overline{C}$ are complex conjugated Clebsh-Gordan coefficients. Using $\kappa'=-\ell$, $t'=s$ and the following formulae given in /1/, p. 129, 131, 132

$$\overline{P_u^{mm'}}(\mu) = (-1)^{m'-m} P_u^{mm'}(\mu), \tag{3.2}$$

$$P_u^{mm'}(\mu) = P_u^{m'm}(\mu), \tag{3.3}$$

$$P_u^{mm'}(-\mu) = (-1)^{u-m-m'} P_u^{m,-m'}(\mu), \text{ i.e. } P_u^{mu}(-\mu) = (-1)^{-m} P_u^{m,-u}(\mu), \tag{3.4}$$

$$P_u^{mu}(\mu) = \frac{i^{u-m}}{2^u}\sqrt{\frac{(2u)!}{(u-m)!\,(u+m)!}}\,(1+\mu)^{\frac{u+m}{2}}(1-\mu)^{\frac{u-m}{2}}, \tag{3.5}$$

i.e.
$$P_u^{m,-u}(\mu) = \frac{i^{u+m}}{2^u}\sqrt{\frac{(2u)!}{(u-m)!\,(u+m)!}}\,(1+\mu)^{\frac{u-m}{2}}(1-\mu)^{\frac{u+m}{2}}, \tag{3.6}$$

we get from (3.1)

$$\overline{C}(s,\ell,u;t,\kappa,\kappa+t)\,C(s,\ell,u;s,-\ell,s-\ell) = \frac{2u+1}{2}\int_{-1}^{1} d\mu\, P_s^{ts}(\mu) P_\ell^{\kappa,-\ell}(\mu)\overline{P_u^{\kappa+t,s-\ell}}(\mu) =$$

$$= \frac{(-1)^{-u+s+\kappa}(2u+1)}{2^{u+\ell+s+1}}\sqrt{\frac{(2\ell)!\,(2s)!\,(u+\kappa+t)!}{(\ell+\kappa)!\,(\ell-\kappa)!\,(s+t)!\,(s-t)!\,(u+\ell-s)!\,(u-\ell+s)!\,(u-\kappa-t)!}}\ \times$$

$$\times \int_{-1}^{1} d\mu\,(1+\mu)^{\ell-\kappa}(1-\mu)^{s-t}\frac{d^{u-\kappa-t}}{d\mu^{u-\kappa-t}}\left[(1+\mu)^{u-\ell+s}(1-\mu)^{u+\ell-s}\right]. \tag{3.7}$$

Using $\kappa=-\ell$, $t=s$ the coefficient $C(s,\ell,u;s,-\ell,s-\ell)$ may be easily found :

$$\overline{C}(s,\ell,u;s,-\ell,s-\ell)\,C(s,\ell,u;s,-\ell,s-\ell) = |C(s,\ell,u;s,-\ell,s-\ell)|^2 =$$

$$= \frac{(-1)^{-u-\ell+s}(2u+1)}{2^{u+\ell+s+1}(u+\ell-s)!}\int_{-1}^{1} d\mu\,(1+\mu)^{2\ell}\frac{d^{u+\ell-s}}{d\mu^{u+\ell-s}}\left[(1+\mu)^{u-\ell+s}(1-\mu)^{u+\ell-s}\right]. \tag{3.8}$$

Integration per-partes $(u+\ell-s)$-times :

$$|C(s,\ell,u;s,-\ell,s-\ell)|^2 = \frac{(-1)^{u+\ell-s}(2u+1)(2\ell)!}{2^{u+\ell+s+1}(u+\ell-s)!\,(\ell+s-u)!}\int_{-1}^{1} d\mu\,(\mu+1)^{2s}(\mu-1)^{u+\ell-s}.$$

Integration per-partes $(2s)$-times :

$$\int_{-1}^{1} d\mu\,(\mu+1)^{2s}(\mu-1)^{n+l-s} = \frac{(-1)^{n+l+s+2}\,2^{n+l+s+1}\,(2s)!}{(n+l-s+1)(n+l-s+2)\cdots(n+l+s+1)} .$$

Thence
$$\left|C(s,l,n;s,-l,s-l)\right| = \sqrt{\frac{(2n+1)(2l)!\,(2s)!}{(l+s-n)!\,(n+l+s+1)!}} . \qquad (3.9)$$

So far we have assumed the $C(s,l,n;s,-l,s-l)$ to be a complex value with the modulus (3.9). The $\overline{C}(s,l,n;t,k,k+t)\,C(s,l,n;s,-l,s-l)$ is a real value which has been shown above. Hence this product does not depend on the arguments of $\overline{C}(s,l,n;t,k,k+t)$, $C(s,l,n;s,-l,s-l)$ respectively. Let φ be the argument of $C(s,l,n;t,k,k+t)$. Hence the argument of the $\overline{C}(s,l,n;t,k,k+t)$ is $-\varphi$ and the argument of $C(s,l,n;s,-l,s-l)$ is φ. The $C(s,l,n;t,k,k+t)$ are not defined unambiguously. For various permitted values of n, l, s, k, t they define a system of concentric circles; the modulus of $C(s,l,n;t,k,k+t)$ alters discontinuously; these circles degenerate into a point (the origin) when $C(s,l,n;t,k,k+t) = 0$. The integral (3.7) is proportional to the product of the moduli of $C(s,l,n;t,k,k+t)$, $C(s,l,n;s,-l,s-l)$. The modulus of the coefficient $C(s,l,n;t,k,k+t)$ is equal to the non-negative value of $C(s,l,n;t,k,k+t)$. For this real value the formula $C(s,l,n;t,k,k+t) = \overline{C}(s,l,n;t,k,k+t)$ is true. Further we respect only this real non-negative value of the C.-G. coefficient $C(s,l,n;t,k,k+t)$. Hence for $k=-l$, $t=s$ may be written according to (3.9) the formula

$$C(s,l,n;s,-l,s-l) = \sqrt{\frac{(2n+1)(2l)!\,(2s)!}{(l+s-n)!\,(n+l+s+1)!}} . \qquad (3.10)$$

For permitted values of n, l, s, k, t there is $C(s,l,n;s,-l,s-l) > 0$. Now the formula (3.7) gives

$$C(s,l,n;t,k,k+t) = \frac{(-1)^{l+s+k+t}}{2^{n+l+s+1}}\sqrt{\frac{(2n+1)(n+k+t)!\,(l+s-n)!\,(n+l+s+1)!}{(l+k)!\,(l-k)!\,(s+t)!\,(s-t)!\,(n+l-s)!\,(n-l+s)!\,(n-k-t)!}} \times$$
$$\times \int_{-1}^{1} d\mu\,(\mu+1)^{l-k}(\mu-1)^{s-t}\,\frac{d^{n-k-t}}{d\mu^{n-k-t}}\left[(\mu+1)^{n-l+s}(\mu-1)^{n+l-s}\right]. \qquad (3.11)$$

Using the Leibnitz formula and integration per-partes we get

$$\int_{-1}^{1} d\mu\,(\mu+1)^{l-k}(\mu-1)^{s-t}\sum_{j=0}^{n-k-t}\binom{n-k-t}{j}\frac{d^{n-k-t-j}}{d\mu^{n-k-t-j}}(\mu+1)^{n-l+s}\,\frac{d^{j}}{d\mu^{j}}(\mu-1)^{n+l-s} =$$

$$= \sum_{j=\max(0,\,l-s-k-t)}^{\min(n-k-t,\,n+l-s)} \frac{(n-k-t)!\,(n-l+s)!\,(n+l-s)!}{j!\,(n-k-t-j)!\,(s-l+k+t+j)!\,(n+l-s-j)!}\int_{-1}^{1} d\mu\,(\mu+1)^{s+t+j}(\mu-1)^{n+l-t-j} .$$

Integration $(s+t+j)$-times per-partes :

$$\int_{-1}^{1} d\mu\,(\mu+1)^{s+t+j}(\mu-1)^{n+l-t-j} = (-1)^{-n+l+t+j}\,2^{n+l+s+1}\,\frac{(s+t+j)!\,(n+l-t-j)!}{(n+l+s+1)!}\,.$$

