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DYNAMICS OF CONTINUA AND PARTICLES FROM GENERAL COVARIANCE OF NEWTONIAN GRAVITATION THEORY

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ABSTRACT : The principle of general covariance, which states that the total action functional in General Relativity is independent of coordinate transformations, is shown to be also applicable to the four-dimensional geometric theory of Newtonian gravitation. It leads to the correct conservation (or balance) equations of continuum mechanics as well as the equations of motion of test particles in a gravitational field. The degeneracy of the "metric" of Newtonian space-time forces us to introduce a "gauge field" which fixes the connection and leads to a conserved current, the mass flow. The particle equations are also derived from an invariant Hamiltonian structure on the extended Galilei group and a minimal interaction principle. One not only finds the same equations of motion but even the same gauge fields.

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1. INTRODUCTION

The covariant space-time formulation of Newtonian gravitation theory goes back to Frank [1] and Cartan [2] and has since been developed by various authors [3-8]. The Newtonian gravitational field equations and the balance equations of continuum mechanics are here formulated in terms of a connection and its curvature on a 4-dimensional space-time exactly as in General Relativity theory, except that the Lorentz metric is replaced by a Galilei structure [7], i.e. essentially a degenerate "metric" that singles out a class of local reference frames related to each other by homogeneous Galilei transformations. Neither the equations for particles, nor for fields were obtained from a covariant variational principle, but rather they were found by analogy with the relativistic theory.

In fact, it seems that such a Lagrangian formulation cannot exist, at least not in terms of the given variables. For example, on the tangent bundle of flat space-time $\mathbb{R}^4$ taken as evolution space E of an isolated 1-particle system, there does not exist a Galilei invariant Lagrangian, while a Lorentz invariant one does [9]. In particle mechanics the problem can be solved in two ways. Either one keeps the original evolution space but gives up the Lagrangian and requires only an invariant Hamiltonian structure to exist (i.e. a presymplectic 2-form σ on $T\mathbb{R}^4$ which defines both equations of motion and Poisson brackets), or one enlarges the evolution space E to some $\tilde{E}$ such that a projection map $\pi : \tilde{E} \longrightarrow E$ exists and attempts to find an invariant 1-form $\tilde{\theta}$ such that $d\tilde{\theta} = \pi^* \sigma$. Such a Cartan 1-form θ is normally obtained from a Lagrangian (e.g. $\theta = \frac{\partial L}{\partial \dot{q}^\alpha} dq^\alpha$ [9,10]), but it may exist in cases where no Lagrangian is defined (like on certain evolution spaces for particles with spin [11,12]) and replaces the latter for almost all purposes. In particular, the minimal interaction principle can be directly applied to θ (though not to σ). In the last section we will illustrate this procedure by constructing a model of particle with spin in a Newtonian gravitational field starting from group theoretical considerations and the minimal interaction principle. The same procedure starting with the Poincaré group would lead to the corresponding relativistic model in a curved space-time.

For a classical field theory, it is not yet too well understood what local differential geometric object corresponds to the Hamiltonian structure in mechanics (for attempts to identify this object see [13]). On the other hand it is always possible to enlarge the number of variables on which the Lagrangian density may depend (i.e. to introduce some sort of "gauge fields"). It might give new insights into the structure of field theories to be able to derive the covariant form of the Newtonian equations of gravitation this way. But this appears to be a non trivial open problem.

What will be solved in this paper is the related problem of how to derive the correct conservation (balance) equations of non relativistic continuum theory from the principle of general covariance. That means we require some action functional to be invariant under arbitrary space-time coordinate transformations as well as gauge transformations for the introduced gauge fields. This approach is well known in General Relativity (see, e.g. [14]). We wish to show that the same principle applies just as well to the classical Newtonian theories.

The procedure is, however, somewhat less straightforward. It cannot be based directly on the Galilei structure and the connection as primary variables. We find it necessary to introduce two "potentials" A_α and u^α that together with the Galilei structure determine the connection but not in a one-to-one way. Thus there appears a "gauge group", invariance with respect to which implies that the corresponding current is conserved. In this way the concept of mass current appears in the Newtonian theory separately from the stress-energy tensor, in contradistinction to the relativistic case, similarly as the mass appears *in a different role in Galilei and Poincaré invariant mechanics*, respectively.

The principle of general covariance has been cast into a particularly elegant and effective form by Souriau [15] which permits it to be directly applied also in the limiting case of matter distribution with support on a single worldline. Then it yields the equations of motion for test particles in a given external field. We obtain this way the equations of a particle with mass, spin and quadrupole moment in a Newtonian gravitational field, and find that they agree with those derived by means of the minimal interaction principle.

Section 2 reviews Souriau's approach to the principle of general covariance and test particle motion in General Relativity; section 3 the basic definitions for Newtonian structure. In the next two sections we derive the balance equations for continua and the equations of motion for particles, respectively, in the Newtonian theory. The last section deals with the Hamiltonian model of spinning particles.

2. PRINCIPLE OF GENERAL COVARIANCE

The Palatini method of considering the stress-energy tensor $T^{\alpha\beta}$ of matter and non gravitational fields as the variational derivative of a Lagrangian density with respect to the space-time metric $g_{\alpha\beta}$ has been restated by Souriau [15] in the following form :

Consider the set S of all Lorentz metrics $g_{\alpha\beta}$ on a given 4-dimensional manifold V and a 1-form $\mathcal{T}$ on the manifold S [(1)] of the form

(1) S is an open subset of the set of sections of a vector bundle and as such has the structure of an infinite dimensional manifold. We will not make this more precise than is usually done in the context of variational calculus used in physical applications (see, e.g., [10]).

$$\mathcal{T}[\delta g] = \int_V \tfrac{1}{2} T^{\alpha\beta} \delta g_{\alpha\beta} \; v\ell_g \qquad (2.1)$$

where the variation δg is interpreted as a tangent vector to S at g and the integral required to exist.

The <u>principle of general covariance</u> now demands that $\mathcal{T}$ be a 1-form not only on S but rather on the quotient "manifold" $H = S/\mathcal{D}iff_0(V)$ (where $\mathcal{D}iff_0(V)$ denotes the group of all C^∞ diffeomorphisms of V with compact support) since any two metrics related by a diffeomorphism are physically equivalent. This means that $\mathcal{T}$ must vanish on all "vectors" δg tangent to the orbit of g under $\mathcal{D}iff_0(V)$ in S, in other words that

$$0 = \mathcal{T}[\mathcal{L}_\xi g] = \int_V \tfrac{1}{2} T^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} \; v\ell_g$$

for all vector fields ξ on V with compact support (where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ). This is well known to be equivalent to the conservation equation of the stress-energy tensor $T^{\alpha\beta}$,

$$\nabla_\alpha T^{\alpha\beta} = 0 \qquad (2.2)$$

Here the connection has been taken to be the unique symmetric Lorentz connection. One could replace S by the set $S' = \{(g_{\alpha\beta}, \Gamma^\gamma_{\alpha\beta}) \mid \nabla_\alpha g_{\beta\gamma} = 0\}$ without requiring that $\Gamma^\gamma_{\alpha\beta}$ be symmetric and (2.1) by

$$\mathcal{T}[\delta(g,\Gamma)] = \int_V \{\tfrac{1}{2} \tilde{T}^{\alpha\beta} \delta g_{\alpha\beta} + \Theta^{\alpha\beta}_\gamma \; \delta\Gamma^\gamma_{\alpha\beta}\} \, v\ell_g \qquad (2.3)$$

