

ON A UNIFIED PRESENTATION OF THE NON-EQUILIBRIUM TWO-PHASE
FLOW MODELS

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ABSTRACT

If the various existing one-dimensional two-phase flow models are consistent, they must appear as particular cases of more general models. It is shown that this is the case if, and only if, the mathematical expressions of the laws of the transfers between the phases are sufficiently general. These transfer laws control the non-equilibrium phenomena.

A convenient general model is a particular form of the two-fluid model. This particular form involves two equations and three dependent variables characterizing the mixture, and three equations and three dependent variables characterizing the differences between the phases (slip, thermal non-equilibrium). The mathematical expressions of the transfer terms present in the above equations involve first-order partial derivatives of the dependent variables.

The other existing models may be deduced from the general model by making assumptions on the fluid motion. Several examples are given. The resulting unified presentation of the existing models enables a comparison of the implicit assumptions made in these models on the transfer laws. It is therefore a useful tool for the appraisal of the existing models and for the development of new models.

NOTATION

- A Cross section area, or point of the t, x plane
- dA/dz
- \mathbf{E} Matrices defined in the text
- $\mathbf{E} + \mathbf{W}_0^T/\mathbf{H}_0$, or column of a matrix
- $\mathbf{E} + \mathbf{I}(\rho\mathbf{W})$, mass velocity
- \mathbf{g} $x=0$ projection on $-oz$ of the gravity acceleration
- h specific enthalpy
- $h^* = h^*_K = dh_{Ksat}/dp$
- \mathbf{H} Interfacial mass transfer term
- $(\mathbf{H}\mathbf{H})$ Interfacial heat and kinetic energy transfer term
- $(\mathbf{H}\mathbf{F})$ Interfacial momentum transfer term
- \mathbf{H} Determinant
- p Pressure
- q Wall heat transfer term
- τ Wall friction term
- t Time
- W velocity
- x Quality
- x_i Any of the dependent variables
- z Abscissa
- α_K Volumetric fraction of phase K ($\alpha_G =$ void fraction)
- \mathbf{V} Velocity
- γ Slip ratio
- Δ Determinant
- $\beta, \lambda, \mu, \nu, \eta, \zeta$ Quantities defined in equations (4) and

table 3.

$\rho = \sum (\rho_K)$ density of the mixture

ρ_K Density of phase K

$$\rho^* p \Delta = \rho^* h^*_K \Delta = (\partial \rho_K / \partial p) \Delta h_K, \quad \rho^* h^*_K = (\partial \rho_K / \partial h_K) p$$

$$\Delta h_K = h_K - h_{Ksat} \text{ thermal non-equilibrium of phase } K$$

$$\Delta(xy) = x_G y_G - x_L y_L$$

$$\Gamma(xy) = x_G y_G + x_L y_L$$

Subscripts

quantities related to

- F Friction or momentum transfer
- G Gas phase
- K Phase K ($K : G, \text{ or } L$)
- L Liquid phase
- M Mass, or mass transfer
- p Pressure
- S slip
- sat Saturation conditions
- T Interfacial transfer
- W Velocity
- x_i Dependent variable x_i
- α Void fraction
- Δh_K Thermal non-equilibrium of phase K
- ΔW Velocity difference $W_G - W_L$
- O Reference, or right-hand side of partial differential equation.
- 2 Right-hand side of equation

Superscripts

- O Right-hand side of partial differential equation
- t $\partial/\partial t$ term
- z $\partial/\partial z$ term

INTRODUCTION : STATEMENT OF THE PROBLEM

A striking fact about the numerous mathematical models used or proposed in literature to describe two-phase flows is their apparent diversity. Since the bases of these models must be, and are generally claimed to be, the conservation principles, their diversity is rather confusing, and questions have been raised about the consistency of existing models (1,2)

For one-dimensional flows, to which this paper is restricted, and within the frame of current assumptions, the complete description of a two-phase flow requires at least the calculation of six dependent variables, as functions of space and time. These variables are, for instance, $\alpha, p, W_G, W_L, h_G, h_L$, and, once they are known, the other quantities of interest (such as G, x, ρ, \dots) can be readily calculated.

In fact, comprehensiveness would require the calculation of seven dependent variables, since a difference may occur between the pressures P_G and P_L of the two phases. The quasi-universal assumption $P_G = P_L$ implies a stronger interaction between the phases than in reality. Therefore it has probably major consequences when interaction phenomena play an important role (transients, non-equilibrium flows, propagation of disturbances). However, it does not affect the method used here and, therefore, may be adopted in this paper without loss of generality.

The number of partial-differential conservation equations (mass, momentum, energy) used in existing one-dimensional models varies between 3 (conservation equations written for the mixture only) and 6 (conservation equations written for each phase separately). The reason for such a variation is that the larger the number of conservation equations, the more detailed the laws of the transfers (between the phases and between each phase and the wall) have to be. Existing models, however, always involve some empirically adjusted quantities, and the number of conservation equations used is in fact the result of a compromise between the necessity to adjust the above quantities and the necessity to detail the above transfer laws. Unfortunately, this compromise does not work when transfer phenomena play an essential part in the process, i.e. for transients, and also when departure from equilibrium is important. (3.4)

Two consequences follow :

1. Since detailed transfer laws are still not available, the current use of six conservation equation models is not yet possible.
2. The six conservation equations have to be satisfied anyway. Therefore, any model, if constant, should appear as a particular case of a six conservation equation model, involving only some complementary assumptions.

To be in a position to decide on the applicability of an available model when some practical problem has to be solved, it is therefore important to check its consistency, and to list the complementary assumptions it implies.