Introducing into (3.11) we get the searched formula

$$\boxed{\begin{aligned} C(s,l,n;t,k,k+t) &= (-1)^{-n+s+k}\sqrt{\frac{(2n+1)(n+k+t)!\,(n-k-t)!\,(l+s-n)!\,(n+l-s)!\,(n-l+s)!}{(n+l+s+1)!\,(l+k)!\,(l-k)!\,(s+t)!\,(s-t)!}}\;\times \\ &\quad\times \sum_{j=\max(0,\,l-s-k-t)}^{\min(n-k-t,\,n+l-s)} \frac{(-1)^j (s+t+j)!\,(n+l-t-j)!}{j!\,(n-k-t-j)!\,(n+l-s-j)!\,(s-l+t+k+j)!}\,. \end{aligned}}$$

(3.12)

We suppose the factorial to be defined only for non-negative integer values (including zero); therefore $l+s-n \geq 0$, $n+l-s \geq 0$, $n-l+s \geq 0$; chosing for instance $s = 0, 1, 2, \ldots$, $l = 0, 1, 2, \ldots$, the values of n are limited; from the first relation follows $n \leq s+l$, from the other two relations follows $n \geq |s-l|$; hence the Clebsh-Gordan coefficients are defined for $|s-l| \leq n \leq s+l$, $|m| \leq n$, $|k| \leq l$, $|t| \leq s$; it may be shown that $C(s, l, n;\ t, k, m) = 0$ for $m \neq k+t$ (see for instance /1/, p.135). Further we are interested in the special formula for $C(s, l, n;\ 0, 0, 0)$; although it may be obtained from (3.12) immediately, it is better to derive it independently as we can get it in a summarized form. Using (3.3) the (3.7) may be written as

$$\overline{C}(s,l,n;t,k,k+t)\,C(s,l,n;s,-l,s-l) = \frac{2n+1}{2}\int_{-1}^{1} d\mu\, P_s^{t,s}(\mu)\,P_l^{k,-l}(\mu)\,\overline{P_n^{s-l,k+t}(\mu)} =$$

$$= \frac{(-1)^{-n+s+t}(2n+1)}{2^{n+l+s+1}}\sqrt{\frac{(2l)!\,(2s)!\,(n-l+s)!}{(l+k)!\,(l-k)!\,(s+t)!\,(s-t)!\,(n-k-t)!\,(n+k+t)!\,(n+l-s)!}}\;\times$$

$$\times\int_{-1}^{1} d\mu\,(1+\mu)^{l-k}(1-\mu)^{l+k}\cdot\frac{d^{n+l-s}}{d\mu^{n+l-s}}\left[(1+\mu)^{n+k+t}(1-\mu)^{n-k-t}\right].$$ Using (3.8):

$$C(s,l,n;t,k,k+t) = \frac{(-1)^{-n+s+k}}{2^{n+l+s+1}}\sqrt{\frac{(2n+1)(n-l+s)!\,(l+s-n)!\,(n+l+s+1)!}{(l+k)!\,(l-k)!\,(s+t)!\,(s-t)!\,(n-k-t)!\,(n+k+t)!\,(n+l-s)!}}\;\times$$

$$\times\int_{-1}^{1} d\mu\,(1+\mu)^{l-k}(1-\mu)^{l+k}\,\frac{d^{n+l-s}}{d\mu^{n+l-s}}\left[(1+\mu)^{n+k+t}(1-\mu)^{n-k-t}\right],$$

$$C(s,l,n;0,0,0) =$$

$$= \frac{(-1)^{s-n}}{2^{n+l+s+1}\,n!\,l!\,s!}\sqrt{\frac{(2n+1)(n-l+s)!\,(l+s-n)!\,(n+l+s+1)!}{(n+l-s)!}}\int_{-1}^{1} d\mu\,(1-\mu^2)^l\,\frac{d^{n+l-s}(1-\mu^2)^n}{d\mu^{n+l-s}}\,.$$

Performing the integral $J^r_{hq} = \int_{-1}^{1} d\mu (1-\mu^2)^h \frac{d^r}{d\mu^r}(1-\mu^2)^q$ we get

$J^0_{hq} = \frac{2^{2h+2q+1}\left[(h-q)!\right]^2}{(2h+2q+1)!}$, $J^1_{hq} = 0$. Finally the following recurent formula will be obtained: $J^r_{hq} = 2q(2q-2)J^{r-2}_{h,q-2} - 2q(2q-1)J^{r-2}_{h,q-1}$. Hence $J^r_{hq} = 0$ for odd values of r and

$$J^{2r}_{hq} = \frac{(-1)^r 2^{2h+2q-2r+1} h!q!(2r)!(h+q-2r)!(h+q-r)!}{r!(h-r)!(q-r)!(2h+2q-2r+1)!} .$$

Thence

$$\int_{-1}^{1} d\mu (1-\mu^2)^l \frac{d^{u+l-s}}{d\mu^{u+l-s}}(1-\mu^2)^u = \frac{(-1)^{\frac{u+l-s}{2}} 2^{u+l+s+1} s!l!u!(u+l-s)!\left(\frac{u+l+s}{2}\right)!}{\left(\frac{u+l-s}{2}\right)!\left(\frac{l+s-u}{2}\right)!\left(\frac{u-l+s}{2}\right)!(u+l+s+1)!}\delta_{u+l+s,2q} =$$

$$= \frac{(-1)^{q-s} 2^{u+l+s+1} u!l!s!q!(u+l-s)!}{(q-u)!(q-l)!(q-s)!(u+l+s+1)!}\delta_{u+l+s,2q} .$$

Finally we get the searched formula

$$\boxed{C(s,l,u;0,0,0) = (-1)^{q-u}\sqrt{\frac{(2u+1)(u-l+s)!(l+s-u)!(u+l-s)!}{(u+l+s+1)!}}\frac{q!}{(q-u)!(q-l)!(q-s)!}\delta_{u+l+s,2q} .}$$

(3.13)

There is $C(s,l,u;0,0,0) = 0$ for odd $u+l+s$.

For practical calculations many useful symmetry formulae are known. To get these formulae it is convenient to use the definition of the well known Wigner's 3j-symbols associated with the Clebsh-Gordan coefficients through the following formula (see /1/, 189; /3/, p. 472; /5/, p. 531; /2/, p. 516; etc.)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{-j_1+j_2+m_3}}{\sqrt{2j_3+1}} C(j_1,j_2,j_3;m_1,m_2,-m_3)\,\delta_{m_3,-m_1-m_2} , \qquad (3.14)$$

i.e.

$$\boxed{\begin{aligned} \begin{pmatrix} s & l & u \\ t & \kappa & m \end{pmatrix} &= \frac{(-1)^{-s+l+m}}{\sqrt{2u+1}} C(s,l,u;t,\kappa,-m)\,\delta_{-m,\kappa+t} , \\ C(s,l,u;t,\kappa,m) &= (-1)^{s-l+m}\sqrt{2u+1}\begin{pmatrix} s & l & u \\ t & \kappa & -m \end{pmatrix} . \end{aligned}}$$

(3.15)

(3.16)

Below a general symmetry rule is given (/1/, p. 190; /3/, p. 473):

Subjoining to $\begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix}$ the matrix $\begin{Vmatrix} -s+l+u & s-l+u & s+l-u \\ s-t & l-k & u-m \\ s+t & l+k & u+m \end{Vmatrix}$, $m=-k-t$,

the following assertions are true: 1) The sum of the values in any line or column is $u+l+s$. 2) Re-arranging of any two parallel rows (lines or columns) changes $\begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix}$ into $(-1)^{s+l+u}\begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix}$. 3) The transpose of the matrix does not change the $\begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix}$.

