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ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

THE BIRTH OF QUANTUM MECHANICS

Werner Heisenberg Memorial Lecture
delivered at the CERN Colloquium on 30 March 1976

by

Jagdish Mehra
Instituts Internationaux de Physique
et de Chimie (Solvay), Bruxelles
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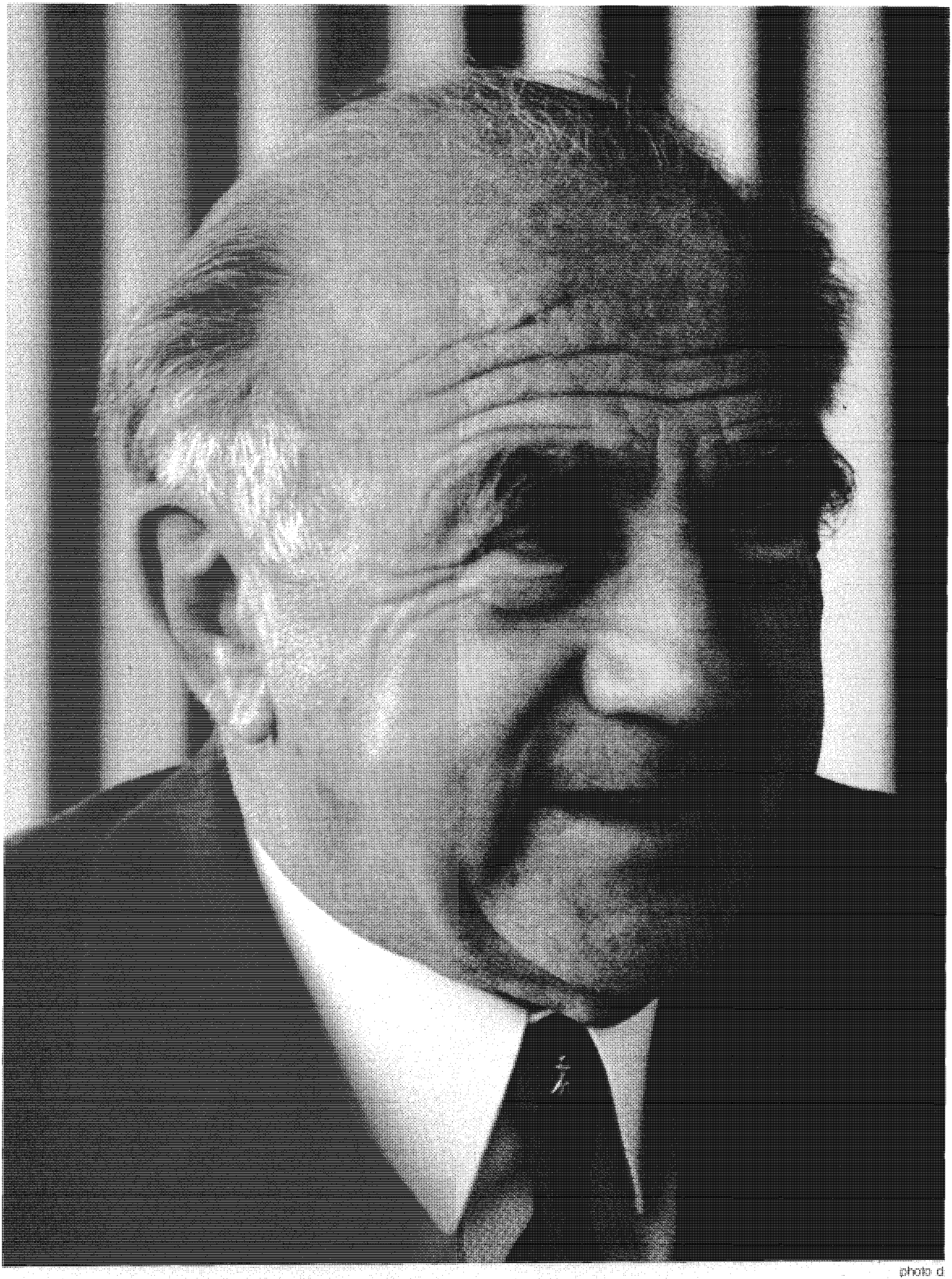
Abstract

In an attempt to give an exact mathematical formulation of Bohr's Correspondence Principle, Heisenberg (June 1925) discovered the rules governing the behaviour of quantum-theoretical magnitudes. In fall 1925 Born, Heisenberg and Jordan and, independently, Dirac, formulated consistent algebraic schemes of quantum mechanics. Early in 1926 Schrödinger developed wave mechanics. In quick succession were discovered : Born's probability interpretation of the wave function, the transformation theory of Dirac, Jordan and F. London, Heisenberg's Uncertainty Relations and Bohr's Principle of Complementarity. By September 1927 the basis of a complete theory of atomic phenomena had been established. Aspects of this development, in which Heisenberg played a central role, are presented here as a tribute to his memory.



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Werner Heisenberg

FOREWORD

Professor Werner HEISENBERG, one of the greatest physicists of all times, died on 1 February 1976. His distinguished career was marked by many scientific contributions and activities in public life.

Werner Heisenberg, born on 5 December 1901 in Würzburg, studied physics in Munich and Göttingen, and took his doctoral degree in 1923 with A. Sommerfeld. He worked in 1924/25 with Niels Bohr in Copenhagen, and was appointed professor at the University of Leipzig in 1927. The 1932 Nobel Prize for Physics was awarded to him in 1933 for his work in quantum mechanics. In 1941 he became Director at the Kaiser Wilhelm Institute for Physics in Berlin and, in 1946, he took over the direction of the successor institution, the Max Planck Institute for Physics in Göttingen, which was transferred to Munich in 1958. He retired in 1971.

After World War II, Werner Heisenberg played a leading role in the reconstruction of science in Germany and in the development of European scientific collaboration, including the first deliberations which led to the establishment of CERN. In 1952 he chaired the Committee which was set up at the first session of the Interim Council, with the task of preparing recommendations on type and energy range of accelerators to be constructed by CERN. In addition to being one of the representatives of the Federal Republic of Germany in various Council sessions Werner Heisenberg was a member of the Scientific

Policy Committee of CERN from its creation in 1954 until 1961. He was the first Chairman of the Committee and served in this capacity until 1957. His last official function at CERN was the inauguration of the ISR in October 1971.

Werner Heisenberg made important contributions to many branches of theoretical physics, in particular to the theory of turbulence, of ferromagnetism, of nuclear forces and, in the last decades of his life, to elementary particle physics. His most fundamental and celebrated work was on the foundation of modern quantum mechanics. This work was reviewed by Professor Jagdish Mehra in a Memorial lecture entitled "The Birth of Quantum Mechanics", delivered at the CERN Colloquium on 30 March 1976. I take great pleasure in presenting this lecture as a CERN Report.

L. Van Hove
Research Director-General

THE BIRTH OF QUANTUM MECHANICS

Professor Van Hove, Ladies and Gentlemen : It gives me a great pleasure to address this distinguished Colloquium, and I find it particularly moving that this occasion is dedicated to the memory of Werner Heisenberg.

Heisenberg made important contributions to the development of many fields of quantum physics : atomic and molecular physics, physics of nuclei and elementary particles, and quantum field theory -- but it is with the discovery of quantum mechanics that his name is inalienably associated. The birth of quantum mechanics presents us with one of the most remarkable episodes in the history of science ; it is as rich, complex, dramatic, and touching as any in the history of human thought.

It shall be presumptuous of me to pretend to invoke more than just a few images dealing with the ideas and events which led to the birth of quantum mechanics, and I invite you to share these images with me.

1. The Wolfskehl Endowment

The birth of quantum mechanics bears a curious relationship with Fermat's Last Theorem of 1637. In this theorem Fermat denied the existence of integers x, y, z, which satisfy the equation, $x^n + y^n = z^n$, for $x, y, z \neq 0$ and $n > 2$. This theorem has not yet been proved, but it is probably the problem for which the greatest number of incorrect 'proofs' has ever been published.

In 1906 the mathematician Paul Wolfskehl from Darmstadt bequeathed a sum of 100,000 Marks to the Royal Academy of Sciences in Göttingen to be given as an award to the first person who, during the next 100 years (i.e. up to 13 September 2007), would publish a complete proof of Fermat's Theorem. In 1908, the Wolfskehl Commission -- consisting of Ehlers, Hilbert, Klein, Minkowski, and Runge -- decided to use the interest on the principal, amounting to 5000 Marks per annum, for the purpose of inviting prominent scientists as guest speakers to Göttingen. There were those who asked Hilbert to submit the proof of Fermat's Theorem himself to win the Wolfskehl Prize, but he laughed it off by saying that, 'One should not kill the goose that lays the golden eggs.'