PURPOSE OF THE PAPER

This paper is devoted to the foregoing problem. Its purpose is to show that a general six conservation equation model can be reduced to most of the existing models, thus proving their consistency and putting in evidence some of their implicit assumptions. The possibility of reduction was demonstrated in (5) for steady-state models. Examples of such reductions, for both steady-state and transient models are given in (6.7). The general problem is dealt with hereunder. For the reduction to be possible, the form of the transfer laws involved in the initial six conservation equation model has to be sufficiently general, as discussed in the next section. The reduced models are considered here in their most general form, i.e. they include terms which are sometimes neglected in the literature. With such a general form, the above implicit complementary assumptions involve the transfer laws.

To achieve the above purpose, it is necessary to write the six conservation equations of the initial model in a particular form such that the equations and the dependent variables used in the reduced models can be easily brought out. In particular, it should include three equations and three dependent variables characterizing the mixture (as in the homogeneous model) and three equations and three dependent variables characterizing the differences between phases

or between each phase and the mixture. (slip, thermal non-equilibria). This transformation, should also preserve the symmetrical role of the two phases. Such a transformation is presented in (11). It enables the system to be written in a dimensionless form.

THE INITIAL SIX CONSERVATION EQUATION MODEL. TRANSFER LAWS

The six conservation equation model consists of the six conservation equations and of the associated transfer laws.

It would be beyond the scope of this paper to give here the derivation of the six conservation equations. In any case, the method used hereunder would be valid even with slightly different equations. Moreover this derivation has been given several times. Its bases are given in (8). The assumptions made and the resulting equations are discussed for instance in (5.9). The conservation equations, written with Q_G , F , W_L , W_G , Δh_L , Δh_G as dependent variables, are recalled tablel. They imply no assumption on the form of the transfer laws.

The transfer terms present in the six conservation equations, i.e. the mass, momentum and energy transfers between the phases as well as the momentum (friction) and energy transfers between each phase and the wall depend on some of the boundary conditions (e.g. wall roughness and temperature,...) and on some of the flow properties (e.g. pressure, velocities...). Therefore, they are ultimately functions of space, time and of the six dependent variables used to describe the flow. Although there are exceptions (10) the transfer terms are generally calculated through algebraic or transcendental equations, which is very restrictive and cannot be substantiated.

It has been shown in (4.9) that this approach induces contradictions such as the absence of any effect of the transfers on critical flow rates and on propagation velocities of small disturbances. It will also be shown hereunder that it renders impossible the reduction of the six conservation equation model to any other model. The transfer terms must be related to the dependent variables by more complicated functions, for instance through integro-differential equations. The example of self-vaporization due to pressure drop in two-phase, one component flows illustrates the necessity of the presence of derivatives, such as the pressure gradient, in the transfer terms. Hereunder, as in (7), the transfer terms are assumed to depend linearly on the first-order partial derivatives of the dependent variables :

$$\left. \begin{aligned}
 N &= N_0 + \sum_{x_i} \left[\lambda_{x_i}^t \frac{\partial x_i}{\partial t} + \lambda_{x_i}^x \frac{\partial x_i}{\partial z} \right], \\
 (NV) &= (NV)_0 + \sum_{x_i} \left[\mu_{x_i}^t \frac{\partial x_i}{\partial t} + \mu_{x_i}^x \frac{\partial x_i}{\partial z} \right], \\
 (M) &= (M)_0 + \sum_{x_i} \left[\nu_{x_i}^t \frac{\partial x_i}{\partial t} + \nu_{x_i}^x \frac{\partial x_i}{\partial z} \right], \\
 T_K &= T_{K0} + \sum_{x_i} \left[\eta_{x_i}^t \frac{\partial x_i}{\partial t} + \eta_{x_i}^x \frac{\partial x_i}{\partial z} \right], \\
 Q_K &= Q_{K0} + \sum_{x_i} \left[\zeta_{x_i}^t \frac{\partial x_i}{\partial t} + \zeta_{x_i}^x \frac{\partial x_i}{\partial z} \right]
 \end{aligned} \right\} (4)$$

the x_i being the six dependent variables used to describe the flow and $N_{C,0}$, $(NW)_{C,0}$, $(NM)_{C,0}$, $T_{L,0}$, $Q_{L,0}$, the λ .

u, v, w, ζ being given functions of the x_i and of x and t . As indicated in equations (4) the interfacial momentum transfer, (NW) may be split in two parts (11) The first part (subscript N) is transferred along with the mass N , so that it depends ultimately on heat fluxes. The second part is due to the interfacial stress. (The same is true for (NM) , but its two parts both depend on heat fluxes, and they do not need to be distinguished hereunder).

All the other assumptions made in the model are customary.

THE OTHER MODELS

When they use, in their practical forms, less than six conservation equations, the models in the literature can be characterized by some restrictions imposed on the evolutions of the dependent variables. These restrictions replace the "missing" equations. Since the minimum number of conservation equations written in consistent models is three (balances of mass, momentum, and energy), the maximum number of restrictions imposed on the evolutions of the dependent variables is

$$6 - 3 = 3$$

Most of the cases met in practice are summarized by table 2. An example is the case of the homogeneous evolutions, defined here as those evolutions with $W_C = W_L$, $\Delta h_C = \Delta h_L = 0$ everywhere and at every time.

Accordingly, only three dependent variables are in this case sought for, and only three conservation equations are written in the so-called homogeneous models. The three dependent variables used, e.g. Q_C , p , W , characterize the mixture, and the three conservation equations written are logically the mixture conservation equations, i.e. for each conservation principle, the sum of the gas and liquid conservation equations.

Nevertheless, as pointed out in the introduction, the six conservation equations have to be satisfied, and an homogeneous evolution is possible if, and only if, $W_C = W_L$, $\Delta h_C = \Delta h_L = 0$ belong to the solution of the complete set. Since the existence of two phases, and of transfers between these two phases, are ignored in the homogeneous models, the mixture conservation equations must not contain any interfacial transfer term: such equations are only obtained by summing up the gas and liquid conservation equations in their primary form (table 1), i.e. prior to any of the commonly-used combinations between the three conservation equations of each phase.