Using this rule we get for instance:

$$\begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix} = \begin{pmatrix} u & s & l \\ m & t & k \end{pmatrix}, \tag{3.17}$$

$$\begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix} = (-1)^{s+l+u}\begin{pmatrix} l & s & u \\ k & t & m \end{pmatrix}, \tag{3.18}$$

$$\begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix} = (-1)^{s+l+u}\begin{pmatrix} s & l & u \\ -t & -k & -m \end{pmatrix}, \tag{3.19}$$

$$\begin{pmatrix} \frac{u+l-m-k}{2} & \frac{u+s-m-t}{2} & \frac{l+s-k-t}{2} \\ \frac{2s-u-l-m-k}{2} & \frac{2l-u-s-m-t}{2} & \frac{2u-l-s-k-t}{2} \end{pmatrix} = (-1)^{s+l+u}\begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix}, \tag{3.20}$$

$$\begin{pmatrix} s & \frac{l+u+t}{2} & \frac{l+u-t}{2} \\ l-u & \frac{-l+u+2k+t}{2} & \frac{-l+u-2k-t}{2} \end{pmatrix} = \begin{pmatrix} s & l & u \\ t & k & m \end{pmatrix}, \tag{3.21}$$

where always $m=-k-t$; the used operations in the matrix were:

(3.17): 1st and 2nd column, 2nd and 3rd column;
(3.18): 1st and 2nd column;
(3.19): 2nd and 3rd line;
(3.20): 1st and 2nd line;
(3.21): transpose operation.

Now we give some special formulae:

According to /1/, p. 190, 192 :

$$\begin{pmatrix} s & l & u \\ s & k & -s-k \end{pmatrix} = (-1)^{l-k}\sqrt{\frac{(u+s+k)!\,(u+l-s)!\,(l-k)!\,(2s)!}{(u+s-k)!\,(l-u+s)!\,(u-l+s)!\,(l+k)!\,(u+l+s+1)!}}, \tag{3.22}$$

$$\begin{pmatrix} s & l & u \\ t & u-t & -u \end{pmatrix} = (-1)^{l-u-t}\sqrt{\frac{(s+l-u)!\,(l+u-t)!\,(s+t)!}{(u+l-s)!\,(u-l+s)!\,(l-u+t)!\,(s-t)!\,(u+l+s+1)!}}, \tag{3.23}$$

$$\begin{pmatrix} s & l & s+l \\ t & k & -k-t \end{pmatrix} = (-1)^{-s+l-k-t}\sqrt{\frac{(s+l+k+t)!\,(s+l-k-t)!\,(2l)!\,(2s)!}{(l+k)!\,(l-k)!\,(s+t)!\,(s-t)!\,(2s+2l)!}}, \tag{3.24}$$

$$\begin{pmatrix} s & l & s-l \\ t & k & -k-t \end{pmatrix} = (-1)^{s+t}\sqrt{\frac{(s+t)!\,(s-t)!\,(2s-2l)!\,(2l)!}{(l+k)!\,(l-k)!\,(s-l+k+t)!\,(s-l-k-t)!\,(2s+1)!}}, \tag{3.25}$$

$$\begin{pmatrix} s & s & u \\ t & t & -2t \end{pmatrix} = (-1)^{u-g} \sqrt{\frac{(u+2t)!\,(u-2t)!\,(2s-u)!}{(2s+u+1)!}} \; \frac{g!}{(g-s-t)!\,(g-s+t)!\,(g-u)!} \; \delta_{2s+u,\,2g} \,, \tag{3.26}$$

according to /3/, p. 473 :

$$\begin{pmatrix} s & l & u \\ s & -s-m & m \end{pmatrix} = (-1)^{-s+l+m} \sqrt{\frac{(u+l-s)!\,(l+s+m)!\,(u-m)!\,(2s)!}{(u-l+s)!\,(l-u+s)!\,(l-s-m)!\,(u+m)!\,(u+l+s+1)!}} \,, \tag{3.27}$$

according to /10/, p. 5 :

$(-1)^{s+m-1} \begin{pmatrix} s & 1 & u \\ m-k & k & -m \end{pmatrix}$:

$u \backslash k$	-1	0	1
$s+1$	$\sqrt{\dfrac{(s-m)(s-m+1)}{(2s+1)(2s+2)(2s+3)}}$	$\sqrt{\dfrac{(s-m+1)(s+m+1)}{(s+1)(2s+1)(2s+3)}}$	$\sqrt{\dfrac{(s+m)(s+m+1)}{(2s+1)(2s+2)(2s+3)}}$
s	$\sqrt{\dfrac{(s-m)(s+m+1)}{(s+1)2s(2s+1)}}$	$\dfrac{m}{\sqrt{s(s+1)(2s+1)}}$	$-\sqrt{\dfrac{(s+m)(s-m+1)}{(s+1)2s(2s+1)}}$
$s-1$	$\sqrt{\dfrac{(s+m)(s+m+1)}{(2s-1)2s(2s+1)}}$	$-\sqrt{\dfrac{(s-m)(s+m)}{s(2s-1)(2s+1)}}$	$\sqrt{\dfrac{(s-m)(s-m+1)}{(2s-1)2s(2s+1)}}$

(3.28)

$$C(s,1,u;\,m-k,k,m) = (-1)^{s+m-1} \sqrt{2u+1} \begin{pmatrix} s & 1 & u \\ m-k & k & -m \end{pmatrix} , \tag{3.29}$$

according to /2/, p. 516 :

$$\begin{pmatrix} s & s & 0 \\ t & -t & 0 \end{pmatrix} = (-1)^{s-t} \frac{1}{\sqrt{2s+1}} \,. \tag{3.30}$$

Many other formulae in both the algebraic and numerical expression are given in the well known tables of Clebsh-Gordan coefficients /10/, where the designation $(j_1\ j_2\ m_1\ m_2 \,|\, j_1\ j_2\ j\ m)$ means

$$(j_1\, j_2\, m_1\, m_2 \,|\, j_1\, j_2\, j\, m) = C(j_1, j_2, j;\, m_1, m_2, m) = (-1)^{j_1-j_2+m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} .$$

Further some recurrence formulae according to /1/, p. 195, 196 are given :

$$\begin{pmatrix} s & l & u \\ t & k & -k-t \end{pmatrix} = -\sqrt{\frac{(s-t)(s+t+1)}{(u-k-t)(u+k+t+1)}} \begin{pmatrix} s & l & u \\ t+1 & k & -k-t-1 \end{pmatrix} - \sqrt{\frac{(l-k)(l+k+1)}{(u-k-t)(u+k+t+1)}} \begin{pmatrix} s & l & u \\ t & k+1 & -k-t-1 \end{pmatrix} =$$

$$= -\sqrt{\frac{(s+t)(s-t+1)}{(u+k+t)(u-k-t+1)}} \begin{pmatrix} s & l & u \\ t-1 & k & -k-t+1 \end{pmatrix} - \sqrt{\frac{(l+k)(l-k+1)}{(u+k+t)(u-k-t+1)}} \begin{pmatrix} s & l & u \\ t & k-1 & -k-t+1 \end{pmatrix} =$$

$$= \sqrt{\frac{(s-t)(s+t+1)(l+k)(l-k+1)}{(s-t)(s+t+1)+(l-k)(l+k+1)-(u-k-t)(u+k+t+1)}} \begin{pmatrix} s & l & u \\ t+1 & k-1 & -k-t \end{pmatrix} + \sqrt{\frac{(s+t)(s-t+1)(l-k)(l+k+1)}{(s-t)(s+t+1)+(l-k)(l+k+1)-(u-k-t)(u+k+t+1)}} \begin{pmatrix} s & l & u \\ t-1 & k+1 & -k-t \end{pmatrix} . \tag{3.31}$$

Finally we give the following formula according to /1/, p. 192 :

$$P_s^{tt'}(\mu)P_l^{\kappa\kappa'}(\mu) = \sum_\alpha C(s,l,\alpha;t,\kappa,\kappa+t)C(s,l,\alpha;t',\kappa',\kappa'+t')P_\alpha^{\kappa+t,\kappa'+t'}(\mu) , \tag{3.32}$$

where $\alpha = max(|s-l|,|\kappa+t|),\dots,s+l$.