Henri Poincaré was the first person to be invited to Göttingen under the new arrangement in April 1909. In his first talk, on 22 April, he spoke on Fredholm's equations in connection with the work of G.W. Hill and Helge von Koch. The relevance of this subject to quantum theory was not recognized until 1925. In his last lecture, on 28 April, on 'La mécanique nouvelle', the only one which he gave in French, Poincaré discussed the Theory of Relativity -- incidentally, without mentioning the name of Einstein.[1]

Hendrik Antoon Lorentz was invited the following year. From 24 to 29 October 1910, he delivered six lectures on 'Old and New Problems of Physics', which were subsequently edited by Max Born and published in the Physikalische Zeitschrift.[2] Lorentz devoted last three of these lectures to the problem of the black-body radiation.

In spring 1913, Hilbert organized the Kinetischen Gas Kongress at Göttingen, the lecturers at which were Planck, Nernst, Debye, Lorentz, and Smoluchowski.[3] In the summer semester of 1914, Sommerfeld gave a series of lectures on problems of mathematical physics, and in 1915 Hilbert invited Einstein to Göttingen. During the next three years the distinguished invitees were, respectively, Marian v. Smoluchowski[4], Gustav Mie[5] and Max Planck[6].

At Hilbert's request extra funds from the Ministry of Education were added to augment the income from the interest on the Wolfskehl endowment, making it possible to invite a prominent scientist as a visiting professor for up to a semester each year in the Faculty of Mathematics and Natural Sciences at Göttingen. The first man to be so honoured after World War I was Niels Bohr. The Wolfskehl Commission invited Bohr to lecture at Göttingen in spring 1921 on the problems of atomic theory.[7] Illness prevented him from doing so in 1921, but he delivered his lectures from 12 to 22 June 1922. Later on that year Bohr was awarded the Nobel Prize for Physics, in which he succeeded Einstein.[8]

2. Bohr's Lectures : Atomic Structure

In seven lectures at Göttingen, which came to be called the Bohr Festival, Niels Bohr covered the full range of the theory of atomic structure, beginning with Ernest Rutherford's (1911) nuclear model of the atom and his own attempt, in 1913, to use quantum theory to explain some of the most important features of the atom.[9] For instance, he discussed the formula connecting the discrete frequencies ν of the spectrum of hydrogen with the parameters determining its constitution :

$$\nu = \frac{2\pi^2 me^4}{h^3} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \tag{1}$$

where m and e are the mass and charge of the electron, respectively, h is Planck's constant, and n_1, n_2 are positive integers with $n_1 < n_2$.

During ten years of exciting work since 1913, outstandingly skillful experimentalists and profound theoreticians -- men like Friedrich Paschen and Arnold Sommerfeld -- had prepared the ground for an extremely successful theory which seemed to explain all the known facts about atoms : this was Bohr's theory of the periodic system of elements, based on detailed considerations of their structure. By the end of 1922 this theory was to receive a brilliant confirmation by the experimental work of Dirk Coster and George de Hevesy. Coster and de Hevesy, at Bohr's Institute in Copenhagen, demonstrated the existence of an element with the atomic number Z = 72, whose chemical properties showed a great similarity to those of zirconium and a decided difference from those of the rare earths. Alexandre Dauvillier, working with Maurice de Broglie in Paris, had, a little earlier, assigned the new element to the rare earths based upon his X-ray studies -- but this assignment seemed to be incompatible with Bohr's theory, and the element was appropriately christened Hafnium in honour of its birthplace.

In his Göttingen lectures, Niels Bohr discussed in detail the principles of his theory of atomic structure and their application to actual configurations. The principal idea was that the mechanical orbits of the electrons, moving in the Coulomb field of the nucleus, were determined by the phase integral subject to the quantum condition,

$$\oint p\,dq = nh, \qquad (2)$$

a condition which can be traced back to Planck's treatment of the heat radiation in 1906. Here p is the momentum and q the position coordinate of an electron performing a periodic motion.

Beginning in 1915, Arnold Sommerfeld examined, in his theory of the hydrogen atom, the systematic use of quantum conditions. For instance, in the non-relativistic approximation, the radial quantum number, n_r, and the azimuthal quantum number, n_ϕ, determine the Kepler ellipses of the electron by the equations,

$$\oint p_r dr = n_r h \quad \text{and} \quad \int_0^{2\pi} p_\phi d\phi = n_\phi h. \qquad (3)$$

The sum of both quantum numbers,

$$n = n_r + n_\phi \qquad (4)$$

plays the rôle of the principal quantum number n which had already made its appearance in Bohr's (1913) theory of the Balmer spectrum.

For non-hydrogenic atoms, i.e. atoms having more than one electron, the stability of the orbits could not be guaranteed by the simple quantum conditions, given by Eqs. (2) and (3), because repulsion from the other electrons in the atom causes serious perturbation of the planar motions. The new degrees of freedom, which thus arise, could be described by other quantum numbers, the hope being that the motions in an atom remain multiply periodic.

By carefully estimating all the interactions between the electrons and the nucleus, and between electrons and electrons, Niels Bohr arrived at the explanation of the structure of atoms, including the existence of finite groups of electrons having very similar energies. The maximum number of electrons in these groups seemed to be 2, 8, 8, 18, 18, 32, in agreement with the observed lengths of periods in the system of chemical elements.

In his lectures at Göttingen, Bohr emphasized the physical principles of the theory, especially the two which were most useful : First, Paul Ehrenfest's Adiabatic Principle which stated that by means of an 'adiabatic' change one could transform an allowed motion into another allowed motion ; second, his own Correspondence Principle which made it possible, in the limit of high quantum numbers, to relate all properties of an atomic system determined by the quantum conditions to analogous properties of the system determined by means of classical mechanics. For instance, this consideration of analogy required that the characteristic frequency of the quantum radiation from the atom should pass, in the correspondence limit, into the frequency of the electron's motion around the atom.

3. Bohr's Audience

Bohr's lectures attracted a large audience. All the physicists and mathematicians, young and old, from Göttingen, including Born, Franck and Hilbert, attended his lectures. Many others came to listen from distant universities, such as Ehrenfest from Leiden and Sommerfeld from Munich. Sommerfeld, next to Bohr the foremost representative of atomic theory, had two of his brightest students there. Bohr's lectures had an historic impact upon Wolfgang Pauli and Werner Heisenberg, and perhaps it is no exaggeration to say that quantum theory was the main beneficiary of Wolfskehl's endowment for the encouragement of the proof of Fermat's Theorem.

At one point during his lectures Bohr discussed the calculation of the quadratic Stark effect that had been made by Hendrik Kramers (1920)[10] on the basis of correspondence considerations. Heisenberg raised a serious objection because the result did not agree with any of the classical frequencies of the atom. On the other hand, the phenomenon of quadratic Stark effect could be related to the dispersion of light of small frequencies by bound electrons in an atom ; moreover, in the existing description of dispersion only the classical frequency of the electron's motion always appeared.

Bohr gave an evasive answer to this objection -- the correct answer was in fact not available until spring 1925 -- but he was very impressed with the incisiveness of the young man who had put him on the spot with the question concerning the validity of the correspondence principle in treating the quadratic Stark effect. He invited the young Heisenberg to go for walks with him on the Hainberg in Göttingen to discuss the problems of atomic theory.

Werner Karl Heisenberg was born on 5 December 1901 in Würzburg in Bavaria. He attended the Maximilian Gymnasium at Munich, where he studied classical languages, mathematics, and a smattering of the sciences. In the spring of 1919, as a young man of seventeen, Heisenberg did voluntary sentry duty with the Cavalry Rifle Command during the revolution involving the 'Räterepublik'. During this service he often spent nights on the roof of the Theological Seminary Building, where he brushed up on his Greek by reading with great fascination Plato's Timaeus.[11] Platonic ideas about the structure of matter left a deep impression on his mind and, at times, guided his later views on atomic and particle physics.

Heisenberg entered the University of Munich in fall 1920. Under the friendly guidance of Sommerfeld he was immediately drawn into research on atomic theory : his task was to explain the frequencies of the lines observed in the so-called anomalous Zeeman effects on the basis of Bohr's theory of atomic constitution -- that is, as differences of energy terms determined by quantum numbers. Heisenberg solved the problem by introducing half quantum numbers which had not occurred before in the description of atomic phenomena. Pauli derided Heisenberg's solution by saying : 'Now you introduce half quantum numbers, then you will introduce quarters and eighths as well, until finally the whole quantum theory will crumble to dust in your capable hands.'[12]

Sommerfeld, himself given to the magic of integral quantum numbers, did not particularly appreciate the half-quantum numbers, but he admired Heisenberg's unconventional approach. He invited Heisenberg in helping him to reformulate Woldemar Voigt's 'phenomenological' theory of the anomalous Zeeman effect, and they published a joint paper in 1922 on the intensities of the anomalous Zeeman lines.[13] The results of this work remained valid even after quantum mechanics was discovered. Heisenberg was thus well acquainted with Bohr's theory before he first met him in Göttingen.