Similar remarks may be made for the other cases listed in table 2.

TRANSFORMATION OF THE SIX CONSERVATION EQUATIONS

A transformation of the six conservation equations, which, to the author's knowledge, is original, is presented in (11). The resulting equations, which are strictly equivalent to the set of table 1, complemented by the transfer laws (4), are recalled in a simplified version in tables 4.1. and 4.2. The important points, in view of the purpose of this paper, are not the details of the equations but:

1. The presence, among the six equations, of three mixture conservation equations, which do not involve the interfacial transfers.
2. The mathematical form of the six equations: thanks to the assumptions made, in particular on the transfer terms (equations 4), they are partial differential equations, linear with respect to the derivatives.

The demonstrations given hereunder are based only on these two points. As already pointed out in the introduction, they remain valid with other sets of six equations fulfilling these two conditions.

Tables 4.1. and 4.2. give the columns C_{ii}^t and C_{ii}^s of the coefficients of the derivatives $\partial_{x_i}/\partial t$ and $\partial_{x_i}/\partial x$ in the equations, and the column C_0^s of their right-hand-side numbers. From top to bottom the equations are three mixture conservation equations of mass, momentum (taking mixture mass equation into account) and energy (taking mixture mass and mixture momentum equations into account) and three equations for the slip, the liquid and the gas thermal non-equilibrium respectively.

The equations are written in terms of the dependent variables $Q_C, p, W, \Delta W, \Delta h_L, \Delta h_G$, where $\Delta W = W_G - W_L$ and $W = \alpha_G W_G + \alpha_L W_L$. W is the center-of-volume velocity, which turns out to be more convenient here than the center-of-mass velocity W_M .

The equations have been made dimensionless, a convenient base consisting of length, density, velocity and enthalpy with characteristic values l_0, ρ_0, W_0, h_0 , as units. Except for the appearance of the ratio $C = W_0^2/h_0$ in several terms of the energy equations, the dimensionless equations are identical to the dimensional ones.

The following quantities appear in tables 4.1. and 4.2.:

$$f_C = W - \Delta W \alpha_C, \quad f_W = W + \Delta W \frac{N(C_0)}{\rho}$$

$$T_0 = T_{L,0} + T_{G,0} + \rho g \cos \theta$$

$$Q_0 = Q_{L,0} + Q_{G,0} + CW_M (T_{L,0} + T_{G,0})$$

$$P_{S,N} = \frac{(NW)_{N,0} - N_0 W_L}{\alpha_L \rho_L} + \frac{(NW)_{N,0} - N_0 W_G}{\alpha_G \rho_G}$$

$$P_{S,p} = \frac{(NW)_{p,0} + T_{L,0}}{\alpha_L \rho_L} + \frac{(NW)_{p,0} - T_{G,0}}{\alpha_G \rho_G}$$

$$R_L = \frac{1}{\alpha_L \rho_L} \left[Q_{L,0} + CW_L T_{L,0} + N_0 \left(h_L - \frac{CW_L^2}{2} \right) + CW_L (NW)_0 - (NM)_0 \right]$$

$$R_G = \frac{1}{\alpha_G \rho_G} \left[Q_{G,0} + CW_G T_{G,0} - N_0 \left(h_G - \frac{CW_G^2}{2} \right) - CW_G (NW)_0 + (NW)_0 \right]$$

The β 's appearing in tables 4.1 and 4.2. stem from the differential terms in the transfer laws (4). Their expressions are given table 3. In the most general case which is probably highly unrealistic, they are 60 distinct functions of t, x and of the six dependent variables. They are defined for any α_C ($0 \leq \alpha_C \leq 1$):

TABLE 3

Differential transfer terms in the transformed equations (written with superscript t . Idem with superscript s)

Mixture Momentum Equation

$$\beta_{x_i, p}^t = \eta_{x_i, C}^t + \eta_{x_i, L}^t$$

Mixture Energy Equation

$$- \beta_{x_i, z}^t = \zeta_{x_i, G}^t + \zeta_{x_i, L}^t + C_{Mz} \left[\eta_{x_i, G}^t + \eta_{x_i, L}^t \right]$$

Slip Equation

$$\beta_{x_i, z}^t = \frac{W_G \lambda_{x_i}^t - \mu_{x_i}^t + \eta_{x_i, G}^t}{\alpha_G \rho_G} + \frac{W_L \lambda_{x_i}^t - \mu_{x_i}^t - \eta_{x_i, L}^t}{\alpha_L \rho_L}$$

Liquid Thermal Non-equilibrium Equation

$$\beta_{x_i, L}^t = - \frac{(h_L - \frac{C_{Mz}^2}{2}) \lambda_{x_i}^t - \nu_{x_i}^t + C_{Mz} \mu_{x_i}^t + \zeta_{x_i, L}^t + C_{Mz} \eta_{x_i, L}^t}{\alpha_L \rho_L}$$

Gas Thermal Non-equilibrium Equation

$$\beta_{x_i, G}^t = \frac{(h_G - \frac{C_{Mz}^2}{2}) \lambda_{x_i}^t - \nu_{x_i}^t + C_{Mz} \mu_{x_i}^t - (\zeta_{x_i, G}^t + C_{Mz} \eta_{x_i, G}^t)}{\alpha_G \rho_G}$$

Since the equations of tables 4.1. and 4.2. are given here only for illustration purpose, the β 's and the set of equations have been simplified by assuming that the η 's and ζ 's are zero, and by neglecting small terms whenever possible.

Incidentally, it may be pointed out that, among the non-equilibrium parameters, it is the velocity difference ΔW which is responsible for most of the complications in the equations.