Using (3.32) and (2.9), (2.4), (2.3) the following formulae may be easily obtained :

$$P_s^t(\mu)P_l^\kappa(\mu) = \sum_\alpha \sqrt{\frac{(s+t)!(l+\kappa)!(\alpha-\kappa-t)!}{(s-t)!(l-\kappa)!(\alpha+\kappa+t)!}}\, C(s,l,\alpha;t,\kappa,\kappa+t)C(s,l,\alpha;0,0,0)P_\alpha^{\kappa+t}(\mu) , \tag{3.33}$$

$$Y_s^t(\vec{\Omega})Y_l^\kappa(\vec{\Omega}) = \sum_\alpha \sqrt{\frac{(s+t)!(l+\kappa)!(\alpha-\kappa-t)!}{(s-t)!(l-\kappa)!(\alpha+\kappa+t)!}}\, C(s,l,\alpha;t,\kappa,\kappa+t)\, C(s,l,\alpha;0,0,0)Y_\alpha^{\kappa+t}(\vec{\Omega}) , \tag{3.34}$$

$$P_s(\mu)P_l(\mu) = \sum_\alpha C(s,l,\alpha;0,0,0)^2\, P_\alpha(\mu) . \tag{3.35}$$

4. Integral formulae of products of spherical functions

At first the following formulae according to /9/, p. 95, 96 are given :

$$\int_0^\pi \sin^p x \cos^q x \, dx = \left[1+(-1)^q\right] \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} , \quad p,q \ge 0 , \tag{4.1}$$

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} , \quad p,q \ge 0 , \tag{4.2}$$

$$\int_{\pi/2}^{\pi} \sin^p x \cos^q x \, dx = (-1)^q \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} , \quad p,q \ge 0 . \tag{4.3}$$

These formulae are needed only for integer values of p, q; hence for the Gamma-function the following is true (/8/, p. 255) :

$$\Gamma(n) = (n-1)! , \quad n = 1,2,3,\dots , \tag{4.4}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \tag{4.5}$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} , \quad n = 1,2,3,\dots , \tag{4.6}$$

where $(2n-1)!! = 1\cdot 3\cdot 5 \cdots (2n-1)$ for $n = 1,2,3,\dots$.

Now we have got a sufficient apparatus to be able to perform the integral formulae of the products of any combination of spherical functions

$$P_n^m(\mu),\ P_n(\mu),\ Y_n^m(\vec{\Omega}),\ Y_n^{m*}(\vec{\Omega}),\ P_n^m(\vec{\Omega}),\ P_n^{m*}(\vec{\Omega}) .$$

i) Orthogonality relations for spherical functions :

Using (2.11), (3.1), (3.16), the general symmetry rule (p.10), (3.30) respectively, and $n=|l-s|,\dots,l+s;\ s=0$ **(hence** $l=n$ **), the following is true:**

$$\int_{-1}^{1} d\mu\, P_l(\mu)P_n(\mu) = \int_{-1}^{1} d\mu\, P_0^{00}(\mu)P_l^{00}(\mu)\overline{P_n^{00}}(\mu) = \frac{2}{2n+1}C(0,n,n;0,0,0)^2\delta_{nl} =$$

$$= 2\begin{pmatrix}0&n&n\\0&0&0\end{pmatrix}^2\delta_{nl} = 2\begin{pmatrix}n&n&0\\0&0&0\end{pmatrix}^2\delta_{nl} = \frac{2}{2n+1}\delta_{nl}\,,$$

$$\int_{-1}^{1} d\mu\, P_l^k(\mu)P_n^k(\mu) = (-1)^k i^{2k}\sqrt{\frac{(l+k)!\,(n+k)!}{(l-k)!\,(n-k)!}}\int_{-1}^{1} d\mu\, P_0^{00}(\mu)P_l^{k0}(\mu)\overline{P_n^{k0}}(\mu) =$$

$$= \frac{(n+k)!}{(n-k)!}\frac{2}{2n+1}C(0,n,n;0,k,k)\,C(0,n,n;0,0,0)\,\delta_{nl} = 2(-1)^k\frac{(n+k)!}{(n-k)!}\begin{pmatrix}0&n&n\\0&k&\end{pmatrix}\begin{pmatrix}0&n&n\\0&0&0\end{pmatrix}\delta_{nl} =$$

$$= 2(-1)^k\frac{(n+k)!}{(n-k)!}\begin{pmatrix}n&n&0\\k&-k&0\end{pmatrix}\begin{pmatrix}n&n&0\\0&0&0\end{pmatrix}\delta_{nl} = \frac{(n+k)!}{(n-k)!}\frac{2}{2n+1}\delta_{nl}\,.$$

Further :

$$\int_{(4\pi)} d\vec{\Omega}\, Y_l^k(\vec{\Omega})Y_n^{m*}(\vec{\Omega}) = \int_0^{2\pi} e^{i(k-m)\psi}d\psi\int_0^{\pi}\sin\vartheta\, d\vartheta\, P_l^k(\cos\vartheta)P_l^m(\cos\vartheta) =$$

$$= 2\pi\,\delta_{mk}\int_{-1}^{1} d\mu\, P_l^k(\mu)P_n^k(\mu) = \frac{(n+k)!}{(n-k)!}\frac{4\pi}{2n+1}\delta_{nl}\delta_{mk}\,,$$

$$\int_{(4\pi)} d\vec{\Omega}\, P_l^k(\vec{\Omega})P_n^{m*}(\vec{\Omega}) = (-1)^{m+k}\sqrt{\frac{(l-k)!\,(n-m)!}{(l+k)!\,(n+m)!}}\int_{(4\pi)} d\vec{\Omega}\, Y_l^k(\vec{\Omega})Y_n^{m*}(\vec{\Omega}) = \frac{4\pi}{2n+1}\delta_{nl}\delta_{mk}\,.$$

Summary :

$$\int_{-1}^{1} d\mu\, P_l(\mu)P_n(\mu) = \frac{2}{2n+1}\delta_{nl}\,, \qquad (4.7)$$

$$\int_{-1}^{1} d\mu\, P_l^k(\mu)P_n^k(\mu) = \frac{(n+k)!}{(n-k)!}\frac{2}{2n+1}\delta_{nl}\,, \qquad (4.8)$$

$$\int_{(4\pi)} d\vec{\Omega}\, Y_l^k(\vec{\Omega})Y_n^{m*}(\vec{\Omega}) = \frac{(n+k)!}{(n-k)!}\frac{4\pi}{2n+1}\delta_{nl}\delta_{mk}\,, \qquad (4.9)$$

$$\int_{(4\pi)} d\vec{\Omega}\, P_l^k(\vec{\Omega})P_n^{m*}(\vec{\Omega}) = \frac{4\pi}{2n+1}\delta_{nl}\delta_{mk}\,. \qquad (4.10)$$

ii) Integrals of products of three spherical functions :

These formulae may be obtained by using the formula (3.1) as in i) or by using the Clebsh-Gordan series (3.33), (3.34), (3.35) together with the orthogonality relations. For $|t|\le s, |k|\le l, |m|=|k+t|\le n$, $\alpha = max(|s-l|, |k+t|), \dots, s+l$

$$\int_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega}) Y_l^k(\vec{\Omega}) Y_n^{m*}(\vec{\Omega}) = 2\pi \delta_{m,k+t} \int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^m(\mu) =$$

$$= \sum_\alpha \sqrt{\frac{(s+t)!(l+k)!(\alpha-k-t)!}{(s-t)!(l-k)!(\alpha+k+t)!}}\, C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0) \int_{(4\pi)} d\vec{\Omega}\, Y_\alpha^{k+t}(\vec{\Omega}) Y_n^{m*}(\vec{\Omega}) =$$

$$= \frac{4\pi}{2n+1} \sqrt{\frac{(s+t)!(l+k)!(n+k+t)!}{(s-t)!(l-k)!(n-k-t)!}}\, C(s,l,n;t,k,k+t)\, C(s,l,n;0,0,0)\, \delta_{m,k+t} .$$

According to (2.6), (2.7) may be written

$$\int_{(4\pi)} d\vec{\Omega}\, P_s^t(\vec{\Omega}) P_l^k(\vec{\Omega}) P_n^{m*}(\vec{\Omega}) = \frac{4\pi}{2n+1} C(s,l,n;t,k,k+t) C(s,l,n;0,0,0)\, \delta_{m,k+t} .$$

Further

$$\int_{-1}^{1} d\mu\, P_s(\mu) P_l(\mu) P_n(\mu) = \sum_{\alpha=|s-l|}^{s+l} C(s,l,\alpha;0,0,0)^2 \int_{-1}^{1} d\mu\, P_\alpha(\mu) P_n(\mu) =$$

$$= \frac{2}{2n+1}\, C(s,l,n;0,0,0)^2 .$$

Finally for $|t|\le s, |k|\le l, |m|\le n, \alpha = max(|s-l|, |k+t|), \dots, l+s$