For Heisenberg's doctoral dissertation, Sommerfeld gave him a problem of classical hydrodynamics : the transition from laminar into turbulent flow. Heisenberg developed his own approximation methods to deal with the non-linear problem, and showed that the Poiseuille flow between two parallel walls would become unstable if the Reynold's number connected with the problem exceeded the value of about 1000.[14] This result, obtained in 1923, was re-confirmed in 1952 by L.H. Thomas.[15] Heisenberg's early success with the problem of turbulence provided him with a certain love of non-linear theories to which he always remained loyal.

Heisenberg's oral examination for the doctorate was a near disaster. In addition to theoretical physics, in which he was examined by Sommerfeld and did very well, he had to take the examination in experimental physics from Willy Wien (of Wien's law fame). Wien asked Heisenberg questions on the theory of storage batteries and the resolving power of microscopes, telescopes, and the Fabry-Perot interferometer -- questions which Heisenberg could not answer. Wien insisted on failing him, but Sommerfeld's intervention saved the day, and Heisenberg got the lowest possible grade of Rite or just sufficient according to the rules.[16] As a dutiful German student, Heisenberg would learn about the resolving power of optical instruments, and would apply these ideas to his thought-experiment with the γ-ray microscope in relation to the understanding of the uncertainty principle in 1927.

4. The Breakdown of Calculations : Helium Problem and Anomalous Zeeman Effects

Right after the examination in July 1923, totally dejected on account of the onslaught of Willy Wien, Heisenberg went to Göttingen to find shelter under Max Born. Born described his coming as follows : 'He looked like a simple peasant boy, with short, fair hair, clear bright eyes and a charming expression. He took his duties as an assistant more seriously than Pauli and was a great help to me. His incredible quickness and acuteness of apprehension has always enabled him to do a colossal amount of work without much effort. Having finished his hydrodynamical thesis, he worked on atomic problems partly alone, partly in collaboration with me, and helped me to direct my research students.'[17]

With Born, Heisenberg embarked upon a systematic study of complex atoms with the help of the perturbation methods of classical mechanics and astronomy. Born & Heisenberg treated the helium atom as a 'multiply periodic' system.[18] By applying quantum conditions of the form of Eq.(2) they found that the energy states which they calculated did not agree with the experimental data ; the ionization energy of parahelium turned out to be 4 volts higher than the observed value. Heisenberg[19] had already found, however, that if he took the azimuthal quantum number of the ground state to be $\frac{1}{2}$, that is,

$$\oint p_\phi d\phi = \frac{1}{2} h, \tag{5}$$

then the experimental value of 24.6 volts for the ionization energy could be reproduced. Again, the half-quantum number made its appearance.

From Copenhagen, Pauli reported Bohr's reaction, and his own, concerning this matter. Bohr thought that it was the mechanics that was wrong and had to be righted, and one could dispense with the half quantum numbers. Bohr had also suggested to Pauli to study the problem of the anomalous Zeeman effects -- upon which Sommerfeld, Landé and Heisenberg had worked earlier, but Bohr disapproved of their approaches -- and Pauli reported to Heisenberg about his preliminary conclusions on this matter.[20] Heisenberg's exasperation was complete. As he wrote to Sommerfeld : 'I am convinced about the incorrectness of Pauli's ideas, but what I find most terrible is the fact that Bohr considers all that is wrong to be right, and all that is right to be wrong.'[21] Heisenberg was beginning to make the acquaintance of Bohr's dialectical thought and of Pauli's criticism, while learning to cope with the difficulties which atomic theory presented.

Pauli, on the other hand, was also not particularly happy with what atomic physics looked like at the time. Five years earlier, in 1918, the eighteen year old Wolfgang Pauli had gone from Vienna to study physics with Sommerfeld. Already before coming to Munich he had completed a paper on the energy tensor in the gravitational field[22], and by December 1920 he had written his masterly review article[23] on Relativity for the Encyklopädie der mathematischen Wissenschaften, a work which Einstein called 'mature and grandly conceived'[24]. Pauli had also actively engaged himself on problems of atomic theory, completing his doctorate in summer 1921 with a thesis on the hydrogen molecule ion, in which he pointed out the difficulties of applying the known methods.[25] In the fall of 1921 Pauli spent a semester in Göttingen, collaborating with Born on the systematic perturbation theory of complex mechanical systems.[26] Afraid that the atmosphere of Göttingen might make a mathematician of him, he accepted to go to Hamburg as the assistant of Wilhelm Lenz, and in fall 1922 he went to Copenhagen at Bohr's invitation to assist him with the German edition of his long memoir on atomic structure.[27] This was, of course, only an excuse to get Pauli to come to Copenhagen ; Bohr, in fact, needed him to tackle numerous difficult problems of atomic theory, including the anomalous Zeeman effect. It was during this stay that, as Pauli recalled many years later, 'A colleague, who met me strolling rather aimlessly in the beautiful streets of Copenhagen, said to me in a friendly manner, "You look very unhappy," whereupon I answered fiercely, "How can one look happy when he is thinking about the anomalous Zeeman effect?"'[28] Within less than two years he will go on to discover the Exclusion Principle which bears his name.[29] However, by the end of 1924 both Pauli and Heisenberg were convinced that an explanation of the anomalous Zeeman effects of spectral lines could not be achieved by simply introducing half quantum numbers, without invoking really new ideas, such as perhaps Bohr's notion of the 'Unmechanischer Zwang' or non-mechanical stress.

In view of the connection[30] between the multiplet structure of spectral lines, such as the relativistic doublet structure of the lines of the Balmer spectrum and the anomalous Zeeman effect, Pauli declared that the existing quantum theory could not even be relied upon to provide the understanding of the hydrogen atom.

Thus, by about the end of 1924, the joyful confidence which had prevailed at the Bohr Festival in June 1922, had been eroded, and the difficulties of atomic theory seemed to be insurmountable.

5. A Fundamental Problem : The Light-Quantum

Even Bohr was upset. Other difficulties had arisen that assailed his conception of atomic phenomena based on the correspondence principle, especially on account of the discovery of the Compton effect in October 1922.[31] This effect was immediately explained by Compton and, independently, by Debye as the directed scattering of individual light-quanta or photons by electrons, with resultant recoil of the electron, thus conserving momentum and energy in individual atomic processes.[32]

The Compton effect was proof positive of the existence of light-quanta, which had been doubted by many serious physicists (including Max Planck) ever since Einstein introduced them in 1905 and explained the photoelectric effect.[33] Bohr himself had used the emission and absorption of light-quanta in his theory of the hydrogen spectrum merely as a heuristic device, without ever believing in their existence.[34] Like Planck, Bohr believed that a merely 'corpuscular theory of light' would lead to enormous difficulties in explaining electrostatic fields, and one would have to sacrifice some of the proudest achievements of Maxwell's electrodynamics. Bohr did not see how the correspondence limit or the analogy between the light-quantum and classical wave radiation could be established, and he had declared : 'Even if Einstein sends me a cable announcing the proof of the light-quantum, the message cannot reach me because it has to be propagated by electromagnetic waves.'[35]

Yet Bohr was extremely bothered by the problem of explaining the Compton effect without the light-quantum. He was therefore very glad when, towards the end of 1923, the young American from Harvard, John Slater, brought to Copenhagen the idea of the 'virtual oscillator' by means of which Slater attempted to reconcile the discrete theory of light-quanta with the continuous wave theory of the electromagnetic field.[36] On the basis of this idea Bohr, Kramers and Slater[37] developed the outline of a new theory of radiation, which Kramers[38] applied to the theory of dispersion.

The quantum theory of dispersion had originated in 1921 when Rudolf Ladenburg[39] made a successful application of the correspondence principle to the translation into the quantum language of the analysis that was used in the classical theory. In place of the classical electrons in motion within the atom, Ladenburg had introduced into the formulas transitions between stationary states, so that instead of the atom being regarded as a Rutherford planetary system of a nucleus and electrons, obeying the laws of classical dynamics, its behaviour with respect to the incident radiation was predicted by means of calculations based on what Bohr, Kramers, and Slater now called the 'virtual oscillators'.