REDUCTION OF THE SIX EQUATION MODELS CHARACTERIZED BY THREE RESTRICTIONS

Since this case corresponds to the maximum number of restrictions (see table 2), it is discussed at length. The demonstration, given here for the homogeneous evolutions, will be generalized afterwards. It is recalled that the homogeneous evolutions are those evolutions with $\Delta W = \Delta h_L = \Delta h_G = 0$ everywhere and at every time.

The solution of the set of six conservation equations is computed in the neighbourhood of A in the t, z plane (fig.1), starting from the knowledge of the dependent variables on a curve Σ going through A: the expressions of the known total derivatives along Σ of the six dependent variables provide six equations which are used together with the six conservation equations to compute the twelve partial derivatives with respect to t and z of $\alpha_G, p, W, \Delta W, \Delta h_L, \Delta h_G$. Eliminations of the $\partial/\partial t$ derivatives, for instance, yields six equations allowing in general, i.e. if Σ is not tangent to a characteristic direction (12), calculation of the $\partial/\partial z$ derivatives.

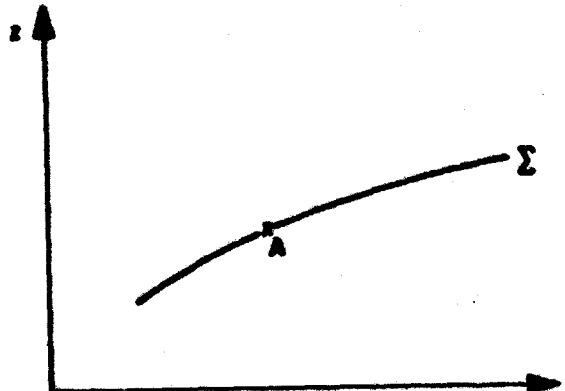


Figure 1.

Assuming $\Delta W = \Delta h_L = \Delta h_G = 0$ on Σ , they remain zero in the neighbourhood of A if and only if

$\partial \Delta W / \partial z = \partial \Delta h_L / \partial z = \partial \Delta h_G / \partial z = 0$ belong to the solution of the set. Let Δ be the determinant of the set and N_{x_i} the determinant obtained on replacing in Δ the column C_{x_i} of the coefficients of $\partial x_i / \partial z$ by the column C_i of the right-hand side numbers. The foregoing conditions are

$$N \Delta y = N_A M_L = N_A M_G = 0 \quad (5)$$

which is equivalent to the existence of three sets of six coefficients l, m, n , (one of the coefficients of each set, at least, being different from 0), such that:

$$\begin{cases} l_0 C_\alpha + l_p C_p + l_W C_W + l_2 C_2 + l_{\Delta M_L} C_{\Delta M_L} + l_{N_A M_G} C_{N_A M_G} = 0 \\ m_0 C_\alpha + m_p C_p + m_W C_W + m_2 C_2 + m_{\Delta M_G} C_{\Delta M_G} = 0 \\ n_0 C_\alpha + n_p C_p + n_W C_W + n_{\Delta W} C_{\Delta W} + n_{\Delta M_L} C_{\Delta M_L} + n_2 C_2 = 0 \end{cases}$$

If two of these sets exist and are distinct, some consequences are the existence of the third, but also

$$\Delta = 0, \quad N_G = N_p = N_W = 0$$

which leads to a particular case (the tangent to Σ at A is a characteristic direction).

Therefore, the general solution is obtained when the three sets of l, m, n are identical, which may be written

$$l_0 C_\alpha + l_p C_p + l_W C_W + l_2 C_2 = 0 \quad (6)$$

It can then be concluded that $\Delta W, \Delta h_L$ and Δh_G remain zero in the neighbourhood of A if and only if the rank of the 6×4 rectangular matrix

$$\begin{bmatrix} C_\alpha & C_p & C_W & C_2 \end{bmatrix} \quad (7)$$

is not larger than 3 (i.e. 3 in general, particular cases being obtained when the rank is less than 3).

Let V be the slope of the tangent to Σ at A and let α', p', W' be respectively the known total derivatives at A along Σ of α_G, p, W . The evolution is homogeneous if, and only if, the above condition is verified for any point A on any curve Σ , i.e. for any value of $V, \alpha_G, p, W, \alpha', p', W'$.

The matrix (7) may be written

$$\begin{bmatrix} c_a^t - v c_a^t & c_p^t - v c_p^t & c_v^t - v c_v^t & c_2^t - v c_2^t & c_a^t - v c_a^t & c_p^t - v c_p^t & c_v^t - v c_v^t \end{bmatrix} \quad (8)$$

It is built with different linear combinations of seven different columns

$$c_a^t \quad c_a^t \quad c_p^t \quad c_p^t \quad c_v^t \quad c_v^t \quad c_2^t \quad (9)$$

The rank of the matrix (8) is never larger than three if, and only if, four distinct linear combinations exist between four columns chosen in (9) in such a way that each column is present at least in one combination.

This condition is necessary: expressing the determinants of the square 4 x 4 submatrices of matrix (8), it is easy to show that, if the rank of matrix (8) is never larger than 3, linear combinations exist, for instance between

$$\begin{cases} c_a^t & c_p^t & c_v^t & c_2^t \\ c_a^t & c_p^t & c_v^t & c_a^t \\ c_a^t & c_p^t & c_v^t & c_p^t \\ c_a^t & c_p^t & c_v^t & c_v^t \end{cases} \text{ and between } \begin{cases} c_a^t & c_p^t & c_v^t & c_2^t \\ c_a^t & c_p^t & c_v^t & c_a^t \\ c_a^t & c_p^t & c_v^t & c_p^t \\ c_a^t & c_p^t & c_v^t & c_v^t \end{cases}$$

On the other hand, if the model is appropriate, it enables actual calculation of the dependent variables to be made: A linear combination between c_a^t , c_p^t , c_v^t

and a linear combination between c_a^t , c_p^t , c_v^t cannot exist simultaneously, since this would preclude such a calculation. Hence, either the linear combinations of the first group or of the second group above are distinct, which establishes the proposition.