then

$$0 = \int_V \{\tfrac{1}{2} \tilde{T}^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} + \Theta^{\alpha\beta}_\gamma \, \mathcal{L}_\xi \Gamma^\gamma_{\alpha\beta}\} \, v\ell_g$$

leads again to equation (2.2) if one defines[2]

$$T^{\alpha\beta} := \tilde{T}^{\alpha\beta} + \nabla_\rho \left(g^{\rho\sigma} \Theta_\sigma^{\alpha\beta} - 2 g^{\sigma(\alpha} \Theta_\sigma^{\beta)\rho} \right) \tag{2.4}$$

and now requires that $\Gamma_{\alpha\beta}^{\gamma}$ and $\Theta_\gamma^{\alpha\beta}$ are symmetric. If, however, torsion is admitted as, for example, in the Einstein-Cartan theory [17], $\Gamma_{\alpha\beta}^{\gamma}$ can be decomposed into the Levi-Civitá connection $\left\{ {\gamma \atop \alpha\beta} \right\}$ and a contorsion tensor. Part of $\Theta_\gamma^{\alpha\beta}$ is again absorbed in $T^{\alpha\beta}$ and the rest interpreted as a spin density which will appear as a source term in (2.2). (The $\Theta_\gamma^{\alpha\beta}$ has been called <u>hypermomentum</u> by Hehl et al. [18]).

Since in this principle only the (global) functional is considered as a physical quantity it is clear that in (2.3) $\tilde{T}^{\alpha\beta}$ and $\Theta_\gamma^{\alpha\beta}$ could only be individually well defined local fields if there were no constraint at all between $g_{\alpha\beta}$ and $\Gamma_{\alpha\beta}^{\gamma}$. If $\Gamma_{\alpha\beta}^{\gamma}$ is to be a symmetric Lorentz connection, then either of $\tilde{T}^{\alpha\beta}$ or $\Theta_\gamma^{\alpha\beta}$ can be dropped and the <u>physical</u> stress-energy tensor defined by (2.4) (with either $\tilde{T}^{\alpha\beta}$ or $\Theta_\gamma^{\alpha\beta}$ equal to zero). Then equation (2.2) will still be a consequence of the general principle of covariance.

The advantage of Souriau's approach is that the same principle also leads very easily to the equations of motion of test particles in the gravitational field (and other external fields). The method is summarized as follows : Suppose the functional $\mathcal{J}$ has its support on a worldline Λ in V and is a distribution of, say, no higher than second order. Then it can be assumed to be of the form

$$\mathcal{J}[\delta g] = \int_\Lambda \left\{ \tfrac{1}{2} T^{\alpha\beta} \delta g_{\alpha\beta} + \Theta_\gamma^{\alpha\beta} \delta \Gamma_{\alpha\beta}^{\gamma} + Q^{\alpha\beta\gamma\delta} \delta R_{\alpha\beta\gamma\delta} \right\} dt \tag{2.5}$$

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(2) ( ) and [ ] denote symmetrization and antisymmetrization respectively.

76/P.849

where t is any curve parameter of Λ and $T^{[\alpha\beta]} = 0$, $\Theta_{\gamma}^{[\alpha\beta]} = 0$ and $Q^{\alpha\beta\gamma\delta}$ has the same symmetries as the curvature of a symmetric Lorentz connection. By the principle of general covariance we have

$$
\begin{aligned}
0 &= \int_{\Lambda} \left\{ \tfrac{1}{2} T^{\alpha\beta} \mathcal{L}_{\xi} g_{\alpha\beta} + \Theta_{\gamma}^{\alpha\beta} \mathcal{L}_{\xi} \Gamma_{\alpha\beta}^{\gamma} + Q^{\alpha\beta\gamma\delta} \mathcal{L}_{\xi} R_{\alpha\beta\gamma\delta} \right\} dt \\
&= \int_{\Lambda} \left\{ \Theta_{\gamma}^{\alpha\beta} \left(\nabla_{\alpha} \nabla_{\beta} \xi^{\gamma} - R^{\gamma}_{\cdot\beta\alpha\rho} \xi^{\rho} \right) + Q^{\alpha\beta\gamma\delta} \xi^{\rho} \nabla_{\rho} R_{\alpha\beta\gamma\delta} \right. \\
&\qquad \left. + \left(T^{\alpha\beta} - Q^{\beta\lambda\mu\nu} R^{\alpha}_{\cdot\lambda\mu\nu} - 4 Q^{\lambda(\alpha\mu)\nu} R_{\lambda\mu\nu\cdot}{}^{\beta} \right) \nabla_{\alpha} \xi_{\beta} \right\} dt \qquad (2.6)
\end{aligned}
$$

for all vector fields ξ^{α} on V with compact support.

In particular, for $\xi^{\alpha} = AB\tilde{\xi}^{\alpha}$ where A and B are C^{∞}-functions vanishing on Λ and $\tilde{\xi}^{\alpha}$ any C^{∞} vector field with compact support equation (2.6) leads to

$$
0 = \int_{\Lambda} \Theta_{\gamma}^{\alpha\beta} \partial_{\alpha} A \, \partial_{\beta} B \, \tilde{\xi}^{\gamma} \, dt
$$

which is equivalent to $\Theta_{\gamma}^{\alpha\beta}$ being of the form

$$
\Theta_{\gamma}^{\alpha\beta} = M^{(\alpha}_{\cdot\gamma} V^{\beta)}
$$

for some $M^{\alpha}_{\cdot\gamma}$, where V^{α} is the tangent vector to Λ with respect to t. Using this and now putting $\xi^{\alpha} = A\bar{\xi}^{\alpha}$ with $A|_{\Lambda} = 0$ we have from (2.6)

$$
0 = \int_{\Lambda} \left\{ T^{\alpha}_{\cdot\beta} - \dot{M}^{\alpha}_{\cdot\beta} - Q_{\beta}^{\cdot\lambda\mu\nu} R^{\alpha}_{\cdot\lambda\mu\nu} - 4 Q_{\lambda}^{\cdot(\alpha\mu)\nu} R^{\lambda}_{\cdot\mu\nu\beta} \right\} \partial_{\alpha} A \, \bar{\xi}^{\beta} \, dt \qquad (2.7)
$$

where $\dot{} := V^{\rho} \nabla_{\rho}$ and integration by parts has been used. Equation (2.7) is equivalent to

$$T^{\alpha}_{\cdot\beta} - \dot{M}^{\alpha}_{\cdot\beta} - Q_{\beta}^{\cdot\lambda\mu\nu} R^{\alpha}_{\cdot\lambda\mu\nu} - 4\, Q_{\lambda}^{\cdot(\alpha\mu)\nu} R^{\lambda}_{\cdot\mu\nu\beta} = P_{\beta} V^{\alpha} \qquad (2.8)$$

for some P_{β} defined along Λ . Substituting again into (2.6) gives

$$\dot{P}_{\alpha} = - M^{\lambda}_{\cdot\mu} R^{\mu}_{\cdot\lambda\rho\alpha} V^{\rho} + Q^{\lambda\mu\nu\rho} \nabla_{\alpha} R_{\lambda\mu\nu\rho} \qquad (2.9)$$

Define

$$S^{\alpha\beta} := 2\, M^{[\alpha}_{\cdot\lambda}\, g^{\beta]\lambda}$$

then (2.9) becomes

$$\dot{P}^{\alpha} = -\frac{1}{2} R^{\alpha}_{\cdot\lambda\mu\nu} V^{\lambda} S^{\mu\nu} + \nabla^{\alpha} R_{\lambda\mu\nu\rho}\, Q^{\lambda\mu\nu\rho} \qquad (2.10)$$

while the antisymmetric part of (2.8) can be brought into the form

$$\frac{1}{2} \dot{S}^{\alpha\beta} = P^{[\alpha} V^{\beta]} - 4\, R^{[\alpha}_{\cdot\lambda\mu\nu}\, Q^{\beta]\lambda\mu\nu} \qquad (2.11)$$

Equations (2.10) and (2.11) are now fairly accepted as equations of motion of a test particle if $S^{\alpha\beta}$ is interpreted as its spin, P^{α} as its 4-momentum and $Q^{\alpha\beta\gamma\delta}$ as its (mass) quadrupole moment (e.g. [19, 15, 20] and references given therein).