Kramers[38] immediately extended Ladenburg's dispersion formula by taking into account <u>both</u> types of dispersion effects of atoms in an <u>arbitrary state n</u> : that is, those connected with the absorption of characteristic frequencies, $\nu_a(n+\alpha,n)$, and their subsequent emission, as well as the ones which had not been considered by Ladenburg, i.e. those connected with the emission of frequencies, $\nu_e(n,n-\beta)$, and their subsequent absorption. These frequencies gave rise to a negative contribution in the dispersion formula for the induced electric moment M,

$$M = E\,\frac{2}{h}\sum_{\alpha}\left[\frac{\Gamma_a(n+\alpha,n)\;\nu_a(n+\alpha,n)}{\nu_a^2(n+\alpha,n)-\nu^2} - \frac{\Gamma_e(n,n-\alpha)\;\nu_e(n,n-\alpha)}{\nu_e^2(n,n-\alpha)-\nu^2}\right], \qquad (6)$$

where E is the electric field of incident wave. Γ_e corresponds to Einstein's <u>induced emission</u>[40], and both Γ_a and Γ_e correspond to the absolute squares of the Fourier coefficients, A_τ, of the electric moment of the unperturbed atom, as Born[41] showed in 1924 in a systematic study of Kramers's dispersion theory.

For the incident light of high frequencies, i.e. $\nu \gg \nu_a, \nu_e$, the electron in the hydrogen atom should behave like a free <u>classical</u> electron. The corresponding classical formula for the scattering of X-rays by one electron had been obtained by J.J. Thomson[42] in 1907 as

$$M = - \frac{e^2 E}{4\pi^2 m} \frac{1}{\nu^2} . \qquad (7)$$

By comparing Eqs.(6) & (7) in the high frequency limit, W. Thomas[43] and W. Kuhn[44], independently, obtained the sum rule,

$$\sum_i p_i = \frac{8\pi^2 m}{e^2 h} \sum \left[\Gamma_a \nu_a - \Gamma_e \nu_e \right] = 1, \qquad (8)$$

where p_i is the number of dispersion electrons. The number on the right-hand side is 1 for hydrogen and 2 for helium.

The results of dispersion theory were indeed encouraging, but the conceptual framework in which Bohr had presented the Bohr-Kramers-Slater radiation theory in 1924 got into serious trouble in spring 1925. In describing the dispersion of light waves by atoms it had been assumed that the processes of emission and absorption in atoms, distant from each other, were statistically independent, and that in individual processes of emission and absorption energy and momentum were not conserved, in contradiction to the explanation of the Compton effect on the basis of Einstein's light-quantum. Niels Bohr believed that energy and momentum were only statistically conserved in atomic processes. Even when this notion was first proposed in 1924, men like Einstein and Pauli, who believed in strict energy-momentum conservation as the divine plan of an orderly universe, regarded Bohr's idea as being nothing short of immoral. This indeed was the beginning of the Einstein-Bohr discussions concerning the statistical interpretation of quantum theory that were yet to come.

In April 1925 Walther Bothe and Hans Geiger obtained the results of their coincidence experiment, showing that the secondary Compton radiation indeed emerged after scattering by a single electron.[45] This simple result represented the demise of the radiation theory of Bohr, Kramers and Slater, and the triumph, not only of virtue in the form of energy-momentum conservation, but of its modern vehicle, the light-quantum. Einstein was convinced that it had to be so, and was glad that it was so. Bohr wrote a touching letter to Rutherford about the terrible difficulties of physics, or of physics as he had conceived it to be, and told him how miserable he was.[46]

In the same month, April 1925, Werner Heisenberg began to ponder about calculating the intensities of hydrogen lines by the 'sharpened' application of a correspondence principle, which had been employed successfully in dispersion theory.

6. Sharpening the Correspondence Principle

Heisenberg had gone for his first visit to Copenhagen at Easter 1924. He had looked forward to criticizing Bohr's methods and results in atomic theory. Before he had had the chance, however, Bohr took him on a walking tour of Denmark, showing him the sights and talking to him about history and philosophy, and finally physics. Heisenberg was charmed. Well, he had known Sommerfeld -- who was a great teacher and a good man, but after all he was a 'Geheimrat'. And Born -- again, Born was a good formalist, a friendly man, but rather distant. Bohr was it. He was friendly, inspiring, kind, and he had thought about the problems of atomic physics like no one else. Heisenberg had gone to Copenhagen to battle against the correspondence principle with the prophet himself ; instead he became its evangelist.

Heisenberg returned to Copenhagen for six months in fall 1924. He worked with Bohr and Kramers on specific problems, in which he sought to formulate the content of the correspondence principle in terms of equations from which new physical results could be derived. For instance, he treated the problem of the polarization of resonance fluorescence light emitted by atoms.[47] Together with Kramers, Heisenberg extended Kramers' dispersion formula, Eq.(6), to the incoherent scattering of light by atoms, that is, to cases in which the frequency ν of the scattered light is changed, and is given by,

$$\nu' = \nu \pm \nu_{qu}, \tag{9}$$

where ν_{qu} is one of the characteristic frequencies of the atom.[48]

The successes thus obtained by what he called the sharpening [Verschärfung] of the correspondence principle increased Heisenberg's confidence in the Copenhagen approach, and he hoped, as he recalled later, that 'Perhaps it would be possible one day, simply by clever guessing, to achieve the passage to a complete mathematical scheme of quantum mechanics.'[49]

In April 1925 Heisenberg returned to Göttingen to take up his duties as Privatdozent during the summer semester.

7. Heisenberg's New Scheme

In Göttingen Heisenberg sought to guess the intensities of the hydrogen lines, but in this specific problem he failed. He concluded that the difficulties arising from the rules of quantization were of a more general nature and had to be treated first. These difficulties were due, not so much to a departure from classical mechanics, but rather to a breakdown of the kinematics underlying this mechanics. In his search for the new kinematics, Heisenberg employed a completely new idea : he assumed that the equation of motion of an electron, say

$$\ddot{x} + f(x) = 0 , \tag{10}$$

could be retained, but the kinematical interpretation of the quantity x as a position depending on time had to be rejected. Now what kind of quantities should be substituted in the equation of motion ?

In a classical periodic motion x(t) can be expanded in a Fourier series,

$$x(t) = \sum_{\alpha=-\infty}^{\infty} a_\alpha \; e^{i\alpha\omega t} . \tag{11}$$

In quantum theory, the coefficients a_α and the frequency ω depend on a quantum number n. Therefore, instead of Eq.(11), Heisenberg wrote x(t) as

$$x(t) = \sum_{\alpha=-\infty}^{\infty} a_\alpha(n) e^{i\alpha\omega_n t}. \qquad (12)$$

He then replaced the terms of the Fourier series in Eq. (12) by a new kind of terms,

$$a(n,n-\alpha) \; e^{i\omega(n,n-\alpha)t}, \qquad (13)$$

which correspond to the transition from n to n-α ; the time factor ω(n,n-α) is 2π times the frequency of light in this transition. For Heisenberg, the main problem was the calculation of the intensity of radiation emitted in a transition P → Q. He knew that this intensity is proportional to Einstein's emission probability, A_{PQ}, and he assumed this probability to be proportional to the absolute square of a, that is

$$A_{PQ} \propto |a(n,n-\alpha)|^2. \qquad (14)$$

He motivated the introduction of a(n,n-α) by saying that the intensities and, therefore, $|a(n,n-\alpha)|^2$, are observable, in contrast to the functions x(t).

From the classical combination law of frequencies,

$$\nu(n,\alpha) = \nu(n,\beta) + \nu(n,\alpha-\beta), \qquad (15)$$

re-interpreted quantum-theoretically by Heisenberg as,

$$\nu(n,n-\alpha) = \nu(n,n-\beta) + \nu(n-\beta,n-\alpha), \qquad (16)$$

it became 'almost inevitable' ['nahezu zwangslaüfig '] to require that the coefficients C(n,n-α) of the product of two re-interpreted Fourier series,

$$x(t)y(t) = \sum_{\alpha} C(n,n-\alpha)\ e^{i\omega(n,n-\alpha)t}, \tag{17}$$

should obey the product rule,

$$C(n,n-\alpha) = \sum_{\beta} A(n,n-\beta)\ .\ B(n-\beta,n-\alpha). \tag{18}$$

By this re-interpretation the correspondence principle was incorporated into the very foundations of his kinematical scheme.

Heisenberg noticed that Eq.(18) introduced a great new difficulty : whereas in classical theory $x(t)y(t)$ is always equal to $y(t)x(t)$, this is not necessarily the case with the definitions (17) & (18). Therefore, he concluded that, in general, it was not clear how to formulate a product of two dynamical variables in quantum theory.