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^m(\mu) =$$

$$= \sum_\alpha \sqrt{\frac{(s+t)!(l+k)!(\alpha-k-t)!}{(s-t)!(l-k)!(\alpha+k+t)!}}\, C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0) \int_{-1}^{1} d\mu\, P_\alpha^{k+t}(\mu) P_n^m(\mu) ;$$

this integral formula is relatively simple only when $m = k+t$; using the orthogonality relation (4.8) we get

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^{k+t}(\mu) = \frac{2}{2n+1} \sqrt{\frac{(s+t)!(l+k)!(n+k+t)!}{(s-t)!(l-k)!(n-k-t)!}}\, C(s,l,n;t,k,k+t)\, C(s,l,n;0,0,0) ;$$

for $m \neq k+t$ the formula (3.33) is to be used twice, whence for $\beta = max(|n-\alpha|, |m+k+t|), \dots, n+\alpha$

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^m(\mu) =$$

$$= \sum_\alpha \sum_\beta \sqrt{\frac{(s+t)!\,(l+k)!\,(n+m)!\,(\beta-m-k-t)!}{(s-t)!\,(l-k)!\,(n-m)!\,(\beta+m+k+t)!}} \times$$

$$\times\, C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0)\, C(\alpha,n,\beta;k+t,m,m+k+t)\, C(\alpha,n,\beta;0,0,0) \int_{-1}^{1} d\mu\, P_\beta^{m+k+t}(\mu) ;$$

the last integral on the right is to be expressed through the integral formula (4.1) :

According to (2.2) the function $P_\beta^{m+k+t}(\mu) = P_\beta^{m+k+t}(\cos\vartheta)$ for integer $m+k+t \geq 0$ may be expressed as

$$P_\beta^{m+k+t}(\cos\vartheta) = \sum_{\nu=0}^{\left[\frac{\beta-m-k-t}{2}\right]} \frac{(-1)^\nu (2\beta-2\nu)!}{2^\beta \nu!(\beta-\nu)!(\beta-m-k-t-2\nu)!} \sin^{m+k+t}\vartheta \cos^{\beta-m-k-t-2\nu}\vartheta \,;$$

as $\beta-m-k-t-2\nu$ is always a non-negative integer value, the formula (4.1) may be used; thence

$$\int_{-1}^{1} d\mu\, P_\beta^{m+k+t}(\mu) = \int_0^\pi \sin\vartheta\, d\vartheta\, P_\beta^{m+k+t}(\cos\vartheta) =$$

$$= \sum_{\nu=0}^{\left[\frac{\beta-m-k-t}{2}\right]} \frac{(-1)^\nu (2\beta-2\nu)!}{2^\beta \nu!(\beta-\nu)!(\beta-m-k-t-2\nu)!} \int_0^\pi \sin^{m+k+t+1}\vartheta \cos^{\beta-m-k-t-2\nu}\vartheta\, d\vartheta =$$

$$= \left[1+(-1)^{\beta-m-k-t}\right] \sum_{\nu=0}^{\left[\frac{\beta-m-k-t}{2}\right]} \frac{(-1)^\nu (2\beta-2\nu)!}{2^\beta \nu!(\beta-\nu)!(\beta-m-k-t-2\nu)!} \frac{\Gamma\left(\frac{m+k+t+2}{2}\right)\Gamma\left(\frac{\beta-m-k-t-2\nu+1}{2}\right)}{2\,\Gamma\left(\frac{\beta-2\nu+3}{2}\right)} \,;$$

hence for $m+k+t$, $\alpha = \max(|s-l|,|k+t|),\dots,s+l$, $\beta = \max(|n-\alpha|,|m+k+t|),\dots,n+\alpha$ may be written

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^m(\mu) =$$

$$= \sum_\alpha \sum_\beta \sum_{\nu=0}^{\left[\frac{\beta-m-k-t}{2}\right]} \left[1+(-1)^{\beta-m-k-t}\right] \sqrt{\frac{(s+t)!(l+k)!(n+m)!(\beta-m-k-t)!}{(s-t)!(l-k)!(n-m)!(\beta+m+k+t)!}} \times$$

$$\times \frac{(-1)^\nu (2\beta-2\nu)!}{2^\beta \nu!(\beta-\nu)!(\beta-m-k-t-2\nu)!} C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0)\, C(\alpha,n,\beta;k+t,m,m+k+t)\, C(\alpha,n,\beta;0,0,0) \times$$

$$\times \frac{\Gamma\left(\frac{m+k+t+2}{2}\right)\Gamma\left(\frac{\beta-m-k-t-2\nu+1}{2}\right)}{2\,\Gamma\left(\frac{\beta-2\nu+3}{2}\right)} \,;$$

we can also write

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^m(\mu) = \sum_{\sigma=0}^{\left[\frac{s-t}{2}\right]} \sum_{\lambda=0}^{\left[\frac{l-k}{2}\right]} \sum_{\nu=0}^{\left[\frac{n-m}{2}\right]} \left[1+(-1)^{n+l+s-m-k-t}\right] \times$$

$$\times \frac{(-1)^{\nu+\lambda+\sigma} (2n-2\nu)!(2l-2\lambda)!(2s-2\sigma)!}{2^{n+l+s} \nu!\lambda!\sigma!(n-\nu)!(l-\lambda)!(s-\sigma)!(n-m-2\nu)!(l-k-2\lambda)!(s-t-2\sigma)!} \times$$

$$\times \frac{\Gamma\left(\frac{m+k+t+2}{2}\right)\Gamma\left(\frac{n+l+s-m-k-t-2\nu-2\lambda-2\sigma+1}{2}\right)}{2\,\Gamma\left(\frac{n+l+s-2\nu-2\lambda-2\sigma+3}{2}\right)}$$

using only the formulae (2.2), (4.1).

For $n=0, m=0$ we get for $k \neq t$.

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) =$$

$$= \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} \sum_{\nu=0}^{\left[\frac{\alpha-k-t}{2}\right]} \left[1+(-1)^{\alpha-k-t}\right] \sqrt{\frac{(s+t)!\,(l+k)!\,(\alpha-k-t)!}{(s-t)!\,(l-k)!\,(\alpha+k+t)!}} \times$$

$$\times \frac{(-1)^\nu (2\alpha-2\nu)!}{2^\alpha \nu!\,(\alpha-\nu)!\,(\alpha-k-t-2\nu)!} C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0)\, \frac{\Gamma\left(\frac{k+t+2}{2}\right)\Gamma\left(\frac{\alpha-k-t-2\nu+1}{2}\right)}{2\,\Gamma\left(\frac{\alpha-2\nu+3}{2}\right)},$$

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) =$$

$$= \sum_{\sigma=0}^{\left[\frac{s-t}{2}\right]} \sum_{\lambda=0}^{\left[\frac{l-k}{2}\right]} \left[1+(-1)^{l+s-k-t}\right] \frac{(-1)^{\lambda+\sigma} (2l-2\lambda)!\,(2s-2\sigma)!}{2^{l+s} \lambda!\, \sigma!\,(l-\lambda)!\,(s-\sigma)!\,(l-k-2\lambda)!\,(s-t-2\sigma)!} \times$$

$$\times \frac{\Gamma\left(\frac{k+t+2}{2}\right)\Gamma\left(\frac{l+s-k-t-2\lambda-2\sigma+1}{2}\right)}{2\,\Gamma\left(\frac{l+s-2\lambda-2\sigma+3}{2}\right)}.$$

Using $Y_n^m(\vec{\Omega}) = (-1)^m Y_n^{-m*}(\vec{\Omega})$, i.e. $\int_{(4\pi)} d\vec{\Omega}\, Y_l^k(\vec{\Omega}) Y_n^m(\vec{\Omega}) = (-1)^{-k} \frac{(n+k)!}{(n-k)!} \frac{4\pi}{2n+1} \delta_{nl} \delta_{m,-k}$,

we can write

$$\int_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega}) Y_l^k(\vec{\Omega}) Y_n^m(\vec{\Omega}) =$$

$$= \sum_\alpha \sqrt{\frac{(s+t)!\,(l+k)!\,(\alpha-k-t)!}{(s-t)!\,(l-k)!\,(\alpha+k+t)!}}\, C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0) \int_{(4\pi)} d\vec{\Omega}\, Y_\alpha^{k+t}(\vec{\Omega}) Y_n^m(\vec{\Omega}) =$$

$$= (-1)^{k+t} \frac{4\pi}{2n+1} \sqrt{\frac{(s+t)!\,(l+k)!\,(n+k+t)!}{(s-t)!\,(l-k)!\,(n-k-t)!}}\, C(s,l,n;t,k,k+t)\, C(s,l,n;0,0,0)\, \delta_{m,-k-t} =$$

$$= 4\pi \sqrt{\frac{(s+t)!\,(l+k)!\,(n+k+t)!}{(s-t)!\,(l-k)!\,(n-k-t)!}} \begin{pmatrix} s & l & n \\ t & k & -k-t \end{pmatrix} \begin{pmatrix} s & l & n \\ 0 & 0 & 0 \end{pmatrix} \delta_{m,-k-t}.$$