It is well known that they do not form a complete set of equations for all dynamical quantities necessary to describe the state of motion of a test particle, unless additional conditions are imposed, the most natural being

$$S^{\alpha\beta} P_{\beta} = 0 \qquad (2.12)$$

(when $Q^{\alpha\beta\gamma\delta} = 0$) [19, 12, 20]. Note that again not all of the coefficients $T^{\alpha\beta}$, $\Theta_{\gamma}^{\alpha\beta}$, $Q^{\alpha\beta\gamma\delta}$ in equation (2.5) can be interpreted physically in an unambiguous way. To obtain the equation of motion of a monopole particle $\Theta^{\alpha\beta}_{\gamma}$ and $Q^{\alpha\beta\gamma\delta}$ can be assumed zero. For the dipole or multipole case the term $T^{\alpha\beta}$ is not necessary.

3. NEWTONIAN STRUCTURES

A Galilei structure on a four-dimensional C^{∞}-manifold V can be given by a pair of tensor fields $(\gamma^{\alpha\beta}, \psi_{\alpha})$ where $\gamma^{\alpha\beta}$ is symmetric and ψ_{α} spans the kernel of $\gamma^{\alpha\beta}$ at every point of V. (See [7] for details and proofs for all the material of this section.)

A vector u^{α} is called unit timelike if $u^{\alpha}\psi_{\alpha} = 1$, it is spacelike if $u^{\alpha}\psi_{\alpha} = 0$. The three-dimensional distribution of hyperplanes S_x formed by all spacelike tangent vectors at each $x \in V$ is completely integrable, in particular, if the 1-form $\psi = \psi_{\alpha}\, dx^{\alpha}$ is closed : $d\psi = 0$. The leaves of this foliation then carry a natural Riemannian metric induced by $\gamma^{\alpha\beta}$.

Linear symmetric Galilei connections (i.e. satisfying

$$\nabla_{\alpha}\, \gamma^{\beta\gamma} = 0 \qquad \text{and} \qquad \nabla_{\alpha}\, \psi_{\beta} = 0 \qquad) \tag{3.1}$$

exist if and only if ψ is closed and are determined up to an arbitrary 2-form $F = \frac{1}{2} F_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}$. In fact, the general solution of (3.1) for symmetric $\Gamma^{\gamma}_{\alpha\beta}$ can be written in the form

$$\Gamma^{\gamma}_{\alpha\beta} = \overset{u}{\Gamma}{}^{\gamma}_{\alpha\beta} + \psi_{(\alpha} F_{\beta)\rho}\, \gamma^{\rho\gamma} \tag{3.2}$$

where u^{α} is an arbitrary given unit timelike vector field and

$\overset{u}{\Gamma}{}^{\gamma}_{\alpha\beta}$ is the unique symmetric Galilei connection such that u^{α} is geodesic and irrotational, i.e. such that ([3, p. 197])

$$u^{\rho}\overset{u}{\nabla}_{\rho}u^{\alpha}=0 \qquad \text{and} \qquad \overset{u}{\nabla}{}^{[\alpha}u^{\beta]}=0 \tag{3.3}$$

(where indices are as always raised by means of $\gamma^{\alpha\beta}$).

The curvature tensor of any symmetric Galilei connection has the properties

$$R^{\alpha}{}_{\cdot\beta(\gamma\delta)}=0\ ,\ R^{\alpha}{}_{\cdot[\beta\gamma\delta]}=0\ ,\ R^{(\alpha\beta)}{}_{\cdot\cdot\gamma\delta}=0\ ,$$

$$\psi_{\rho}R^{\rho}{}_{\cdot\beta\gamma\delta}=0$$

We will only consider a subclass of symmetric Galilei connections, called Newtonian connections which are characterized by satisfying in addition

$$R^{\alpha}{}_{\cdot\beta}{}^{\gamma}{}_{\cdot\delta}=R^{\gamma}{}_{\cdot\delta}{}^{\alpha}{}_{\cdot\beta} \tag{3.4}$$

or, equivalently, by

$$dF=0 \tag{3.5}$$

for the $F_{\alpha\beta}$ in (3.2). Equation (3.4) is evidently satisfied for any Galilei connection that is obtained as a limit of symmetric Lorentz connections [21]. It means that we confine ourselves to conservative forces. We will call the triple $(\gamma^{\alpha\beta}, \psi_{\alpha}, \Gamma^{\gamma}_{\alpha\beta})$ on V a Newtonian structure if (3.4) holds.

A unit timelike vector field u^{α} (which can be considered as an observer field) is also necessary to define a "generalized inverse" of $\gamma^{\alpha\beta}$ by means of

$$\overset{u}{\gamma}_{\alpha\rho}u^{\rho}=0 \qquad \text{and} \qquad \overset{u}{\gamma}_{\alpha\rho}\gamma^{\rho\beta}=\delta^{\beta}_{\alpha}-\psi_{\alpha}u^{\beta} \tag{3.6}$$

Any orientable Galilei manifold carries a canonical volume element vol which can be written as $\psi \wedge vol_3$ where vol_3 is the Riemannian volume element of S_x .

It has been shown in [7] that an asymptotically Euclidean Newtonian space-time satisfying the field equations

$$R_{\alpha\beta} := R^{\varsigma}{}_{\cdot\alpha\varsigma\beta} = 4\pi k \rho\, \psi_\alpha \psi_\beta$$

is completely equivalent to the classical Newtonian gravitation theory (with a potential u satisfying the Poisson equation $\Delta u = 4\pi k\rho$ in Euclidean 3-space).