Rather than being discouraged by the unusual situation which had never before occurred in physics, and which he did not comprehend, Heisenberg looked for an example in which he could employ his quantum-theoretical re-interpretation [‘Umdeutung’] of classical mechanical quantities by avoiding the new difficulty concerning the product. He chose the anharmonic oscillator, described in classical theory by,

$$\ddot{x} + \omega_0^2 x + \lambda x^3 = 0, \tag{19}$$

in which the perturbation term, λx^3, involves only the products of $x(t)$'s. Assuming the perturbation term to be a small correction, he decided to employ the classical perturbation method, that is, he used the Ansatz,

$$x(t) = a_1 \cos(\omega t) + \lambda a_3 \cos(3\omega t) + \lambda^2 a_5 \cos(5\omega t) + \ldots, \tag{20}$$

and re-interpreted it quantum-theoretically as,

$$x(t) = a(n,n-1)\cos\left[\omega(n,n-1)t\right] + \lambda a(n,n-3)\left[\cos\omega(n,n-3)t\right] +\ldots. \quad (21)$$

He also expanded the frequencies in a power series in λ as

$$\omega(n,n-1) = \omega_0(n,n-1) + \lambda\omega_1(n,n-1) + \ldots\text{etc.} \quad (22)$$

By substituting these assumptions, i.e. Eqs.(21) and (22), in Eq.(19), he obtained, for $\lambda = 0$,

$$\left[\omega_0^2 - \omega^2(n,n-1)\right] a(n,n-1) = 0, \quad (23)$$

the harmonic oscillator solution, and, in the first approximation,

$$\left[\omega_0^2 - \omega^2(n,n-3)\right] a(n,n-3) + a(n,n-1)a(n-1,n-2)a(n-2)(n-3) = 0 \quad (24)$$

for the anharmonic oscillator.

Heisenberg found that the $a(n,n-\alpha)$'s, the '<u>transition amplitudes</u>' as he called them, as solutions of the equations of motion (23) or (24), were determined only upto a constant, and he did not know what to do with this constant. This was the beginning of June 1925, and his programme was stuck.

8. Helgoland and the Joy of Discovery

With the coming of spring in 1925 Heisenberg had developed a case of severe hay fever which would just not leave him, and he decided to take a week or ten days off in June 1925 at the rocky island of Helgoland in the North Sea. At Helgoland, not only did he cure his hay fever but wiped the nose clean of the chronic colds of erstwhile problems of atomic mechanics. It was of this discovery that Dirac later said : 'We were both young men at the same time, working on the same problem. He succeeded where I failed.'[50]

At Helgoland Heisenberg divided his time in taking long walks, reading Goethe's West-Östlicher Divan, and seeking to give his vague ideas on quantum mechanics a more definite shape.[51] There he solved two problems.

First, he had to obtain the quantum condition in the new scheme that would be equivalent to Eq.(2). In one dimension, Eq.(2) can be written as

$$\int m\dot{x}\, dx = J = nh, \qquad (25)$$

integrated over a full period of the motion. Substituting the Fourier series, Eq.(12), for x, he obtained,

$$nh = 2\pi m \sum_{\alpha=-\infty}^{\infty} \left|a_\alpha(n)\right|^2 \alpha^2\omega_n. \qquad (26)$$

Heisenberg replaced this formula by the one obtained by differentiation w.r.t. n, that is,

$$h = 2\pi m \sum_{\alpha=-\infty}^{\infty} \alpha \frac{d}{dn} (\alpha\omega_n \left|a_\alpha\right|^2), \qquad (27)$$

where the expression within the parenthesis is defined for n integer only. Heisenberg regarded this replacement as being more natural from the viewpoint of the correspondence principle. Using Eq.(27) as an intermediate step, he replaced the derivative with a difference and obtained,

$$h = 4\pi m \sum_{\alpha=0}^{\infty} \{\left|a(n,n+\alpha)\right|^2 \omega(n+\alpha,\alpha) - \left|a(n,n-\alpha)\right|^2 \omega(n,n-\alpha)\}. \qquad (28)$$

Eq.(28) is Heisenberg's quantum condition, and is equivalent to the sum rule of Kuhn and Thomas (Eq.(8)). Since for the ground state no transition is possible, one has to put

$$a(n,n-\alpha) = 0, \tag{29}$$

if n is the quantum number of the ground state. In Eq.(14) Heisenberg had already assumed the squares, $|a(n,n-\alpha)|^2$, as being proportional to the probabilities of transitions $n \rightarrow n-\alpha$.

The derivation of the quantum condition (28) and the consequent determination of the transition amplitudes was thus the first problem that was solved. The second problem, still nagging Heisenberg, was whether energy conservation will hold in the new scheme. After all, this question had become vital after the outcome of the Bothe-Geiger experiment.

In classical mechanics, the conservation of energy immediately follows from the equation of motion of the anharmonic oscillator, Eq.(19), which, when multiplied by $m\ddot{x}$ can be written as,

$$\frac{d}{dt} H = \frac{d}{dt}\left[\frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega_0^2 x^2 + \frac{1}{4} \lambda x^4\right] = 0, \tag{30}$$

where H, the quantity within the bracket, is evidently the energy and is conserved. Such a relation need not be satisfied when Eq.(30) is re-interpreted quantum-theoretically, and Heisenberg went through the calculation of the terms up to the second order in λ, making errors along the way and rechecking them. He found that no time-dependent terms remained in the kinematically re-interpreted Hamiltonian.

The example of the anharmonic oscillator showed him that a dynamical problem in quantum theory could be solved with the help of his scheme. As he recalled : 'It was almost three o'clock in the morning before the final result of my computations lay before me. The energy principle had held for all the terms, and I could no longer doubt the mathematical consistency and coherence of the kind of quantum mechanics to which my calculations pointed. At first, I was deeply alarmed. I had the feeling that, through the surface of atomic phenomena, I was looking at a strangely beautiful interior, and felt almost giddy at the thought that I now had to probe this wealth of mathematical structures nature had so generously spread out before me. I was far too excited to sleep, and so, as a new day dawned, I made for the southern tip of the island, where I had been longing to climb a rock jutting out into the sea. I now did so without too much trouble, and waited for the sun to rise.'[52] Heisenberg was happy. With this happiness, the blissful experience of new knowledge, though still not recognizing fully that in his quantum-theoretical scheme the key had been found to solve atomic problems consistently, Heisenberg returned to Göttingen.

On the way back he stopped to see Pauli in Hamburg. Pauli was his critical genius, and he had learned to respect Pauli's critical faculties since their first encounter in Sommerfeld's Seminar in Munich in 1920. Pauli encouraged him to go on. During the next couple of weeks Heisenberg exchanged several letters with him, and on 9 July 1925 sent him the manuscript of the finished paper. Pauli's opinion was favourable : 'It was the Morgenröte ,' the beginning of the dawn in quantum theory, he said.

Having received Pauli's favourable verdict, Heisenberg gave the paper[53], around the middle of July, to Max Born, and asked him to do with it what he thought was right.

9. The Matrix Formulation of Quantum Mechanics

When Born read Heisenberg's paper he was, as he has said, just 'fascinated'. 'I began to ponder about his symbolic multiplication,' he said, 'and was soon involved in it that I thought the whole day and could hardly sleep at night... In the morning I suddenly saw the light : Heisenberg's symbolic multiplication was nothing but the matrix calculus, well-known to me since my student days from the lectures of Rosanes in Breslau.'[54]

Born put Heisenberg's quantum condition, Eq.(28), into the matrix notation as

$$\sum_{k} \left[p(nk)q(kn) - q(nk)p(kn) \right] = \frac{h}{2\pi i}, \qquad (31)$$

and determined that the two matrix products pq and qp were not identical. Born guessed that the non-diagonal elements of the matrix pq - qp were zero, and the quantum condition could be written, in general, as

$$pq - qp = \frac{h}{2\pi i}\, 1 \qquad (32)$$

(where 1 is the unit matrix), but it was only a guess and he could not prove it. The proof was given, independently, by Jordan and Dirac.

A couple of days later, on 19 July 1925, Born travelled from Göttingen to Hanover to attend a meeting of the German Physical Society, where Pauli also came from Hamburg. At the railway station he told Pauli about the matrices and his difficulties in finding the value of the non-diagonal elements. Born invited Pauli to collaborate with him, to which Pauli gave a sarcastic refusal by saying to Born : 'Yes, I know that you are fond of tedious and complicated formalism. You are only going to spoil Heisenberg's physical ideas by your futile mathematics.'[55]

Pauli genuinely believed that the new quantum theory was Knabenphysik, or the youngman's game, such as Heisenberg and himself, and that Born should better stick to his work on crystal lattices. He never really forgave Born for what he believed was 'poaching' on the new territory. To Kronig Pauli wrote : 'Heisenberg's mechanics has again given me hope and joy in living [Mind you, here is a young man just turned twenty five]. It has not yet brought the solution of the riddle, but I believe that it is again possible to go forward. One must first seek to liberate Heisenberg's mechanics from Göttingen's [i.e. Max Born's] deluge of formal learning [formalen Gelehrsamkeitsschwall] and better expose its physical essence.'[56] Pauli himself decided not to interfere in the development of Heisenberg's ideas and plans.[57]

On his return from Hanover, Born immediately persuaded Jordan to help him in his work, which led to Born and Jordan's matrix formulation of quantum mechanics, being completed on 27 September 1925.[58] This paper contained a résumé of matrix methods, the interpretation of Heisenberg's symbolic multiplication, the proof of the formula for the product difference of pq and qp, Eq.(32), proof of energy conservation, and the proof of Bohr's frequency condition. It already contained an attempt, made entirely by Jordan, at the quantization of the electromagnetic field by regarding its components as matrices. This paper was the systematic formulation of matrix mechanics, in which the Göttinger Gelehrsamkeit served in providing method to madness.