This condition is sufficient: If four distinct linear combinations, as defined above, exist, then any other possible set of four columns chosen in (9) verifies a linear combination and the rank of the matrix (8) is never larger than three.

Finally, since each linear combination of four columns of six elements is equivalent to three relationships between these elements, a homogeneous evolution implies $4 \times 3 = 12$ relationships between the elements of the columns (9). These relationships involve the β 's present in the columns (9).

Taking into account the fact that the three mixture conservation equations, and therefore the three first lines of columns (9), must be linearly independent, the 12 relationships can be written by using only one of the three last lines of columns (9) at a time: The 12 relationships are linear (but not homogeneous) with respect to the 18 β 's present in these last lines (see tables 4.1. and 4.2). Therefore

1. There are always an infinity of solutions, for the β 's.
2. All the β 's cannot be taken simultaneously as zero, in general.

In other words, the reduction of the six conservation equation model to the homogeneous model is possible through an appropriate choice of the coefficients of derivatives of the dependent variables in the expressions (4) of the transfer terms. The reduction is not possible if all these coefficients (λ , μ , ν , η , ζ) are taken as zero.

Any three-restriction case can be dealt with in the same way as the homogeneous evolutions at the expense of a change of variable. For instance, the slip ratio $\gamma = W_c/W_l$ and the dependent variable ΔW are related through

$$\Delta W = \frac{(\gamma - 1)W}{1 + (\gamma - 1)\alpha_c}$$

If γ is an imposed function of α_c , P , W , the evolutions with thermal equilibrium and an imposed slip-ratio correspond, by the change of variable

$$\Delta W = \frac{(\gamma - 1)W}{1 + (\gamma - 1)\alpha_c} - \gamma,$$

$$\text{to } \gamma = \Delta h_l = \Delta h_g = 0$$

When the dependent variable γ replaces the dependent variable ΔW , the demonstration given above for the case of the homogeneous evolutions remains valid: its conclusions may be applied to any three-restriction evolution.

REDUCTION OF THE SIX-EQUATION MODEL TO MODELS CHARACTERIZED BY TWO RESTRICTIONS

The demonstration, illustrated here for the thermal-equilibrium evolutions ($\Delta h_l = \Delta h_g = 0$) parallels the foregoing.

$$\text{Equations (5) are replaced by } N \Delta h_l = N \Delta h_g = 0 \quad (10)$$

Equation (6) is replaced by:

$$L_1 c_a^t + L_2 c_p^t + L_3 c_v^t + L_4 c_{\Delta W}^t + L_5 c_2^t = 0 \quad (11)$$

and the matrix (7) by

$$\begin{bmatrix} c_a^t & c_p^t & c_v^t & c_{\Delta W}^t & c_2^t \end{bmatrix} \quad (12)$$

The evolution is a thermal-equilibrium one if, and only if, the rank of the 6 x 5 rectangular matrix (12) is not larger than 4. The matrix (12) is built with different linear combinations of nine different columns

$$c_a^t \quad c_a^t \quad c_p^t \quad c_p^t \quad c_v^t \quad c_v^t \quad c_{\Delta W}^t \quad c_{\Delta W}^t \quad c_2^t \quad (13)$$

The rank of the matrix (12) is never larger than four if, and only if, five distinct linear combinations exist between five columns chosen in (13) in such a way that each column is present at least in one combination.

Finally, since each linear combination of five columns of six elements is equivalent to two relationships between these elements, a two-restriction evolution implies $5 \times 2 = 10$ relationships between the β 's present in the columns (13). These 10 relationships are linear, (but not homogeneous), with respect to the 18 β 's, present in two of the three last lines of columns (13). These are always an infinity of solutions for the β 's but they cannot be taken simultaneously as zero, in general.

REDUCTION OF THE SIX-EQUATION MODEL TO MODELS CHARACTERIZED BY ONE RESTRICTION

The demonstration, illustrated here for the evolutions with $\Delta h_g = 0$, follows the same path. The matrices (7) and (12) are replaced by:

$$\begin{bmatrix} c_a^t & c_p^t & c_v^t & c_{\Delta W}^t & c_{\Delta h_l}^t & c_2^t \end{bmatrix} \quad (14)$$

The evolution is a zero- Δh_c evolution if, and only if, the rank of the 6×6 square matrix (14) is not larger than 5 i.e. if, and only if, six distinct linear combinations exist between the 11 columns

$$\begin{matrix} C_{\alpha}^t & C_{\alpha}^x & C_p^t & C_p^x & C_v^t & C_v^x \\ C_{\Delta W}^t & C_{\Delta W}^x & C_{\Delta hL}^t & C_{\Delta hL}^x & C_2^t & C_2^x \end{matrix} \quad (15)$$

in such a way that each column is present at least in one combination.

Finally, since each linear combination of six columns of six elements is equivalent to one relationship between these elements, a one-restriction evolution implies 6 relationships between the β 's present in the columns (15). These 6 relationships are linear (but not homogeneous), with respect to the 10 β 's present in one of the three last lines of columns (15). There are always an infinity of solutions for the β 's but they cannot be taken simultaneously as zero in general.

EXAMPLES OF REDUCTION

The derivation of the above relationships is long and sometimes tedious, but it does not raise any mathematical difficulty. It is possible to choose, among the infinity of possibilities, a single solution for the β 's, and consequently for the λ 's, μ 's, ν 's... of equations (4), by assuming that the effects of cross-section area variation, stresses and heat fluxes do not interact.

Some results are given hereunder, for illustration purpose, with the same simplifications as in tables 4.1. and 4.2., and in particular with $T_K = T_{K,0}$ and $Q_K = Q_{K,0}$. Complete calculations for several models of practical interest and discussion of the results will be made available later as a special report.