We end this section by deriving some formulae that will later be needed. Covariant derivatives will always refer to $\Gamma^{\gamma}_{\alpha\beta}$ given by (3.2) except when specified otherwise. From (3.2) we have

$$\nabla_\beta u^\alpha = \overset{u}{\nabla}_\beta u^\alpha - \tfrac{1}{2}\psi_\beta F^{\alpha}_{\cdot\varsigma} u^\varsigma - \tfrac{1}{2} F^{\alpha}_{\cdot\beta}$$

thus, in view of (3.3),

$$u^\varsigma \nabla_\varsigma u^\alpha = - F^{\alpha}_{\cdot\rho}\, u^\rho \tag{3.7}$$

and

$$\nabla^{[\alpha} u^{\beta]} = \tfrac{1}{2} F^{\alpha\beta} . \tag{3.8}$$

Using (3.6) we can derive from these equations

$$F_{\alpha\beta} = -2 \overset{u}{\gamma}_{\lambda[\alpha} \nabla_{\beta]} u^\lambda \tag{3.9}$$

The general variation of $\overset{u}{\gamma}_{\alpha\beta}$ is obtained from (3.6) directly as

$$\delta \overset{u}{\gamma}_{\alpha\beta} = - \overset{u}{\gamma}_{\alpha\lambda} \overset{u}{\gamma}_{\beta\mu} \delta\gamma^{\lambda\mu} - 2\, \psi_{(\alpha} \overset{u}{\gamma}_{\beta)\lambda}\, \delta u^{\lambda} \quad , \tag{3.10}$$

a particular case of which is

$$\nabla_{\gamma} \overset{u}{\gamma}_{\alpha\beta} = - 2\, \nabla_{\gamma} u^{\lambda}\, \overset{u}{\gamma}_{\lambda(\alpha} \psi_{\beta)} \; . \tag{3.11}$$

4. BALANCE EQUATIONS FOR CONTINUA IN THE NEWTONIAN THEORY

Taking S now to be the set of all Galilei structures $(\gamma^{\alpha\beta}, \psi_{\alpha})$ on V , one might think of replacing (2.1) by

$$\mathcal{T}[\delta(\gamma,\psi)] = \int_V \left\{ -\tfrac{1}{2} P_{\alpha\beta}\, \delta\gamma^{\alpha\beta} + Q^{\alpha} \delta\psi_{\alpha} \right\} vol$$

But as shown in [21] the balance equations following from the general covariance of this functional lack the acceleration term of the classical equations. This is not surprising because $(\gamma^{\alpha\beta}, \psi_{\alpha})$ do not fully determine the connection. Even if we require the connection to be symmetric, we must therefore take this functional to be

$$\mathcal{T}[\delta(\gamma,\psi,\Gamma)] = \int_V \left\{ -\tfrac{1}{2} P_{\alpha\beta}\, \delta\gamma^{\alpha\beta} + Q^{\alpha} \delta\psi_{\alpha} + \Theta_{\gamma}^{\alpha\beta}\, \delta\Gamma^{\gamma}_{\alpha\beta} \right\} vol \tag{4.1}$$

where $P_{[\alpha\beta]} = 0$ and $\Theta_{\gamma}^{[\alpha\beta]} = 0$. In the relativistic theory the $\delta\Gamma^{\gamma}_{\alpha\beta}$ can be expressed in terms of $\delta g_{\alpha\beta}$. For a Newtonian connection $\Gamma^{\gamma}_{\alpha\beta}$ compatible with the Galilei structure we have by (3.2,5)

$$\Gamma^{\gamma}_{\alpha\beta} = \overset{u}{\Gamma}{}^{\gamma}_{\alpha\beta} + \psi_{(\alpha} \left[\partial_{\beta)} A_{\rho} - \partial_{\rho} A_{\beta)} \right] \gamma^{\rho\gamma}$$

where $\overset{u}{\Gamma}{}^{\gamma}_{\alpha\beta}$ is characterized by (3.3) in terms of derivatives of $(\gamma^{\alpha\beta}, \psi_\alpha)$, and a unit vector field u^α. A_α is an arbitrary 1-form. The variation $\delta\Gamma^\gamma_{\alpha\beta}$ can thus be expressed in terms of $\delta\gamma^{\alpha\beta}$, $\delta\psi_\alpha$ and δA_α while u^α could be considered fixed. However, there is no canonical choice for u^α and it seems therefore best to regard the connection fixed (in a somewhat redundant way) by $(\gamma^{\alpha\beta}, \psi_\alpha)$ and the pair (A_α, u^α). Thus (4.1) can be replaced by

$$\mathcal{J}[\delta(\gamma,\psi,\Gamma)] = \int_V \{-\tfrac{1}{2} R_{\alpha\beta}\delta\gamma^{\alpha\beta} + Q^\alpha \delta\psi_\alpha + J^\alpha \delta A_\alpha + K_\alpha \delta u^\alpha\} \, v\,d \qquad (4.2)$$

We now require that $\mathcal{J}$ be a 1-form on the manifold of equivalence classes of Newtonian structures. Thus $\mathcal{J}$ must in particular be left invariant by any "gauge transformation", i.e. a change $(\gamma^{\alpha\beta}, \psi_\alpha, A_\alpha, u^\alpha) \to (\bar\gamma^{\alpha\beta}, \bar\psi_\alpha, \bar A_\alpha, \bar u^\alpha)$ that does not change the Newtonian structure $(\gamma^{\alpha\beta}, \psi_\alpha, \Gamma^\gamma_{\alpha\beta})$. To find the infinitesimal gauge transformation suppose therefore that $\delta\gamma^{\alpha\beta} = \delta\psi_\alpha = \delta\Gamma^\gamma_{\alpha\beta} = 0$. Then $\psi_\alpha \delta u^\alpha = 0$ and varying equation (3.9) gives (in view of (3.10, 11))

$$\begin{aligned} \delta F_{\alpha\beta} &= -2\,\delta\overset{u}{\gamma}_{\lambda[\alpha} \nabla_{\beta]} u^\lambda - 2\overset{u}{\gamma}_{\lambda[\alpha} \nabla_{\beta]} \delta u^\lambda - 2\overset{u}{\gamma}_{\lambda[\alpha}\, \delta\Gamma^\lambda_{\beta]\rho} u^\rho \\ &= 2\,\psi_{[\alpha} \nabla_{\beta]} u^\lambda \overset{u}{\gamma}_{\lambda\rho} \delta u^\rho - 2\overset{u}{\gamma}_{\rho[\alpha} \nabla_{\beta]} \delta u^\rho \\ &= 2\nabla_{[\alpha} \left(\overset{u}{\gamma}_{\beta]\rho} \delta u^\rho \right). \end{aligned}$$

But $\delta F_{\alpha\beta} = 2\,\delta\,\partial_{[\alpha} A_{\beta]} = 2\nabla_{[\alpha} \delta A_{\beta]}$, therefore we find

$$\nabla_{[\alpha} \left(\delta A_{\beta]} - \overset{u}{\gamma}_{\beta]\rho} \delta u^\rho \right) = 0$$

In other words, variations such that $\delta\gamma^{\alpha\beta} = 0$, $\delta\psi_\alpha = 0$ and

$$\delta A_\alpha = \overset{u}{\gamma}_{\alpha\rho} \delta u^\rho + \partial_\alpha \chi \qquad (4.3)$$

for any δu^ρ and χ are pure gauge variations.

Therefore $\mathcal{J}$ must vanish for all such variations and we have from (4.2)

$$0 = \int_V \left\{ \left(J^\rho \overset{u}{\gamma}_{\rho\alpha} + K_\alpha \right) \delta u^\alpha + J^\alpha \partial_\alpha \chi \right\} \mathrm{vol}$$

for arbitrary spacelike δu^α and χ with compact support. This implies that

$$\nabla_\alpha J^\alpha = 0 \tag{4.4}$$

and

$$K_\alpha = - \overset{u}{\gamma}_{\alpha\rho} J^\rho + \kappa \psi_\alpha \tag{4.5}$$

where κ is an arbitrary scalar.