Further development towards the completion of the matrix scheme of quantum mechanics began immediately afterwards in the collaboration of Born, Heisenberg and Jordan. This collaboration began when Jordan wrote a letter to Heisenberg early in September 1925 -- Heisenberg was in Copenhagen for a few weeks before he returned to Göttingen for the winter semester -- with Heisenberg and Born and Jordan, all contributing their bits. The general editing of the paper was done by Jordan, and the leading introduction was written by Heisenberg. This paper by Born, Heisenberg and Jordan, was thus the third paper in the series after Heisenberg's discovery[53], and it gave a logically consistent exposition of matrix mechanics. It was completed by the end of October 1925, and is usually called the 'Drei-Männer Arbeit'.[59]

It was really a learned paper, bringing in all the Gelerh-samkeitsschwall of Göttingen -- eigenvalues and eigenvectors, canonical transformations, principal axis transformation, Hilbert's quadratic forms in an infinite number of variables, general commutation relations, and physical applications -- including the quantization of the electromagnetic field and the calculation of fluctuations in this field by Jordan[60]. This paper contained essentially the entire apparatus of modern matrix mechanics, and is one of the most learned papers in scientific literature.

For all we know, Pauli might have wished that he had participated in this game ; as it was, he remained a spectator of this development until he appointed himself its referee. As a critic of the formal approach, he sent blasts of sarcasm in letters to Heisenberg and Kronig, then at Copenhagen, against the methods of Born, Heisenberg and Jordan.[56] On reading these, Heisenberg , for the first time irritated by Pauli, wrote to him in the middle of October 1925 : 'Your eternal reviling of Copenhagen and Göttingen is a flagrant scandal. You will have to allow that, in any case, we are not seeking to ruin physics out of malicious intent. When you reproach us that we are such big donkeys that we have never produced anything new physically, it may well be true. But then, you are also an equally big jackass because you have not accomplished it either.'[61]

These words probably touched Pauli deep enough. He took up the problem of the hydrogen atom and solved it within the next few weeks by means of matrix methods, employing all the formal mathematical learning against which he had complained earlier. He made an ingenious application of the integration method which Wilhelm Lenz had used earlier for determining the effect of crossed electric and magnetic fields on the energy states of the hydrogen atom in the Bohr-Sommerfeld theory.[62] With the help of the 'Lenz vector' Pauli obtained the Balmer formula and showed how the situation with respect to the forbidden orbits could now be understood naturally.[63] It was exactly two years since Pauli had first seriously doubted Bohr's theory of the hydrogen atom, and now one had come round a full circle. This was indeed a triumphant moment for the new quantum mechanics, and Niels Bohr celebrated it by writing another letter to Rutherford, informing him that the reasons for his misery in the previous spring had now disappeared.[64]

10. Non-Commutation and the Poisson Bracket : Dirac's Discovery

Just before the Born, Heisenberg, Jordan paper was published in the Zeitschrift für Physik in January 1926, another paper, containing the complete scheme of quantum mechanics, made its appearance in the Proceedings of the Royal Society.[65]

Let us go back to the moment in July 1925 when Heisenberg gave his paper on quantum-theoretical kinematics to Max Born.

Immediately after depositing his paper with Born, Heisenberg left for Leyden and Cambridge.

In Leyden he stayed and discussed physical problems with Paul Ehrenfest. Ehrenfest was a perceptive man, more inquiring and critical in his attitude to physics than creative. In his first encounter with Pauli, Ehrenfest had declared : 'Herr Pauli, I happen to like your papers better than I like you', to which Pauli responded : 'Strange ; with me, regarding you, it is just the opposite.' Ehrenfest cared greatly for the progress of his research students who, at that time, included Uhlenbeck and Goudsmit. Heisenberg discussed all about spectroscopic problems with them. Earlier that spring he had completed another paper on the anomalous Zeeman effect[66], and Uhlenbeck and Goudsmit were very excited as to what Pauli's exclusion principle and the assignation of a new magnetic quantum number to the electron, in addition to Heisenberg's work on the anomalous Zeeman effect, meant for the organization of spectroscopic terms. They would soon postulate the hypothesis of electron spin[67], thereby giving meaning to the half quantum numbers which had first made their appearance in Heisenberg's work on the anomalous Zeeman effect. The spin also explained what Pauli had called the 'two-valuedness of the electron' not describable classically.[68]

From Leyden Heisenberg went to Cambridge, where he stayed with R.H. Fowler with whom he had become acquainted in Copenhagen. On 28 July 1925, Heisenberg addressed the Kapitza Club, the membership of which was limited to the students and friends of Peter Kapitza. The subject of his talk was 'Term Zoology and Zeeman Botany', dealing with the enormous difficulties of masterminding the details of atomic spectroscopy with the help of ad hoc rules.[69] It is remarkable that Heisenberg chose to speak on this subject, even though he seems to have found the solution of the quantum riddle recently. He was apparently not certain that the solution was really in hand. However, he did talk privately to Fowler about his new scheme.

Paul Dirac, then a research student of Fowler's in Cambridge, probably attended Heisenberg's seminar, and he himself gave a talk at the Kapitza Club one week later.[70] Fowler received the proof-sheets of Heisenberg's paper[53] at the beginning of September 1925, found it interesting, but was a bit uncertain about it and wanted to know what Dirac's reaction would be. Dirac has said : 'I was so impressed then with the Hamiltonian formalism as the basis of atomic physics, that I thought anything not connected with it would not be much good. I thought there was not much in it [i.e. in Heisenberg's paper] and I put it aside for a week or so.'[71]

When Dirac went back to it, it suddenly became clear to him that Heisenberg's idea had provided the key to the 'whole mystery'. During the following weeks Dirac tried to connect Heisenberg's quantum-theoretical re-interpretation of kinematical quantities with the action-angle variables of the Hamilton-Jacobi theory. 'I worked on it intensively from September 1925,' Dirac said. 'During a long walk on a Sunday it occurred to me that the commutator might be the analogue of the Poisson bracket, but I did not know very well then what a Poisson bracket was. I had just read a bit about it, and forgotten most of what I had read. I wanted to check up on this idea, but I could not do so because I did not have any book at home which gave Poisson brackets, and all the libraries were closed. So I had just to wait impatiently until Monday morning when the libraries were open and check on what Poisson bracket really was. Then I found that they would fit, but I had one impatient night of waiting.'[72]

From the very beginning Dirac's clarification of the relationship between Heisenberg's variables and the classical variables made the formulation look more classical, and at the same time it very cleanly isolated the small point at which the reformulation had to make a break with the classical theory.

From the quantum conditions expressed in angular variables Dirac found the correspondence between Heisenberg's commutation brackets and the classical Poisson brackets for the variables X and Y,

$$XY - YX = i\hbar\sum_{r}\left\{\frac{\partial X}{\partial q_r}\frac{\partial Y}{\partial p_r} - \frac{\partial Y}{\partial q_r}\frac{\partial X}{\partial p_r}\right\}, \qquad (33)$$

where q_r and p_r can be regarded as the action-angle variables (w_r and J_r).

Dirac was now safely back on Hamiltonian ground. He showed his new results to Fowler who fully appreciated their importance. Fowler knew what was going on in Copenhagen and Göttingen, and he realized that there would be competition from these places.

He thought that the results obtained in England in this field had to be published at once, and urged the Proceedings of the Royal Society to give immediate priority to the publication of Dirac's paper on 'The Fundamental Equations of Quantum Mechanics.'[65] Sir James Jeans, who was then editor of the Proceedings and Secretary of the Royal Society, was ready and willing to oblige. All of Dirac's papers from 1925 to 1933 were thus published very fast.