Summary :

$$\int_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega}) Y_l^k(\vec{\Omega}) Y_n^{m*}(\vec{\Omega}) =$$

$$= \frac{4\pi}{2n+1}\sqrt{\frac{(s+t)!(l+k)!(n+k+t)!}{(s-t)!(l-k)!(n-k-t)!}}\, C(s,l,n;t,k,k+t)\, C(s,l,n;0,0,0)\,\delta_{m,k+t} =$$

$$= (-1)^{k+t}\, 4\pi \sqrt{\frac{(s+t)!(l+k)!(n+k+t)!}{(s-t)!(l-k)!(n-k-t)!}} \begin{pmatrix} s & l & n \\ t & k & -k-t \end{pmatrix}\begin{pmatrix} s & l & n \\ 0 & 0 & 0 \end{pmatrix} \delta_{m,k+t} \, , \qquad (4.11)$$

$$\int_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega}) Y_l^k(\vec{\Omega}) Y_n^{m}(\vec{\Omega}) =$$

$$= (-1)^{k+t}\frac{4\pi}{2n+1}\sqrt{\frac{(s+t)!(l+k)!(n+k+t)!}{(s-t)!(l-k)!(n-k-t)!}}\, C(s,l,n;t,k,k+t)\, C(s,l,n;0,0,0)\,\delta_{m,-k-t} =$$

$$= 4\pi \sqrt{\frac{(s+t)!(l+k)!(n+k+t)!}{(s-t)!(l-k)!(n-k-t)!}} \begin{pmatrix} s & l & n \\ t & k & -k-t \end{pmatrix}\begin{pmatrix} s & l & n \\ 0 & 0 & 0 \end{pmatrix} \delta_{m,-k-t} \, , \qquad (4.12)$$

$$\int_{(4\pi)} d\vec{\Omega}\, P_s^t(\vec{\Omega}) P_l^k(\vec{\Omega}) P_n^{m*}(\vec{\Omega}) =$$

$$= \frac{4\pi}{2n+1} C(s,l,n;t,k,k+t)\, C(s,l,n;0,0,0)\,\delta_{m,k+t} =$$

$$= (-1)^{k+t}\, 4\pi \begin{pmatrix} s & l & n \\ t & k & -k-t \end{pmatrix}\begin{pmatrix} s & l & n \\ 0 & 0 & 0 \end{pmatrix} \delta_{m,k+t} \, , \qquad (4.13)$$

$$\int_{(4\pi)} d\vec{\Omega}\, P_s^t(\vec{\Omega}) P_l^k(\vec{\Omega}) P_n^{m}(\vec{\Omega}) =$$

$$= (-1)^{k+t}\frac{4\pi}{2n+1} C(s,l,n;t,k,k+t)\, C(s,l,n;0,0,0)\,\delta_{m,-k-t} =$$

$$= 4\pi \begin{pmatrix} s & l & n \\ t & k & -k-t \end{pmatrix}\begin{pmatrix} s & l & n \\ 0 & 0 & 0 \end{pmatrix} \delta_{m,-k-t} \, , \qquad (4.14)$$

$$\int_{-1}^{1} d\mu\, P_s(\mu) P_l(\mu) P_n(\mu) =$$

$$= \frac{2}{2n+1} C(s,l,n;0,0,0)^2 = 2\begin{pmatrix} s & l & n \\ 0 & 0 & 0 \end{pmatrix}^2 , \qquad (4.15)$$

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^{k+t}(\mu) =$$

$$= \frac{2}{2n+1}\sqrt{\frac{(s+t)!(l+k)!(n+k+t)!}{(s-t)!(l-k)!(n-k-t)!}}\, C(s,l,n;t,k,k+t)\, C(s,l,n;0,0,0) =$$

$$= (-1)^{k+t}\, 2\sqrt{\frac{(s+t)!(l+k)!(n+k+t)!}{(s-t)!(l-k)!(n-k-t)!}} \begin{pmatrix} s & l & n \\ t & k & -k-t \end{pmatrix}\begin{pmatrix} s & l & n \\ 0 & 0 & 0 \end{pmatrix}, \qquad (4.16)$$

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_n^m(\mu) =$$

$$= \sum_{\alpha}\sum_{\beta}\sum_{\nu=0}^{\left[\frac{\beta-m-k-t}{2}\right]} \left[1+(-1)^{\beta-m-k-t}\right] \sqrt{\frac{(s+t)!(l+k)!(n+m)!(\beta-m-k-t)!}{(s-t)!(l-k)!(n-m)!(\beta+m+k+t)!}} \times$$

$$\times \frac{(-1)^{\nu}(2\beta-2\nu)!}{2^{\beta}\nu!(\beta-\nu)!(\beta-m-k-t-2\nu)!} C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0)\, C(\alpha,n,\beta;k+t,m,m+k+t)\, C(\alpha,n,\beta;0,0,0) \times$$

$$\times \frac{\Gamma\left(\frac{m+k+t+2}{2}\right)\Gamma\left(\frac{\beta-m-k-t-2\nu+1}{2}\right)}{2\,\Gamma\left(\frac{\beta-2\nu+3}{2}\right)} =$$

$$= \sum_{\alpha}\sum_{\beta}\sum_{\nu=0}^{\left[\frac{\beta-m-k-t}{2}\right]} \left[1+(-1)^{\beta-m-k-t}\right] \sqrt{\frac{(s+t)!(l+k)!(n+m)!(\beta-m-k-t)!}{(s-t)!(l-k)!(n-m)!(\beta+m+k+t)!}} \times$$

$$\times \frac{(-1)^{m+\nu}(2\alpha+1)(2\beta+1)(2\beta-2\nu)!}{2^{\beta}\nu!(\beta-\nu)!(\beta-m-k-t-2\nu)!} \begin{pmatrix} s & l & \alpha \\ t & k & -k-t \end{pmatrix}\begin{pmatrix} s & l & \alpha \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha & n & \beta \\ k+t & m & -m-k-t \end{pmatrix}\begin{pmatrix} \alpha & n & \beta \\ 0 & 0 & 0 \end{pmatrix} \times$$

$$\times \frac{\Gamma\left(\frac{m+k+t+2}{2}\right)\Gamma\left(\frac{\beta-m-k-t-2\nu+1}{2}\right)}{2\,\Gamma\left(\frac{\beta-2\nu+3}{2}\right)} =$$

$$= \sum_{\sigma=0}^{\left[\frac{s-t}{2}\right]}\sum_{\lambda=0}^{\left[\frac{l-k}{2}\right]}\sum_{\nu=0}^{\left[\frac{n-m}{2}\right]} \left[1+(-1)^{n+l+s-m-k-t}\right] \times$$

$$\times \frac{(-1)^{\nu+\lambda+\sigma}(2n-2\nu)!(2l-2\lambda)!(2s-2\sigma)!}{2^{n+l+s}\nu!\lambda!\sigma!(n-\nu)!(l-\lambda)!(s-\sigma)!(n-m-2\nu)!(l-k-2\lambda)!(s-t-2\sigma)!} \times$$

$$\times \frac{\Gamma\left(\frac{m+k+t+2}{2}\right)\Gamma\left(\frac{n+l+s-m-k-t-2\nu-2\lambda-2\sigma+1}{2}\right)}{2\,\Gamma\left(\frac{n+l+s-2\nu-2\lambda-2\sigma+3}{2}\right)}, \qquad (4.17)$$