Example 1 : Homogeneous model

With the assumption that $M_0 = Q_0/\Delta H$, and as long as T_0 and Q_0 are not zero, the six equation model reduces to the homogeneous model when the following transfer laws are adopted :

$$\begin{cases} M = M_0 - \frac{\Sigma(\alpha \rho H') - C}{\Delta H} \frac{D_p}{D_t} \\ (MV) = (MV)_0 + W(M - M_0) + \frac{\alpha_G \rho_L \rho_G \rho_L}{\rho} \left[\frac{B_{s,F}}{T_0} + \Delta \left(\frac{1}{\rho} \right) \frac{\partial p}{\partial z} + \frac{\rho B_{s,F}}{T_0} \frac{\partial W}{\partial t} \right] \\ (MH) = (MH)_0 + (M - M_0) H_T - \frac{\rho_G \rho_L}{\rho} \frac{\alpha_G \rho_G \rho_G}{Q_0} \Delta H \frac{D \rho_G}{D t} \end{cases}$$

In these expressions $\frac{D}{D t}$ stands for $\frac{\partial}{\partial t} + W \frac{\partial}{\partial z}$ and H_T stands for :

$$H_T = \Delta H \left\{ \frac{\alpha_L \alpha_L H'_L - \alpha_L C}{\Sigma(\alpha \rho H') - C} - \frac{\alpha_L \rho_L B_L}{Q_0} \left[\frac{\alpha_G \alpha_L \rho_G \rho_L \Delta H}{\rho [\Sigma(\alpha \rho H') - C] \Delta} \left(\frac{\rho' \rho \Delta}{\rho} \right) + 1 \right] \right\}$$

which is also equal to

$$- \Delta H \left\{ \frac{\alpha_G \rho_G H'_G - \alpha_G C}{\Sigma(\alpha \rho H') - C} - \frac{\alpha_G \rho_G B_G}{Q_0} \left[\frac{\alpha_G \alpha_L \rho_G \rho_L \Delta H}{\rho [\Sigma(\alpha \rho H') - C] \Delta} \left(\frac{\rho' \rho \Delta}{\rho} \right) + 1 \right] \right\}$$

since $\alpha_G \rho_G B_G = -\alpha_L \rho_L B_L$

Example 2 : Thermal-equilibrium model

With the assumption that $M_0 = Q_0/\Delta H$, and as long as Q_0 is not zero, the six equation model reduces to the thermal-equilibrium model when the following transfer laws are adopted :

$$\begin{cases} M = M_0 - \frac{\Sigma(\alpha \rho H') - C}{\Delta H} \frac{\partial p}{\partial z} - \frac{\Sigma(\alpha \rho H') - CW}{\Delta H} \frac{\partial p}{\partial z} \\ MV = (MV)_0 + \frac{\alpha_L \rho_L V_G + \alpha_G \rho_G W_L}{\rho} (M - M_0) \\ (MH) = (MH)_0 - \left[\frac{\Sigma(\alpha \rho H') - C}{\Delta H} H_L - \alpha_L \rho_L B_{p,L}^t \right] \frac{\partial p}{\partial z} \\ - \left[\frac{\Sigma(\alpha \rho H') - CW}{\Delta H} H_L - \alpha_L \rho_L B_{p,L}^x \right] \frac{\partial p}{\partial z} \end{cases}$$

with

$$\begin{cases} B_{p,L}^t = - \left(H'_L - \frac{C}{\rho_L} \right) \\ + \frac{H_L}{Q_0} \left[\frac{\alpha_G \alpha_L \rho_G \rho_L}{\rho} \frac{W_L V_G}{W_H} \Delta \left(\frac{\rho' \rho \Delta}{\rho} \right) \Delta H + \Sigma(\alpha \rho H') - C \right] \\ B_{p,L}^x = - \left(H'_L - \frac{C}{\rho_L} \right) H_L \\ + \frac{H_L}{Q_0} \left[\frac{\alpha_G \alpha_L \rho_G \rho_L}{\rho} \frac{W_L V_G}{W_H} \Delta \left(\frac{\rho' \rho \Delta}{\rho} \right) \Delta H + \Sigma(\alpha \rho H') - CW \right] \end{cases}$$

Example 3 : Model with equilibrium in the vapor only

In this case, the symmetry between the two phases is lost and the expression for M is complicated. However it is interesting to notice that

$$\begin{cases} MV = (MV)_0 + \frac{\alpha_L \rho_L V_G + \alpha_G \rho_G W_L}{\rho} (M - M_0) \text{ and} \\ MH = (MH)_0 + (M - M_0) H_L \end{cases}$$

Discussion

It is emphasized again that, whenever some reduced model is used, implicit assumptions are made on the interfacial transfer laws, as exemplified above for three kinds of models.

Not unexpectedly, a restriction on one enthalpy (example 3) essentially affects the mass transfer law, restrictions on both enthalpies (example 2) essentially affect both the mass and energy transfer laws and a restriction on slip essentially affects the momentum transfer law (as shown by the comparison between examples 1 and 2)

A complete discussion of the foregoing examples would be beyond the scope of this paper. However, a few remarks can be made :

1. Simplified forms of the mass transfer law of the homogeneous model are often used in the literature. As a matter of fact, an expression for M which is analogous to the above expression can be directly deduced from conservation equations (11).

The first and second terms correspond respectively to the vapor production by heat and by flashing.

The momentum and energy transfer laws are more complicated. In the second terms on the right-hand side of both expressions, W and W_p may be regarded respectively as a velocity and an enthalpy characterizing the mass which is transferred. In practice W_p is comprised between W_L and W_G . The momentum transfer is affected by the pressure and velocity gradients, which could have been expected, while the energy transfer is affected by the void fraction gradient.