In his fundamental paper[65] Dirac first gave a summary of Heisenberg's ideas, simplifying the mathematics and making it at once more elegant. He anticipated all the essential results of the papers of Born and Jordan[58] and Born, Heisenberg and Jordan[59]. He developed a quantum algebra, derived Heisenberg's quantization rules, and obtained the canonical equations of motion for quantum systems. In the same paper, Dirac introduced an early form of creation and annihilation operators, pointing out their analogues in classical theory.

Dirac quickly followed this paper by another a few weeks later.[73] In it he developed the algebra of q-numbers, that is, the dynamical variables which satisfy all the rules of normal numbers except their product is not necessarily commutative. He gave detailed theorems on the operations with q-numbers, and applied the rules he had obtained to multiply-periodic systems in close analogy with the old quantum rules.

Dirac's aim was to apply his scheme to the hydrogen atom. He wrote its Hamiltonian by simply replacing position and momentum variables in the classical Hamiltonian by q-numbers, and proceeded to obtain the Balmer formula in order to show that this abstract scheme could give results closely related to the experiments. Dirac, however, did not go into the details of this calculation as Pauli[63] (in his paper published during the same month, March 1926) had already shown that this could be done, and Dirac mentioned it in a footnote.[74] He then went on to calculate the various features of the splitting and intensities of spectral lines in a magnetic field (including Zeeman effect) in agreement with the experiments.

With all this work on the principles of quantum mechanics Dirac was awarded the Ph.D. degree in May 1926 at Cambridge.[75]

With the concept of spin now available, and acceptable even to Pauli who had resisted it for long, Heisenberg and Jordan, in spring 1926, solved the problem of the anomalous Zeeman effect in the matrix scheme of quantum mechanics.[76] There was at least a symbolic satisfaction in doing so, for Heisenberg had taken his first unsure steps in quantum theory by working on the problem of the anomalous Zeeman effect, and now it seemed all right to show that one could cover the same steps by running.

This sequence of events relates only to a part of the birth of quantum mechanics, for it was a twin birth. It would, however, be inappropriate for me to embark on the development of wave mechanics this afternoon. Yet the occasion calls for a summary of the events in order to conclude our story. Besides, Schrödinger's wave mechanics and ideas related to it completed the formulation of the rational theory of atomic phenomena which began with Heisenberg's discovery of a quantum kinematics.

11. Quantization As An Eigenvalue Problem

Since 1921 Erwin Schrödinger had been at the University of Zurich, where he occupied the chair of theoretical physics which Einstein had once, albeit briefly, warmed[77]. Schrödinger was a Viennese and a man of vast personal culture that included the study of Greek literature and philosophy in the original and the writing of poetry. A distinguished physicist by any measure, Schrödinger traced his scientific lineage to Boltzmann through his teacher Fritz Hasenöhrl, but he had himself not yet set the world aglow although he had done excellent work on problems of Brownian motion, specific heat and quantum statistics, and of general relativity. By summer 1925 Schrödinger had become tired of his stay in Zürich because, as he wrote to Sommerfeld, 'die Schweizer sind gar zu ungemütlich' ['the Swiss are just too uncogenial'] and he wanted to go back home to Austria.[78] He was negotiating for the chair at Innsbruck, but since the University of Innsbruck sought to dicker about the salary he let the matter, rather the chair, drop in favour of Arthur March. Within eighteen months Schrödinger would be appointed as Max Planck's successor at the University of Berlin.

In the fall of 1925 Schrödinger suffered not only from the lack of congeniality of his colleagues in Zurich, but the work of Heisenberg and of Born and Jordan on matrix mechanics added to his discomfort, for he remarked :'...I was discouraged ("abgeschreckt"), if not repelled ("abgestossen"), by what appeared to me a rather difficult method of transcendental algebra, defying any visualization ("Anschaulichkeit")'.[79] He decided to sublimate his social and scientific unhappiness by conceiving and delivering a scheme of atomic mechanics which not only seemed to be a genuine alternative to the matrix or q-number mechanics of Heisenberg, Born, Jordan, and Dirac, but helped in completing the edifice of quantum mechanics and in inaugurating the discussions that led to its physical and philosophical interpretation.

In four communications to the Annalen der Physik, submitted from the end of January to end of June 1926, Schrödinger developed his theory of wave mechanics, entitled 'Quantization as an Eigenvalue Problem'. Without any ado he presented his fundamental equation,

$$H\left(q, \frac{h}{2\pi i} \frac{\partial}{\partial q}\right) \psi(q) = E\psi(q) \qquad (34)$$

and solved the problem of the spectrum of the hydrogen atom.[80] In the mathematical aspects of some of his work he had invaluable help from Hermann Weyl, then also in Zurich at the E.T.H., and Schrödinger acknowledged it.[81] Weyl's 1908 thesis[82] under Hilbert had dealt with integral equations, eigenvalue problems, orthogonal functions, etc., and it was a fortuitous combination of circumstances that brought Schrödinger and Weyl together. Weyl had been invited in 1917 to become Felix Klein's successor in Göttingen, but he had refused ; in 1930 he would not be able to refuse the call to become Hilbert's successor in Göttingen and leave Zurich and the E.T.H. which, in parting, he called 'Wartesaal erster Klasse' /a first class waiting room/. In the altered political circumstances of 1933 the distinguished intellectual stations of Continental Europe would be emptied in favour of Cambridge, Dublin, Princeton, and a hundred other more modest distant waiting rooms of scholarship. But in 1926 it was good to be in Zurich and witness the rapid development of wave mechanics.

In his communications Schrödinger provided the basis of treating all those problems of atomic physics that had been impossible to handle in the Bohr-Sommerfeld theory. In Schrödinger's work the fundamental ideas of Einstein and Louis de Broglie[83] found a natural place. Schrödinger soon recognized that in spite of fundamental disparities the two approaches, his own and Heisenberg and Born's, did not clash but rather complemented each other. In fact, in the early spring of 1926, prior to the publication of his third communication, Schrödinger discovered what he called 'a formal, mathematical identity' of wave mechanics and matrix mechanics.[84] The same formal equivalence was demonstrated, independently, by Carl Eckart[85] in the United States and by Pauli[86] in a letter to Jordan.

Now Heisenberg had believed throughout that the solution of the problems of atomic mechanics would lead to one, unique, general mathematical scheme, and when he discovered his scheme, well, that was it. With the arrival of Schrödinger's theory, Heisenberg was unhappy, and he believed, indeed hoped, that it was wrong.[87] When in June 1926 Born[88] applied the Schrödinger method to treat collision problems, a work which led to the statistical interpretation of Schrödinger's wave function, Heisenberg reproached him for going over 'to the enemy camp'.[89] And to Pauli, he wrote : 'The more I ponder about the physical part of Schrödinger's theory, the more disgusting ["desto abscheulicher"] it appears to me.'[90] This was, of course, not a matter of just wishing things away. Heisenberg genuinely believed that Schrödinger's claim, that the absolute square of the wave function, $|\psi|^2$, described the charge distribution of the electron in space, was absolutely wrong. Still, Heisenberg did not feel particularly unhappy when, during June and July 1926, he successfully applied Schrödinger's theory to treat the helium problem.[91]

On 23 July 1926 Schrödinger gave a lecture in Munich on 'The Fundamental Ideas of a Wave-Theoretical Atomic Physics'[92], in which he presented his views in detail. Heisenberg, who attended this lecture, discussed many questions with Schrödinger and frankly told him that, on the basis of his interpretation, he (Schrödinger) would not be able to derive even Planck's law ; whereupon Willy Wien, who was also present, told Heisenberg sharply that while 'he understood my regrets that quantum mechanics was finished, and with it all such nonsense as quantum jumps, etc., the difficulties I had mentioned would undoubtedly be solved by Schrödinger in the very near future.'[93]

12. Uncertainty and Complementarity

With Born's statistical interpretation[88] of the wave function in hand in July 1926, serious and prolonged discussions began about the fundamental physical meaning of quantum mechanics as represented by the two schemes. Their equivalence was established rigorously by the transformation theory of Dirac[94], Jordan[95] and Fritz London[96] by late fall 1926, and the question of physical interpretation became paramount. Bohr, Heisenberg, Pauli and Schrödinger, primarily, took part in these discussions.

The problem of the interpretation of quantum theory had occupied Niels Bohr increasingly since 1923 when the question of the nature of radiation became crucial for the understanding of the Compton effect. For Heisenberg, who had eagerly pressed forward by abandoning the use of classical concepts such as electron orbits in atoms, the problem of the interpretation arose late in 1925 when he thought about the simultaneous existence of the discrete energy spectrum of electrons bound in atoms and the continuous spectrum of free electrons moving along well-defined paths. It now occurred to him that, in some sense which was not yet clear, a space-time description should also be possible for the electrons in atoms.