Using (4.5) we now state that $\mathcal{J}$ must vanish on variations that are infinitesimal diffeomorphisms with compact support, that is

$$\begin{aligned} 0 &= \int_V \left\{ -\tfrac{1}{2} P_{\alpha\beta} \mathcal{L}_\xi \gamma^{\alpha\beta} + Q^\alpha \mathcal{L}_\xi \psi_\alpha + J^\alpha \mathcal{L}_\xi A_\alpha + \left(- \overset{u}{\gamma}_{\alpha\rho} J^\rho + \kappa \psi_\alpha \right) \mathcal{L}_\xi u^\alpha \right\} \mathrm{vol} \\ &= \int_V \Big\{ \left(P_{\alpha\rho} \gamma^{\rho\beta} + Q^\beta \psi_\alpha + A_\alpha J^\beta + \overset{u}{\gamma}_{\alpha\rho} J^\rho u^\beta - \kappa \psi_\alpha u^\beta \right) \nabla_\beta \xi^\alpha \\ &\qquad + \left(J^\beta \nabla_\alpha A_\beta - \overset{u}{\gamma}_{\beta\rho} J^\rho \nabla_\alpha u^\beta \right) \xi^\alpha \Big\} \mathrm{vol} \\ &= \int_V \left\{ - \nabla_\beta \tilde{T}^\beta{}_\alpha + 2 J^\beta \nabla_{[\alpha} A_{\beta]} - \overset{u}{\gamma}_{\beta\rho} J^\rho \nabla_\alpha u^\beta \right\} \xi^\alpha \mathrm{vol} \end{aligned} \tag{4.6}$$

by Stokes' theorem and (4.4), where

$$\tilde{T}^\beta{}_\alpha := P_{\alpha\rho} \gamma^{\rho\beta} + Q^\beta \psi_\alpha + \overset{u}{\gamma}_{\alpha\rho} J^\rho u^\beta - \kappa \psi_\alpha u^\beta . \tag{4.7}$$

Since (4.6) must hold for all vector fields ξ^α with compact support, we have

$$\nabla_\beta \tilde{T}^\beta{}_{\cdot\alpha} = -\nabla_\alpha u^\beta \overset{u}{\gamma}_{\beta\rho} J^\rho + F_{\alpha\beta} J^\beta . \tag{4.8}$$

Equation (4.4) states that J^α is a conserved current which is most naturally interpreted as the mass flow. Assume therefore that

$$J^\alpha = \rho V^\alpha \quad , \quad V^\alpha \psi_\alpha = 1 \tag{4.9}$$

and confine considerations to a region where the mass density ρ is positive. Now, with the help of (3.9) expression (4.8) can be written as

$$\nabla_\beta \tilde{T}^\beta{}_{\cdot\alpha} = -\rho \overset{u}{\gamma}_{\alpha\lambda} \dot{u}^\lambda \tag{4.10}$$

with $\dot{} := V^\rho \nabla_\rho$. But we wish to replace all references to the arbitrary field u^α by one to the physical matter flow V^α. It is easily seen that

$$\overset{u}{\gamma}_{\alpha\beta} = \overset{V}{\gamma}_{\alpha\beta} + 2\psi_{(\alpha} V_{\beta)} - V^2 \psi_\alpha \psi_\beta$$

where

$$V_\alpha := \overset{u}{\gamma}_{\alpha\beta} V^\beta = -\overset{V}{\gamma}_{\alpha\beta} u^\beta + V^2 \psi_\alpha \quad , V^2 := V^\alpha V_\alpha \ . \tag{4.11}$$

Then from (4.10)

$$\begin{aligned}
\nabla_\beta \tilde{T}^\beta{}_{\cdot\alpha} &= \rho (\overset{u}{\gamma}_{\alpha\beta})^{\cdot} u^\beta \\
&= \rho u^\beta \left(\overset{V}{\gamma}_{\alpha\beta} + 2\psi_{(\alpha} V_{\beta)} - V^2 \psi_\alpha \psi_\beta \right)^{\cdot} \\
&= -2 \overset{V}{\gamma}_{\lambda(\alpha} \psi_{\beta)} \dot{V}^\lambda u^\beta + \rho \dot{V}_\alpha + \rho \psi_\alpha \left(u^\beta \dot{V}_\beta - (V^2)^{\cdot} \right) \\
&= -\rho \overset{V}{\gamma}_{\alpha\beta} \dot{V}^\beta + \rho \dot{V}^\alpha - \tfrac{1}{2} \rho (V^2)^{\cdot} \psi_\alpha
\end{aligned}$$

where (3.11) has been used as well as

$$(v^2)^{\cdot} = 2\, v_\alpha \left(\dot{v}^\alpha - \dot{u}^\alpha \right)$$

But by (4.4) and (4.9) this can be written as

$$\nabla_\beta T^\beta_{\cdot\,\alpha} = \rho \, \overset{\vee}{\gamma}_{\alpha\beta} \, \dot{v}^\beta \tag{4.12}$$

provided one defines

$$T^\beta_{\cdot\,\alpha} := -\tilde{T}^\beta_{\cdot\,\alpha} + \rho V^\beta V_\alpha - \tfrac{1}{2} \rho V^2 \psi_\alpha V^\beta \tag{4.13}$$

Equation (4.12) is the closest analogue to the relativistic conservation law (2.2) one can find in the Newtonian theory, since it is not possible to absorb the right hand side in the divergence. The relativistic stress energy tensor $T^{\alpha\beta}$ therefore decouples in the Newtonian theory in a tensor $T^\beta_{\cdot\,\alpha}$ describing stresses and energy flow, and the matter flow vector $J^\alpha = \rho V^\alpha$, where $T^\beta_{\cdot\,\alpha}$ has the property

$$T^{[\alpha}_{\cdot\,\rho} \, \gamma^{\beta]\rho} = 0 \tag{4.14}$$

which follows at once from (4.7) and (4.13) since $P_{[\alpha\beta]} = 0$.
This description seems quite satisfactory since it is rather analogous to the situation in particle mechanics. Contravariant symmetric tensors $T^{\alpha\beta}$ have also been used in Newtonian theory (e.g. [8]), but they do not describe stresses and energy (flow).

Just as in the relativistic theory the tensor $T^\beta_{\cdot\,\alpha}$ can be decomposed in different reference frames and then interpreted physically. The most canonical one is the "matter frame". There, in view of (4.14)

$$T^\beta_{\cdot\,\alpha} = \rho\, \epsilon\, V^\beta \psi_\alpha + q^\beta \psi_\alpha + \gamma^{\beta\rho} p_{\rho\alpha} \tag{4.15}$$

where $q^{\alpha}\psi_{\alpha} = 0$ and $p_{[\alpha\beta]} = 0 = p_{\alpha\rho} V^{\rho}$, and one can interpret ϵ as the specific (non mechanical) internal energy density, q^{α} as the energy (or heat) flow and $p_{\alpha\beta}$ as the stress tensor. This decomposition corresponds to the well known one in Relativity where a matter flow vector ρV^{α} is introduced ad hoc, namely to

$$T^{\alpha\beta} = \rho(c^2 + \epsilon)V^{\alpha}V^{\beta} + 2\, q^{(\alpha}V^{\beta)} + p^{\alpha\beta}$$

with $q^{\alpha}V_{\alpha} = 0 \;, \quad p^{\alpha\beta}V_{\beta} = 0$.