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) =$$

$$= \sum_{\alpha} \sum_{\nu=0}^{\left[\frac{\alpha-k-t}{2}\right]} \left[1+(-1)^{\alpha-k-t}\right] \sqrt{\frac{(s+t)!\,(l+k)!\,(\alpha-k-t)!}{(s-t)!\,(l-k)!\,(\alpha+k+t)!}} \times$$

$$\times \frac{(-1)^{\nu}(2\alpha-2\nu)!}{2^{\alpha}\nu!(\alpha-\nu)!(\alpha-k-t-2\nu)!} C(s,l,\alpha;t,k,k+t)\,C(s,l,\alpha;0,0,0)\, \frac{\Gamma\left(\frac{k+t+2}{2}\right)\Gamma\left(\frac{\alpha-k-t-2\nu+1}{2}\right)}{2\Gamma\left(\frac{\alpha-2\nu+3}{2}\right)} =$$

$$= \sum_{\alpha} \sum_{\nu=0}^{\left[\frac{\alpha-k-t}{2}\right]} \left[1+(-1)^{\alpha-k-t}\right] \sqrt{\frac{(s+t)!\,(l+k)!\,(\alpha-k-t)!}{(s-t)!\,(l-k)!\,(\alpha+k+t)!}} \times$$

$$\times \frac{(-1)^{k+t+\nu}(2\alpha+1)(2\alpha-2\nu)!}{2^{\alpha}\nu!(\alpha-\nu)!(\alpha-k-t-2\nu)!} \begin{pmatrix} s & l & \alpha \\ t & k & -k-t \end{pmatrix} \begin{pmatrix} s & l & \alpha \\ 0 & 0 & 0 \end{pmatrix} \frac{\Gamma\left(\frac{k+t+2}{2}\right)\Gamma\left(\frac{\alpha-k-t-2\nu+1}{2}\right)}{2\Gamma\left(\frac{\alpha-2\nu+3}{2}\right)} =$$

$$= \sum_{\sigma=0}^{\left[\frac{s-t}{2}\right]} \sum_{\lambda=0}^{\left[\frac{l-k}{2}\right]} \left[1+(-1)^{l+s-k-t}\right] \frac{(-1)^{\lambda+\sigma}(2l-2\lambda)!\,(2s-2\sigma)!}{2^{l+s}\lambda!\,\sigma!\,(l-\lambda)!\,(s-\sigma)!\,(l-k-2\lambda)!\,(s-t-2\sigma)!} \times$$

$$\times \frac{\Gamma\left(\frac{k+t+2}{2}\right)\Gamma\left(\frac{l+s-k-t-2\lambda-2\sigma+1}{2}\right)}{2\Gamma\left(\frac{l+s-2\lambda-2\sigma+3}{2}\right)} . \qquad (4.18)$$

Here $\alpha = \max(|s-l|,|k+t|),\dots,s+l$, $\beta = \max(|u-\alpha|,|m+k+t|),\dots,u+\alpha$ and

$\Gamma(x) = (x-1)!$ for $x = 1,2,3,\dots$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(x+\frac{1}{2}) = \frac{1\cdot 3\cdot 5\cdots(2x-1)}{2^x}\sqrt{\pi}$ for $x = 1,2,3,\dots$

iii) General formulae :

Using the Clebsh-Gordan series and the integral formulae given above we are able to perform the integrals of the product of any combination of spherical functions. We give for instance the following formulae :

$$\int_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega}) Y_l^k(\vec{\Omega}) Y_u^m(\vec{\Omega}) Y_p^q(\vec{\Omega}) =$$

$$= \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} \sqrt{\frac{(s+t)!(l+k)!(\alpha-k-t)!}{(s-t)!(l-k)!(\alpha+k+t)!}}\, C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0) \int_{(4\pi)} d\vec{\Omega}\, Y_\alpha^{k+t} Y_u^m Y_p^q =$$

$$= \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} \quad \sum_{\beta=\max(|u-\alpha|,|m+k+t|)}^{u+\alpha} \sqrt{\frac{(s+t)!(l+k)!(u+m)!(\beta-m-k-t)!}{(s-t)!(l-k)!(u-m)!(\beta+m+k+t)!}} \times$$

$$\times C(s,l,\alpha;t,k,k+t)\,C(s,l,\alpha;0,0,0)\,C(\alpha,u,\beta;k+t,m,m+k+t)\,C(\alpha,u,\beta;0,0,0)\int_{(4\pi)} d\vec{\Omega}\, Y_\beta^{m+k+t}\, Y_p^q =$$

$$= (-1)^{m+k+t}\,\frac{4\pi}{2p+1} \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} \sqrt{\frac{(s+t)!(l+k)!(u+m)!(p+m+k+t)!}{(s-t)!(l-k)!(u-m)!(p-m-k-t)!}} \times$$

$$\times C(s,l,\alpha;t,k,k+t)\,C(s,l,\alpha;0,0,0)\,C(\alpha,u,p;k+t,m,m+k+t)\,C(\alpha,u,p;0,0,0)\,\delta_{q,-m-k-t} \,.$$

We could also use the formula (4.12) to get the same result. Using successively the Clebsh-Gordan series for various combinations of two functions $Y_j^i\, Y_{j'}^{i'}$ we get formally different formulae, which, of course, are identical. Below we give a list of some integrals :

$$\int_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega})\, Y_l^k(\vec{\Omega})\, Y_u^m(\vec{\Omega})\, Y_p^q(\vec{\Omega}) =$$

$$= (-1)^{m+k+t}\,\frac{4\pi}{2p+1} \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} \sqrt{\frac{(s+t)!(l+k)!(u+m)!(p+m+k+t)!}{(s-t)!(l-k)!(u-m)!(p-m-k-t)!}} \times$$

$$\times C(s,l,\alpha;t,k,k+t)\,C(s,l,\alpha;0,0,0)\,C(\alpha,u,p;k+t,m,m+k+t)\,C(\alpha,u,p;0,0,0)\,\delta_{q,-m-k-t} =$$

$$= (-1)^{k+t}\,4\pi \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} (2\alpha+1) \sqrt{\frac{(s+t)!(l+k)!(u+m)!(p+m+k+t)!}{(s-t)!(l-k)!(u-m)!(p-m-k-t)!}} \times$$

$$\times \begin{pmatrix} s & l & \alpha \\ t & k & -k-t \end{pmatrix} \begin{pmatrix} s & l & \alpha \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & u & p \\ k+t & m & -m-k-t \end{pmatrix} \begin{pmatrix} \alpha & u & p \\ 0 & 0 & 0 \end{pmatrix} \delta_{q,-m-k-t} \,, \qquad (4.19)$$

$$\int_{-1}^{1} d\mu\, P_s(\mu)\, P_l(\mu)\, P_u(\mu)\, P_p(\mu) =$$

$$= \frac{2}{2p+1} \sum_{\alpha=|s-l|}^{s+l} C(s,l,\alpha;0,0,0)^2\, C(\alpha,u,p;0,0,0)^2 =$$

$$= 2 \sum_{\alpha=|s-l|}^{s+l} (2\alpha+1) \begin{pmatrix} s & l & \alpha \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} \alpha & u & p \\ 0 & 0 & 0 \end{pmatrix}^2 \,, \qquad (4.20)$$

$$\int_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega}) Y_l^k(\vec{\Omega}) Y_u^m(\vec{\Omega}) Y_p^{q*}(\vec{\Omega}) =$$

$$= \frac{4\pi}{2p+1} \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} \sqrt{\frac{(s+t)!(l+k)!(u+m)!(p+m+k+t)!}{(s-t)!(l-k)!(u-m)!(p-m-k-t)!}}\, \times$$

$$\times\, C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0)\, C(\alpha,u,p;k+t,m,m+k+t)\, C(\alpha,u,p;0,0,0)\, \delta_{q,m+k+t} =$$

$$= (-1)^m 4\pi \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} (2\alpha+1) \sqrt{\frac{(s+t)!(l+k)!(u+m)!(p+m+k+t)!}{(s-t)!(l-k)!(u-m)!(p-m-k-t)!}}\, \times$$

$$\times \begin{pmatrix} s & l & \alpha \\ t & k & -k-t \end{pmatrix} \begin{pmatrix} s & l & \alpha \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & u & p \\ k+t & m & -m-k-t \end{pmatrix} \begin{pmatrix} \alpha & u & p \\ 0 & 0 & 0 \end{pmatrix} \delta_{q,m+k+t} \,, \qquad (4.21)$$