In the fall of 1926 Heisenberg returned to the question of the space-time description of electron's behaviour in the atom. Pauli pointed out to him that Schrödinger's wave function could be considered in momentum space, as $\psi(p)$, just as well as in coordinate space, as $\psi(q)$, to which Heisenberg responded : 'The fundamental equivalence of p and q pleases me very much. Thus, in the wave-formulation, the equation $pq - qp = \frac{h}{2\pi i}$ always corresponds to the fact that it makes no sense to speak of a monochromatic wave at a definite moment (or in a very small time-interval).'[97] At this place, in the margin of the letter, Pauli noted : 'It also makes no sense to speak of a state (energy) in a time-interval which is small compared to the period [because the state or the energy can be defined only over the entire period].' Heisenberg continued : 'If the [spectral] line may be taken as being not too sharp, i.e. the time-interval is not too small, that of course makes sense. Analogously, there is no point in talking about the position of a particle of a definite velocity. However, it makes sense if one does not consider the velocity and the position too accurately. It is quite clear that, macroscopically, it is meaningful to talk about the position and velocity of a body.'[97]

Thus far, Heisenberg had only vaguely formulated his ideas about a 'coarse' space-time description, reflecting his new understanding based upon wave mechanics. In fall 1926 Heisenberg was in Copenhagen, where he had taken up his new duties as a lecturer as the successor of Kramers who had been appointed to a professorship in Utrecht. Bohr, with whom he discussed daily, had been developing his own approach to the problem of the interpretation by emphasizing the duality of the wave and particle pictures in quantum theory. Heisenberg preferred to abide by the quantum mechanical scheme, as formulated by Born, Heisenberg, and Jordan, and by Dirac ; he believed that the wave features should be brought in only by means of the transformation theory which Dirac[94] had worked out in Copenhagen in fall 1926.

Dirac had shown conclusively that the matrix S, employed in solving the problem of the principal axis transformation in the case of a Hermitian Hamiltonian function H(p,q), could be identified with Schrödinger's wave function. In other words, for each column vector, there exists the identity

$$S_{q,E} = \psi_E(q), \tag{35}$$

where E is the discrete or continuous eigenvalue of the energy matrix. In order to handle the problem of continuous indices Dirac introduced the delta-function, δ, as

$$1(\alpha'\alpha'') = \delta(\alpha'-\alpha''), \tag{36}$$

having the property that

$$\int d\alpha'' \ \delta(\alpha'-\alpha'') \quad f(\alpha'') = f(\alpha'), \tag{37}$$

and its derivative, $\delta'(\alpha'-\alpha'')$, defined as

$$\int d\alpha'' \ \delta'(\alpha'-\alpha'') \quad f(\alpha'') = \frac{\partial f(\alpha')}{\partial \alpha'} . \tag{38}$$

The momentum p, conjugate to a continuous position variable q, could then be written formally as

$$p(q',q'') = \frac{h}{2\pi i} \delta'(q'-q'') = \frac{h}{2\pi i} \ \frac{\partial}{\partial q'} , \tag{39}$$

and the Born-Jordan matrix equation for diagonalizing the Hamiltonian H,

$$H(q,p)\ S_E(q) = E\ .\ S_E(q), \tag{40}$$

could thus be transformed into Schrödinger's wave equation,

$$H(q, \frac{h}{2\pi i}\ \frac{\partial}{\partial q})\ \psi_E(q) = E\psi_E(q). \qquad (34),(41)$$

The discussions, sometimes stormy, between Bohr and Heisenberg about the physical interpretation of quantum mechanics continued during the winter months of 1926-27. About mid-February 1927 Bohr left for a skiing vacation in Norway, while Heisenberg stayed on in Copenhagen. He made an effort to bring some order into his thoughts and results of the past few months, and on 23 February 1927 he wrote a long letter to Pauli in which he dealt with the problem of observing simultaneously the position and momentum of atomic systems. He stated that the 'commutation relation', $pq - qp = \frac{h}{2\pi i}$, has the following physical interpretation : Given the exact momentum p of an electron in an atom, its position is then completely undetermined, and vice versa.[98] To support this point of view, and to render it more visual, Heisenberg briefly discussed the Gedankenexperiment for the observation of an electron by means of a γ-ray microscope, an analogy which occurred to him from his doctoral oral examination under Wien several years before. Then he turned to the exact calculation of the accuracy involved in the observation of p and q.

The probability amplitude of the position of an object, which lies within the space-interval $q_0 - q_1 < q < q_0 + q_1$, is given by

$$S(q) = \text{const. } \exp\left[\frac{-(q-q_0)^2}{2q_1^2} - \frac{2\pi i p_0 (q-q_0)}{h}\right], \qquad (42)$$

where the first term represents a Gaussian distribution and the second, the general wave function. From S(q), he determined S(p) with the help of the transformation equation,

$$S(p) = \int dq\ S(q)\ e^{2\pi i pq/h} = \text{const. } \exp\left[-\frac{2\pi^2 q_1^2 (p-p_0)^2}{h^2} + \frac{2\pi i}{h}(p-p_0)q_0\right] \qquad (43)$$

Hence, for a given uncertainty $\delta q = q_1$ in the position, the probability distribution, $|S(p)|^2$, of the momentum p is non-zero in the region $p_0 - p_1 < p < p_0 + p_1$, such that

$$\frac{4\pi^2 q_1^2 p_1^2}{h^2} \simeq 1 \qquad (44)$$

The simultaneous measurement of the position and momentum of an electron is thus limited by the Uncertainty Relation,

$$\delta p \, . \, \delta q \simeq \frac{h}{2\pi} \, . \qquad (45)$$

Heisenberg asked Pauli for his severe criticism ['unnachsichtige Kritik'].[98] However, Pauli at once approved Heisenberg's ideas on the uncertainty principle, and thought that this interpretation endowed quantum mechanics with a coherent physical meaning. On returning from his vacation in Norway Bohr was not immediately satisfied with Heisenberg's formulation of the 'intuitive content of the quantum-theoretical kinematics and mechanics', but Heisenberg was not willing to make any changes in his paper.[99] However, in a postscript added in proof, he incorporated Bohr's suggestions.[100]

In his Special Relativity (1905), Einstein[101] had emphasized the fundamental importance of employing 'observable magnitudes' only in the construction of a physical theory. This conception of Einstein's had guided Heisenberg in his discovery of a quantum-theoretical kinematics.[53] But when in spring 1926 Heisenberg met Einstein in Berlin, the latter had told him : '...it may be heuristically useful to keep in mind what one has actually observed. But on principle, it is quite wrong to try founding a theory on observable magnitudes alone. In reality the very opposite happens. It is the theory which decides what we can observe.'[102] Heisenberg had employed this turnabout of Einstein's original conception in the derivation of the physical interpretation from the mathematical formalism of quantum mechanics. This interpretation, embodied in the uncertainty principle, Eq.(45), could be generalized to any pair of conjugate dynamical variables, and was soon accepted as 'the real core of the new theory'.[103]

After his discussions with Schrödinger in Copenhagen in September 1926, during which the question of 'quantum jumps' often came up, Bohr reflected deeply upon the meaning of the fundamental equations of quantum theory. His goal was to construct a general philosophical guideline for the physical interpretation, independent of the mathematics used. Bohr came to the conclusion that the situation in atomic physics could only be described in terms of dual, complementary pictures which, in classical physics, exclude each other. The uncertainty relations ensure that no contradiction will arise in the exercise of the Principle of Complementarity in nature ; they exclude the possibility of situations occurring that exhibit both the wave and particle aspects of a phenomenon simultaneously. In his address to the International Congress of Physics in Como in September 1927, on the occasion of the centenary of Alessandro Volta's death, Niels Bohr presented his views on complementarity.[104]

At the fifth Solvay Conference in Brussels, from 24 to 29 October 1927, quantum mechanics, together with the 'Copenhagen interpretation', was publicly presented as a complete and final theory of atomic phenomena by its numerous protagonists whose leader was Niels Bohr. Einstein expressed some of his reservations about the new theory, thereby beginning a discussion with Bohr about deterministic description versus statistical causality. During one of the lectures at the Solvay Conference, Paul Ehrenfest passed on a note to Einstein, saying 'Don't laugh ! There is a special section in purgatory for professors of quantum theory, where they will be obliged to listen to lectures on classical physics ten hours every day.' To which Einstein replied, 'I laugh only at their naïveté. Who knows who would have the laugh in a few years ?'[105]

The Einstein-Bohr discussions, on the question whether the quantum mechanical description of physical reality is 'complete', were continued at the sixth Solvay Conference in October 1930. Since then the discussion on the interpretation and epistemology of quantum mechanics has been joined in by a growing number of participants. However, quantum theory, the Knabenphysik, the young man's game, continues to grow through its manifestations in all regions of physical experience.

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