2. In the thermal-equilibrium model, the mass transfer law is analogous to that of the homogeneous model (it reduces to the homogeneous model mass transfer law when ΔW tends to zero). The velocity which "characterizes" the mass transferred is

$$\frac{\alpha_L \rho_L W_G + \frac{\alpha}{C} \frac{\rho}{C} W_L}{\alpha_L \rho_L + \alpha_G \rho_G}$$

It varies from W_G for $\alpha_G = 0$ to W_L for $\alpha_G = 1$. The energy transfer law is complicated. It must be noticed however that the lack of symmetry in (10) is only apparent.

3. The models which assume thermal-equilibrium in one phase only (which are of common use) imply a very complicated mass transfer law and the lack of symmetry between the two phases appears to have serious disadvantages.

CONCLUSION

It has been recalled that any one-dimensional two-phase flow model, if consistent, should appear as a particular case of a more general model, which, within the frame of current assumptions, is a six-conservation-equation model.

It has been shown that such is the case, provided the form of the transfer terms present in the six-equation model is sufficiently general. In particular, it has been shown that the form proposed elsewhere (7), in which the transfer terms depend linearly on the first-order partial derivatives of the dependent variables, enables the reduction of the six-equation model to any other model to be made. On the contrary, a form in which the transfer terms depend only on the dependent variables themselves is not acceptable. The presence of partial derivatives of the dependent variables in the transfer terms corresponds to small history and neighbourhood effects. (such effects are sometimes taken into account in other fields, such as single-phase-flow turbulence).

A form of the six-equation model, which is convenient for reduction, has been given. It contains three mixture conservation equations which do not involve the interfacial transfers.

A reduced model can be characterized by the restrictions imposed on the evolutions of the dependent variables. The maximum number of such restrictions is three. The demonstration of the possibility of reduction, and examples of reductions have been given. Three-restriction, two-restriction and one-restriction models imply respectively twelve, ten, and six conditions on the transfer laws. It is felt that the knowledge of these conditions which have never been given before, is a useful tool for the appraisal of the existing models and a clue to a more thorough understanding of two-phase flows.

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TABLE 1

Primary Conservation Equations (1)

Mass

$$\rho_k \left[\frac{\partial \rho_k}{\partial t} + u_k \frac{\partial \rho_k}{\partial x} \right] + \rho_k v_{kD} \left[\frac{\partial \rho_k}{\partial t} + u_k \frac{\partial \rho_k}{\partial x} \right] + \rho_k v_{kD} \left[\frac{\partial (\Delta \rho_k)}{\partial t} + u_k \frac{\partial (\Delta \rho_k)}{\partial x} \right] = 0 \quad (1_1)$$

Momentum

$$\rho_k u_k \left[\frac{\partial u_k}{\partial t} + u_k \frac{\partial u_k}{\partial x} \right] + \rho_k v_{kD} u_k \left[\frac{\partial u_k}{\partial t} + u_k \frac{\partial u_k}{\partial x} \right] + \rho_k v_{kD} \left[\frac{\partial u_k}{\partial t} + u_k \frac{\partial u_k}{\partial x} \right] + \rho_k v_{kD} \left[\frac{\partial (\Delta u_k)}{\partial t} + u_k \frac{\partial (\Delta u_k)}{\partial x} \right] = 0 \quad (1_2)$$

Energy

$$\rho_k \left(\rho_k + \frac{1}{2} \rho_k^2 \right) \left[\frac{\partial \rho_k}{\partial t} + u_k \frac{\partial \rho_k}{\partial x} \right] + \rho_k \left(\rho_k + \frac{1}{2} \rho_k^2 v_{kD} + \rho_k v_{kD} \right) \left[\frac{\partial \rho_k}{\partial t} + u_k \frac{\partial \rho_k}{\partial x} \right] + \rho_k v_{kD} \left[\frac{\partial \rho_k}{\partial t} + u_k \frac{\partial \rho_k}{\partial x} \right] + \rho_k v_{kD} \left[\frac{\partial (\Delta \rho_k)}{\partial t} + u_k \frac{\partial (\Delta \rho_k)}{\partial x} \right] = 0 \quad (1_3)$$

(1) The upper sign is valid for $k = G$, and the lower sign is valid for $k = L$.

TABLE 2

One-dimensional Two-Phase Flow Models

Number of Restrictions	Restrictions adopted in practice	Remaining dependent Variables	Conservation Equations written in practice.
3	$u_G = u_L, \Delta h_L = \Delta h_G = 0$ (Homogeneous Evolutions)	ρ_k, P, W	3 Mixture Conservation Equations
	$u_G = \gamma u_L, \Delta h_L = \Delta h_G = 0$	ρ_k, P, u_L	- 1d -
2	$\Delta h_L = \Delta h_G = 0$ (Thermal-equilibrium Evolutions)	ρ_k, P, u_L, u_G	3 Mixture Conservation Equations + 1 Phase (Momentum) Conservation Eq.
	$u_G = u_L, \Delta h_G \text{ (or } \Delta h_L) = 0$	$\rho_k, P, W, \Delta h_L \text{ (or } \Delta h_G)$	3 Mixture Conservation Equations + 1 Phase (Energy) Conservation Eq.
	$u_G = \gamma u_L, \Delta h_G \text{ (or } \Delta h_L) = 0$	$\rho_k, P, u_L, \Delta h_L \text{ (or } \Delta h_G)$	- 1d -
1	$\Delta h_G \text{ (or } \Delta h_L) = 0$	$\rho_k, P, u_L, u_G, \Delta h_L \text{ (or } \Delta h_G)$	3 Mixture Conservation Equations + 1 Phase (Momentum) Conservation Eq. + 1 Phase (Energy) Conservation Eq.
	$u_G = u_L$	$\rho_k, P, W, \Delta h_L, \Delta h_G$	3 Mixture Conservation Equations + 1 Phase (Mass) Conservation Eq. + 1 Phase (Energy) Conservation Eq.
	$u_G = \gamma u_L$	$\rho_k, P, u_L, \Delta h_L, \Delta h_G$	- 1d -
0		$\rho_k, P, u_L, u_G, \Delta h_L, \Delta h_G$	3 Mixture Cons. Eq. + 3 phase cons. Eq. or 6 phase conservation eq.