It was shown in [21] that equations (4.12) for the decomposition (4.15) are equivalent in a flat Galilei space-time to the classical balance equations of continuum mechanics, namely to

$$\rho\left(\partial_0 v^A + v^B\partial_B v^A\right) = \partial_B p^{AB} + \rho\, b^A$$

$$\rho\left(\partial_0 \epsilon + v^B\partial_B \epsilon\right) = p^{AB}\partial_A v_B - \partial_A q^A$$

where $V^{\alpha} = (1, v^A)$, $A = 1, 2, 3$ and $b^A = -\Gamma^A_{00} - 2\Gamma^A_{0B}v^B$ is a body force density expressed in terms of the only non-vanishing components of $\Gamma^{\gamma}_{\alpha\beta}$ for a Newtonian connection in Cartesian coordinates.

5. EQUATIONS OF MOTION OF TEST PARTICLES

We now show that the principle of general covariance applied to the 1-form $\mathcal{J}$ on the manifold of Newtonian structures on V also implies the correct equations of motion of a test particle just as in the relativistic theory. This could again be done for a monopole or any multipole particle, but we will perform the calculation only for a particle having mass, spin and a quadrupole moment.

The starting point is thus the Newtonian equivalent of equation (2.5), that is, in view of (4.2),

$$\mathcal{J}[\delta(\gamma,\psi,\Gamma)] = \int_\Lambda \Big\{ -\tfrac{1}{2} \Sigma_{\alpha\beta}\,\delta\gamma^{\alpha\beta} + \varphi^\alpha \delta\psi_\alpha + J^\alpha \delta A_\alpha + K_\alpha \delta u^\alpha + \Theta^{\alpha\beta}_{\ \ \gamma}\,\delta\Gamma^\gamma_{\alpha\beta} + Q_\alpha^{\ \cdot\beta\gamma\delta}\,\delta R^\alpha_{\ \cdot\beta\gamma\delta} \Big\}\, dt \tag{5.1}$$

where Λ is again a worldline and t any curve parameter. In contradistinction to the relativistic case it is not possible to drop all the "potentials" $\delta\gamma^{\alpha\beta}$, $\delta\psi_\alpha$, δA_α and δu^α without loosing the mass term in the equations of motion. This is related to the different role the mass plays in nonrelativistic and relativistic theories and will be discussed from another point of view in the next section.

First we exploit again the gauge invariance (4.3) of $\mathcal{J}$, namely

$$0 = \int_\Lambda \Big\{ (J^\rho \overset{u}{\gamma}_{\rho\alpha} + K_\alpha)\delta u^\alpha + J^\alpha \partial_\alpha \chi \Big\}\, dt \tag{5.2}$$

for arbitrary spacelike δu^α and χ with compact support. For $\delta u^\alpha = 0$ and $\chi = A\tilde{\chi}$, $A|_\Lambda = 0$ we have

$$0 = \int_\Lambda J^\alpha \partial_\alpha A\, \tilde{\chi}\, dt \qquad \text{for all } \tilde{\chi}$$

and therefore

$$J^\alpha = m V^\alpha$$

if V^α is the tangent vector to Λ with respect to t, which we will now choose such that $V^\alpha \psi_\alpha = 1$, and where m is a scalar.

$$\text{Now } 0 = \int_\Lambda m V^\alpha \partial_\alpha \chi\, dt = \int_\Lambda m\,\dot{\chi}\, dt = -\int_\Lambda \dot{m}\,\chi\, dt$$

for all χ with compact support where $\dot{} := V^\alpha \nabla_\alpha$ is the derivative with respect to worldtime along Λ. Thus $\dot{m} = 0$ and m is a constant of the motion, obviously to be interpreted as the mass of the particle. Choosing $\chi = 0$ and δu^α arbitrary spacelike we find from (5.2), similarly as in (4.5)

$$K_\alpha = -mV_\alpha + \kappa\psi_\alpha$$

for an arbitrary scalar κ (where (4.11) has been used).

The invariance under diffeomeorphisms now gives

$$0 = \int_\Lambda \Big\{ -\tfrac{1}{2} \mathcal{P}_{\alpha\beta} \mathcal{L}_\xi \gamma^{\alpha\beta} + Q^\alpha \mathcal{L}_\xi \psi_\alpha + J^\alpha \mathcal{L}_\xi A_\alpha + (-mV_\alpha + \kappa\psi_\alpha)\mathcal{L}_\xi u^\alpha + \Theta^{\alpha\beta}_\gamma \mathcal{L}_\xi \Gamma^\gamma_{\alpha\beta} + Q_\alpha^{\cdot\beta\gamma\delta} \mathcal{L}_\xi R^\alpha_{\cdot\beta\gamma\delta} \Big\} dt$$

$$= \int_\Lambda \Big\{ \Theta^{\alpha\beta}_\gamma \nabla_\alpha \nabla_\beta \xi^\gamma + \big(T^\beta_{\cdot\alpha} + mA_\alpha V^\beta - Q_\alpha^{\cdot\lambda\mu\nu} R^\beta_{\cdot\lambda\mu\nu} - 4Q_\lambda^{\cdot(\beta\mu)\nu} R^\lambda_{\cdot\mu\nu\alpha}\big)\nabla_\beta \xi^\alpha + \big(-mV^\lambda \nabla_\alpha A_\lambda - mV_\lambda \nabla_\alpha u^\lambda - \Theta^{\rho}_{\lambda} R^\lambda_{\cdot\mu\nu\alpha} + Q_\rho^{\cdot\lambda\mu\nu} \nabla_\alpha R^\rho_{\cdot\lambda\mu\nu}\big)\xi^\alpha \Big\} dt$$

for all ξ^α with compact support, where

$$T^\beta_{\cdot\alpha} := \mathcal{P}_{\alpha\rho}\gamma^{\rho\beta} + \psi_\alpha\left(Q^\beta - \kappa u^\beta\right) + mV_\alpha u^\beta$$

As in section 2 we successively choose $\xi^\alpha = AB\tilde{\xi}^\alpha$, $\xi^\alpha = A\bar{\xi}^\alpha$, ξ^α arbitrary with $A|_\Lambda = B|_\Lambda = 0$ and find

$$\Theta^{\alpha\beta}_\gamma = M^{(\alpha}_{\cdot\gamma} V^{\beta)}$$

for some $M^\alpha_{\cdot\beta}$, then

$$T^\alpha_{\cdot\beta} - \dot{M}^\alpha_{\cdot\beta} - Q_\beta^{\cdot\lambda\mu\nu} R^\alpha_{\cdot\lambda\mu\nu} - 4Q_\lambda^{\cdot(\alpha\mu)\nu} R^\lambda_{\cdot\mu\nu\beta} = \tilde{\mathcal{P}}_\beta V^\alpha \tag{5.3}$$

for some $\tilde{\mathcal{P}}_\beta$, and finally

$$\dot{\tilde{\mathcal{P}}}_\alpha = -m\overset{u}{\gamma}_{\alpha\lambda}\dot{u}^\lambda - M^\lambda_{\cdot\mu} R^\mu_{\cdot\lambda\rho\alpha} V^\rho + Q_\rho^{\cdot\lambda\mu\nu} \nabla_\alpha R^\rho_{\cdot\lambda\mu\nu} \tag{5.4}$$

if also (3.9) is used.