$$\int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_u^m(\mu) P_p^{m+k+t}(\mu) =$$

$$= \frac{2}{2p+1} \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} \sqrt{\frac{(s+t)!(l+k)!(u+m)!(p+m+k+t)!}{(s-t)!(l-k)!(u-m)!(p-m-k-t)!}}\, \times$$

$$\times\, C(s,l,\alpha;t,k,k+t)\, C(s,l,\alpha;0,0,0)\, C(\alpha,u,p;k+t,m,m+k+t)\, C(\alpha,u,p;0,0,0)\, \delta_{q,m+k+t} =$$

$$= (-1)^m 2 \sum_{\alpha=\max(|s-l|,|k+t|)}^{s+l} (2\alpha+1) \sqrt{\frac{(s+t)!(l+k)!(u+m)!(p+m+k+t)!}{(s-t)!(l-k)!(u-m)!(p-m-k-t)!}}\, \times$$

$$\times \begin{pmatrix} s & l & \alpha \\ t & k & -k-t \end{pmatrix} \begin{pmatrix} s & l & \alpha \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & u & p \\ k+t & m & -m-k-t \end{pmatrix} \begin{pmatrix} \alpha & u & p \\ 0 & 0 & 0 \end{pmatrix} \delta_{q,m+k+t} \,. \qquad (4.22)$$

The (4.20) may be also obtained directly from (4.19) or (4.21) for $t = 0$, $k = 0$, $m = 0$, $q = 0$; the (4.22) may be obtained directly from (4.21), as for $|t| \le s$, $|k| \le l$, $|m| \le u$, $|q| \le p$

$$\int_{(4\pi)} d\vec{\Omega}\, Y_s^t(\vec{\Omega}) Y_l^k(\vec{\Omega}) Y_u^m(\vec{\Omega}) Y_p^{q*}(\vec{\Omega}) =$$

$$= \int_0^{2\pi} e^{i(m+k+t-q)\psi} d\psi \int_0^{\pi} \sin\vartheta\, d\vartheta\, P_s^t(\cos\vartheta) P_l^k(\cos\vartheta) P_u^m(\cos\vartheta) P_s^t(\cos\vartheta) =$$

$$= 2\pi\, \delta_{q,m+k+t} \int_{-1}^{1} d\mu\, P_s^t(\mu) P_l^k(\mu) P_u^m(\mu) P_p^{m+k+t}(\mu) \,.$$

iiii) Some special integrals of Legendre polynomials :

Here the integral formulae of the product of two Legendre polynomials in the intervals $\langle -1, 0\rangle$, $\langle 0, 1\rangle$ are given. We use the Legendre equation

$$[(1-\mu^2)u']' + n(n+1)u = 0 \; , \quad n = 0, 1, 2, \dots \qquad (4.23)$$

Substituting $P_l(\mu)$ into (4.23), multiplying by $P_n(\mu)$ and substituting $P_n(\mu)$ into (4.23) and multiplying by $P_l(\mu)$ we get two equations, the difference of which is

$$\left\{(1-\mu^2)\left[P_l'(\mu)P_n(\mu) - P_l(\mu)P_n'(\mu)\right]\right\}' + (l-n)(l+n+1)P_l(\mu)P_n(\mu) = 0 \; .$$

Integrating this equation in the interval $\langle 0, 1\rangle$ we get for $l \neq n$

$$\int_0^1 d\mu\, P_l(\mu)P_n(\mu) = \frac{1}{(l-n)(l+n+1)}\left[P_l'(0)P_n(0) - P_l(0)P_n'(0)\right] \; .$$

Using the following values according to /42/, chapter IV (Orthogonal polynomials), § 4.6

$$P_{2m}(0) = (-1)^m \frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(m+1)} \; , \; P_{2m+1}(0) = 0 \; , \; P_{2m}'(0) = 0 \; , \; P_{2m+1}'(0) = (-1)^m \frac{2\Gamma\left(m+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(m+1)} \; ,$$

the integral is evidently equal to zero for $l \neq n$ and even values of l, n or odd values of l, n ; for even l and odd n we get

$$\int_0^1 d\mu\, P_l(\mu)P_n(\mu) = \frac{(-1)^{\frac{1}{2}(l+n-1)}\, 2\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+1\right)}{\pi(l-n)(l+n+1)\Gamma\left(\frac{l}{2}+1\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} =$$

$$= \frac{(-1)^{\frac{1}{2}(l+n-1)}\, l!\, n!}{2^{l+n-1}(l-n)(l+n+1)\left[\left(\frac{l}{2}\right)!\left(\frac{n-1}{2}\right)!\right]^2} \; ;$$

for odd l and even n the l, n must be interchanged and the exponent of -1 changes into $\frac{1}{2}(l+n+1)$; we get of course nothing new. Using the formula (4.7) we can write for $l \neq n$ and even l , odd n

$$\int_{-1}^0 d\mu\, P_l(\mu)P_n(\mu) = \int_{-1}^1 d\mu\, P_l(\mu)P_n(\mu) - \int_0^1 d\mu\, P_l(\mu)P_n(\mu) = -\int_0^1 d\mu\, P_l(\mu)P_n(\mu) =$$

$$= \frac{(-1)^{\frac{1}{2}(l+n+1)}\, l!\, n!}{2^{l+n-1}(l-n)(l+n+1)\left[\left(\frac{l}{2}\right)!\left(\frac{n-1}{2}\right)!\right]^2} \; .$$ For $n = l$ may be written

$$\int_{-1}^1 \left[P_l(\mu)\right]^2 d\mu = 2\int_0^1 \left[P_l(\mu)\right]^2 d\mu = 2\int_{-1}^0 \left[P_l(\mu)\right]^2 d\mu = \frac{2}{2l+1} \; ,$$ whence

$$\int_{-1}^0 d\mu\, P_l(\mu)P_l(\mu) = \int_0^1 d\mu\, P_l(\mu)P_l(\mu) = \frac{1}{2l+1} \; .$$

Summary :

$$\int_0^1 d\mu\, P_l(\mu) P_n(\mu) = \begin{cases} \dfrac{1}{2l+1} & l = n \\ 0 & l \neq n,\ l+n \ \text{even} \\ \dfrac{(-1)^{\frac{1}{2}(l+n-1)}\, l!\, n!}{2^{l+n-1}(l-n)(l+n+1)\left[\left(\frac{l}{2}\right)!\left(\frac{n-1}{2}\right)!\right]^2} & l \neq n,\ l \ \text{even},\ n \ \text{odd} \end{cases} \tag{4.24}$$

$$\int_{-1}^{1} d\mu\, P_l(\mu) P_n(\mu) = \begin{cases} \dfrac{1}{2l+1} & l = n \\ 0 & l \neq n,\ l+n \ \text{even} \\ \dfrac{(-1)^{\frac{1}{2}(l+n+1)}\, l!\, n!}{2^{l+n-1}(l-n)(l+n+1)\left[\left(\frac{l}{2}\right)!\left(\frac{n-1}{2}\right)!\right]^2} & l \neq n,\ l \ \text{even},\ n \ \text{odd} \end{cases} \tag{4.25}$$

These results correspond with those given in /7/, p. 306; /11/, p. 24. For even n and odd l changes, of course, nothing.

5. Assymptotic formula for Clebsh-Gordan coefficients

In accordance with /1/, p. 24, the following assymptotic formula for large values of s, l, n, i.e. Ns, Nl, Nn, $N \to \infty$, is true :

$$C(Ns, Nl, Nn; t, k, k+t) \to \begin{cases} (-1)^{N(q-n)} \left[\dfrac{2n}{\pi N \sqrt{4s^2 l^2 - (n^2 - s^2 - l^2)^2}}\right]^{1/2} \\ \text{for} \quad |s-l| < n < s+l \\ 0 \quad \text{in the contrary} \end{cases} \tag{5.1}$$

where $2q = s + l + n$. This result does not depend on t, k, $k+t$; hence for large values of N

$$C(Ns, Nl, Nn; 0, 0, 0) \to C(Ns, Nl, Nn; t, k, k+t) . \tag{5.2}$$

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