TABLE 4.1 Transformed Conservation Equations (Dimensionless)

C_1^c	C_2^c	C_3^c	C_4^c	C_5^c	C_6^c	C_7^c	C_8^c
ρ	$\rho_0 \rho^*$	$\rho (\rho_0 \rho^*)^2$	$\rho (\rho_0 \rho^*)^3$	0	0	0	$\rho_0 \rho^*$
0	$-(\rho_0)^2 \alpha_1 \rho^*$	$\Delta \rho (\rho_0 \rho^*)^2$	$-\rho_0 \alpha_2 (\rho_0 \rho^*)^3$	0	ρ_0^2	$\rho_0 \rho^*$	$\rho_0 \rho^*$ $\rho_0 \rho^* \rho^*$
$\frac{\rho_0 \rho^*}{\rho} =$	$\frac{\rho_0 \rho^*}{\rho} \rho_0 =$	$\frac{\rho_0 \rho^* \rho^*}{\rho} \left(\frac{\rho_0 \rho^*}{\rho} \right) =$	$\frac{\rho_0 \rho^* \rho^*}{\rho} \left(\frac{\rho_0 \rho^*}{\rho} \right) =$	0	0	$\rho_0 \frac{\rho_0 \rho^* \rho^*}{\rho}$	$\frac{\rho_0 \rho^* \rho^*}{\rho} =$
$\rho_{0,0}^c$	$-(\rho_0)^2 \cdot \rho_{0,0}^c$	$\rho_{0,0}^c$	$\left(\frac{\rho_0}{\rho} \right) \cdot \rho_{0,0}^c$	$\rho_{0,0}^c$	$\rho_0 \cdot \rho_{0,0}^c$	$1 \cdot \rho_{0,0}^c$	$\rho_0 \cdot \rho_{0,0}^c$
$\rho_{0,L}^c$	$\rho_{0,L}^c$	$\rho_{0,L}^c - \frac{c}{\rho_0} \cdot \rho_{0,L}^c$	$\rho_{0,L}^c \left(\rho_0^* - \frac{c}{\rho_0} \right) \cdot \rho_{0,L}^c$	$\rho_{0,L}^c$	$\rho_{0,L}^c$	$\rho_{0,L}^c$	$\rho_{0,L}^c$
$\rho_{0,C}^c$	$\rho_{0,C}^c$	$\rho_{0,C}^c - \frac{c}{\rho_0} \cdot \rho_{0,C}^c$	$\rho_{0,C}^c \left(\rho_0^* - \frac{c}{\rho_0} \right) \cdot \rho_{0,C}^c$	$\rho_{0,C}^c$	$\rho_{0,C}^c$	$\rho_{0,C}^c$	$\rho_{0,C}^c$

TABLE 4.2. Transformed Conservation Equations (Dimensionless) - Continued

$C_{\Delta M}^c$	$C_{\Delta M}^c$	$C_{\Delta C}^c$	$C_{\Delta C}^c$	C_7^c
$\rho_L \rho^* \rho_{LP}$	$\rho_L \rho^* \rho_{LP} \rho_L$	$\rho^* \rho_{CP}$	$\rho^* \rho_{CP} \rho_C$	$-\rho_{0,0}^c \frac{\rho^*}{\rho}$
$-\Delta \rho \rho_L^2 \rho^* \rho_{LP}$	$-\Delta \rho \rho_L^2 \rho^* \rho_{LP} \rho_L$	$\Delta \rho \rho_C^2 \rho^* \rho_{CP}$	$\Delta \rho \rho_C^2 \rho^* \rho_{CP} \rho_C$	$-\rho_{0,0}^c - \Delta \rho (\rho_0 \rho^*) \frac{\rho^*}{\rho}$
$\rho_L \rho^* \rho_L$	$\rho_L \rho^* \rho_L \rho_L$	$\rho^* \rho_C$	$\rho^* \rho_C \rho_C$	
$-\frac{\rho_0 \rho^* \rho_0 \rho^* \rho^*}{\rho} \frac{\rho^* \rho_{LP}}{\rho_L} \Delta \rho$	$-\frac{\rho_0 \rho^* \rho_0 \rho^* \rho^*}{\rho} \frac{\rho^* \rho_{LP}}{\rho_L} \rho_L \Delta \rho$	$-\frac{\rho_0 \rho^* \rho_0 \rho^* \rho^*}{\rho} \frac{\rho^* \rho_{CP}}{\rho_C} \Delta \rho$	$-\frac{\rho_0 \rho^* \rho_0 \rho^* \rho^*}{\rho} \frac{\rho^* \rho_{CP}}{\rho_C} \rho_C \Delta \rho$	$\rho_{0,0}^c - \frac{\rho_0 \rho^* \rho_0 \rho^* \rho^*}{\rho} \Delta \rho \frac{\rho^*}{\rho}$
$\rho_{\Delta M,0}^c$	$\rho_{\Delta M,0}^c$	$\rho_{\Delta C,0}^c$	$\rho_{\Delta C,0}^c$	$\rho_{\Delta M,0}^c = \rho_{\Delta C,0}^c$
$1 \cdot \rho_{\Delta M,L}^c$	$\rho_L \cdot \rho_{\Delta M,L}^c$	$\rho_{\Delta C,L}^c$	$\rho_{\Delta C,L}^c$	ρ_L
$\rho_{\Delta M,C}^c$	$\rho_{\Delta M,C}^c$	$1 \cdot \rho_{\Delta C,C}^c$	$\rho_C \cdot \rho_{\Delta C,C}^c$	ρ_C