Equations (5.3) and (5.4) correspond to (2.8) and (2.9), respectively, but they still contain the gauge field u^α . To eliminate u^α we proceed also by raising indices (by means of $\gamma^{\alpha\beta}$), which gives

$$-\dot{M}^{[\alpha\beta]} - m\, V^{[\alpha} u^{\beta]} - 4\, R^{[\alpha}_{\cdot\,\lambda\mu\nu}\, Q^{\beta]\lambda\mu\nu} = V^{[\alpha}\, \tilde{P}^{\beta]}$$

where (3.4) was used and the symmetries

$$Q^{\rho\lambda\mu\nu} = Q^{[\rho\lambda]\mu\nu} = Q^{\mu\nu\rho\lambda}$$

were assumed for the quadrupole tensor. This suggests that we define

$$S^{\alpha\beta} := 2\, M^{[\alpha\beta]} = 2\, M^{[\alpha}_{\cdot\,\rho}\, \gamma^{\beta]\rho}$$

and

$$P^\alpha := m\, u^\alpha + \gamma^{\alpha\rho}\, \tilde{P}_\rho \tag{5.5}$$

then we find

$$\frac{1}{2}\dot{S}^{\alpha\beta} = P^{[\alpha} V^{\beta]} - 4\, R^{[\alpha}_{\cdot\,\lambda\mu\nu}\, Q^{\beta]\lambda\mu\nu} \tag{5.6}$$

which is formally identical with (2.11). But from (5.5) together with (5.4) we find

$$\dot{P}^\alpha = -\frac{1}{2} R^{\alpha}_{\cdot\,\lambda\mu\nu}\, V^\lambda S^{\mu\nu} + \nabla^\alpha R^{\rho}_{\cdot\,\lambda\mu\nu}\, Q_\rho^{\cdot\,\lambda\mu\nu} \tag{5.7}$$

which also agrees formally with the relativistic equation (2.10).

Note that all references to the gauge fields A_α and u^α has disappeared in (5.6) and (5.7). It is thus reasonable to regard P^α and $S^{\alpha\beta}$ as physical quantities describing 4-momentum and spin of the particle.

To get a deterministic system of equations for the spinning particle (with $\dot{Q}_{\alpha}{}^{\beta\gamma\delta} = 0$) we need again an auxiliary condition. But there is less ambiguity than in the relativistic case and the simplest one,

$$S^{\alpha\beta}\,\psi_{\beta} = 0 \tag{5.8}$$

that comes to mind is already satisfactory. Indeed, contracting (5.6) with ψ_{β} and using (5.8) and (5.5) gives

$$I^{\alpha} = m\,V^{\alpha}$$

Thus (5.7) and (5.6) reduce to

$$m\dot{V}^{\alpha} = -\tfrac{1}{2}\,R^{\alpha}{}_{\cdot\lambda\mu\nu}\,V^{\lambda}\,S^{\mu\nu} \qquad \text{and} \qquad \dot{S}^{\alpha\beta} = 0 \tag{5.9}$$

respectively, for a particle with mass and spin in a Newtonian gravitational field. They agree with those for a rigid test body model in Newtonian space-time obtained by Baxter [22] who also showed that they reduce to the classical three-dimensional equations in a flat Galilei space-time.

6. HAMILTONIAN STRUCTURE FOR THE SPINNING PARTICLE(3)

In this final section we derive the equations of motion of a particle with mass and intrinsic spin only in a Newtonian gravitational field, not as a limit case of the conservation laws of continuous matter but strictly in the framework of Hamiltonian particle mechanics. We will

(3) To keep this section reasonably short, details of calculations that are similar to those in [11] or [16] are left out.

obtain equations (5.9) (together with a symplectic structure on the phase space) by applying a minimal interaction principle to the equations in flat space . This cannot be done on the level of the equations of motion or the symplectic structure, rather it must be applied to a Lagrangian or the Cartan 1-form. But in the space-time picture such a Galilei invariant 1-form does not exist which means we have to introduce the extended Galilei group. In this way we will understand, at least from one point of view, why the gauge fields A_α and u^α had to be introduced.

Let G be the proper Galilei group and $\tilde{G}$ its non trivial 11-dimensional extension, parametrized as follows (e.g. [23])

$$\tilde{G} = \{ g = (\tau, b, a, R, \lambda) \mid R \in SO(3) ; a, b \in \mathbb{R}^3 ; \tau, \lambda \in \mathbb{R} \}$$

with the composition law

$$(\tau, b, a, R, \lambda)(\tau', b', a', R', \lambda') = (\tau + \tau', b + \tau' a + R b', a + R a', R R', \lambda'')$$

where

$$\lambda'' = \left(\tfrac{1}{2} \tau' a^T a + a^T R b' \right) + \lambda + \lambda' .$$

Then $h : \tilde{G} \to G : (\tau, b, a, R, \lambda) \longmapsto (\tau, b, a, R)$ is a surjective homomorphism, the extension map.

Define also $p : G \to V : (\tau, b, a, R) \mapsto (\tau, b)$ and $\pi = p \circ h$. They are surjective maps onto the space V of translations.

We will now identify V with the (flat) space-time and consider G and $\tilde{G}$ as fibre bundles over V . They are in fact principal (frame) bundles with, as standard fibre the homogeneous groups G^h and $\tilde{G}^h$, respectively, acting on the right [24]. To make

this more explicit we will write[4] $y=(x^\alpha, e_0^\Gamma, e_B^\Gamma, x)\in\tilde{G}$ when considered as bundle space (where $(x^\alpha)=(x^0,x^\Gamma)$, (e_0^Γ) , (e_B^Γ) correspond to (τ, b) , a and R , respectively).
The right action of $\tilde{G}^h=\{g=(a,R,\lambda)\}$ is then

$$r_g y = \left(x^\alpha ,\ e_0^\Gamma + e_B^\Gamma a^B ,\ e_C^\Gamma R^C_{\cdot B} ,\ x+\lambda\right)$$

which on G is exactly the correct action if (e_a^α) is interpreted as a Galilei frame [7].

Define

$$\theta^a := \theta^a_\alpha\, dx^\alpha \tag{6.1}$$

where $\theta^a_\alpha\, e^\alpha_b = \delta^a_b$. Then (θ^a) is the fundamental $\mathbb{R}^4$-valued 1-form on the bundle of Galilei frames G . We can pull it back to $\tilde{G}$ by means of h . To introduce also a connection form we first let $\{f_0, f_A, F_A, F_{AB}, f_*\}$ $(A<B)$ be a basis of the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ such that the non vanishing brackets are

$$[f_0, F_A] = -f_A \quad , [f_A, F_B] = -\delta_{AB} f_* \quad ,$$

$$[f_A, F_{BC}] = 2\,\delta_{A[B}\,\delta^K_{C]}\, f_K \quad ,[F_A, F_{BC}] = 2\,\delta_{A[B}\,\delta^K_{C]}\, F_K \quad ,$$

$$[F_{AB}, F_{CD}] = 2\,\delta_{A[C}\, F_{D]B} - 2\,\delta_{B[C}\, F_{D]A}$$

$$(F_{(AB)}=0) .$$

If we now write a $\tilde{\mathfrak{g}}^h$-valued 1-form ω on G as

(4) Capital Greek and Latin indices run from 1 to 3, the former denote space components, the latter frame components and are raised and lowered by means of δ^{AB} and δ_{AB}

$$\omega = \omega^A F_A + \frac{1}{2}\omega^{AB} F_{AB} + \omega^* f_*$$

and impose the conditions that it be a connection form, i.e. satisfy [24]

$$r_g^* \omega = ad_{g^{-1}} \omega$$

and

$$\omega(\underset{\sim}{A}) = A$$

for every $A \in \tilde{\mathcal{G}}^h$, where $\underset{\sim}{A}$ is the corresponding generator of the right action on $\tilde{G}$, then

$$\omega^A = \theta^A_\rho \left(de^\rho_o + \Gamma^\rho_{\lambda\mu} e^\mu_o \, dx^\lambda \right)$$

$$\omega^{AB} = \delta^{BR} \theta^A_\rho \left(de^\rho_R + \Gamma^\rho_{\lambda\mu} e^\mu_R \, dx^\lambda \right) \tag{6.2}$$

$$\omega^* = d\chi - A_\lambda dx^\lambda$$

where $\Gamma^\rho_{\lambda\mu}$ and A_λ are arbitrary functions of x^α only and the $\Gamma^\rho_{\lambda\mu}$ are the components of a Galilei connection compatible with the standard flat Galilei structure on $V = \mathbb{R}^4$. In particular, choosing $\Gamma^\rho_{\lambda\mu}$ and A_λ equal to zero, we have interpreted $\tilde{G}$ as a principal fibre bundle over $V = \mathbb{R}^4$ with its standard flat connection.

Now the left action of $\tilde{G}$ on itself takes the form

$$\ell_g(x^\alpha, e^\alpha_a, \chi) = \Big(x^o + \tau,\ R^\Gamma_{\cdot\Delta} x^\Delta + a^\Gamma x^o + b^\Gamma,\ R^\Gamma_{\cdot\Delta} e^\Delta_o + a^\Gamma,$$
$$R^\Gamma_{\cdot\Delta} e^\Delta_B,\ \chi + \lambda + \frac{1}{2} a^T a\, x^o + a^\Gamma \delta_{\Gamma\Delta} R^\Delta_{\cdot\Sigma} x^\Sigma \Big)$$

which is recognized as the standard action of the Galilei group on flat space-time and its natural lift to the frame bundle. We will now interpret the extended frame bundle as an evolution space for the Galilean spinning particle motivated by the relativistic model [11]. For the free particle we thus wish to choose an $\ell_{\tilde{G}}$ -invariant Cartan 1-form θ whose exterior derivative will define the Hamiltonian structure.

It is easily verified that θ^{a} and ω^{A}, ω^{AB} defined by (6.1) and (6.2) are $\ell_{\tilde{G}}$ -invariant 1-forms on $\tilde{G}$ and it takes only a little more work to see that the same is true for

$$\gamma := \omega^{*} - \frac{1}{2}\delta_{\Gamma\Delta} e^{\Gamma}_{o} e^{\Delta}_{o} \theta^{o} - \delta_{\Gamma\Delta} e^{\Gamma}_{o} e^{\Delta}_{C} \theta^{C} \tag{6.3}$$

The most general $\ell_{\tilde{G}}$ -invariant 1-form on $\tilde{G}$ is thus

$$\theta = P_{a}\theta^{a} + G_{A}\omega^{A} + \frac{1}{2} M_{AB}\omega^{AB} + m\gamma$$

where P_{a}, G_{A}, $M_{AB} = M_{[AB]}$ and m are eleven arbitrary constants.

Two such invariant 1-forms θ and $\bar{\theta}$ can be called *equivalent (in the sense of the Hamiltonian structure they define)* if there exists a diffeomorphism $\phi : \tilde{G} \longrightarrow \tilde{G}$ such that

(i) $\phi^{*}\bar{\theta} = \theta$, and

(ii) $\phi \circ \ell_{g} = \ell_{g} \circ \phi$ for all $g \in \tilde{G}$.

This implies immediately that ϕ must be any right translation. Under the latter the constants change as follows

$$\begin{aligned}
\bar{P}_{o} &= P_{o} + \tfrac{1}{2} m a^{T} a - a_{K} R^{K}_{\cdot L} P^{L}, \\
\bar{P}^{A} &= R^{A}_{\cdot B} P^{B} - m a^{A}, \\
\bar{M}^{AB} &= R^{A}_{\cdot K} R^{B}_{\cdot L} M^{KL} + 2 R^{[A}_{\cdot K}\left(b^{B]} P^{K} + a^{B]} G^{K}\right) - 2m\, a^{[A} b^{B]}, \\
\bar{G}^{A} &= R^{A}_{\cdot K}\left(G^{K} + \tau P^{K}\right) + m\left(b^{A} - \tau a^{A}\right), \\
\bar{m} &= m
\end{aligned} \tag{6.4}$$

The constant m is thus an invariant under such changes, and if $m \neq 0$ the only other invariants turn out to be $S^2 = \frac{1}{2} S^{AB} S_{AB}$ where $S^{AB} := M^{AB} + 2m^{-1} G^{[A} P^{B]}$ which can be interpreted as the (intrinsic) spin of the particle and $P_0 - (2m)^{-1} P^K P_K$, the "internal energy" [23]. One recovers here the classification of Galilei invariant symplectic structures obtained by Souriau [9] using the orbits of the co-adjoint representation.

We will assume that $m > 0$ and use the transformations (6.4) to bring θ into the form

$$\theta = P_0 \theta^0 + \frac{1}{2} S_{AB} \omega^{AB} + m \Upsilon \tag{6.5}$$

which is now only preserved under space rotations. The exterior derivative of θ then becomes (see equation (6.9) below)

$$\sigma := d\theta = m\, \delta_{AB}\, \theta^A \wedge \omega^B - \frac{1}{2} S_{AB}\, \omega^{AC} \wedge \omega_C{}^B \ .$$

It is left-invariant, of course, and thus of constant rank, whence presympelctic. (Note that σ is exact on $\tilde{C}$, but not if considered as a form on G.)

By computing the kernel of σ (e.g. by the method of [11] and [16]) it is found that a curve $\tau \mapsto (x^\alpha(\tau), e^\alpha_\alpha(\tau), \chi(\tau))$ is tangent to $\ker \sigma$ if and only if

$$dx^0/d\tau = v^0 \ , \ dx^\Gamma/d\tau = v^0 e_0^\Gamma \ , \ de_0^\Gamma / d\tau = 0 \ , \tag{6.6}$$

$$de_A^\Gamma / d\tau = \lambda\, e_B^\Gamma S^B{}_A \ , \ d\chi / d\tau = \zeta \tag{6.7}$$

where v^0, λ and ζ are three arbitrary parameters. Therefore $\ker \sigma$ has dimension 3 and the phase space dimension 8 as it should for a particle with spin. Equation (6.6) means that the worldline is a straight line. From (6.7) we can derive (recalling that for a standard flat Galilei frame $e^{0}_{\alpha} = \delta^{0}_{\alpha}$) that the space-time components of the spin $S^{\alpha\beta} := e^{\alpha}_{A} e^{\beta}_{B} S^{AB}$ are constants of the motion.

Remark :
Right translations on $\tilde{G}$, while they map a Cartan 1-form into an equivalent one as far as the induced (pre)-symplectic structure is concerned, do not leave the fibre structure of $\tilde{G}$ over space-time V invariant. Thus if other choices of the constants are made then in (6.5) the space-time motion will be different, it can in fact be unrecognizable. In a sense, we have here incorporated an auxiliary condition corresponding to (2.12) or (5.8).

We propose now to apply a "minimal interaction principle" to the Cartan 1-form Θ given by (6.5). This means that we now consider V as a space-time equipped with a possibly curved Newtonian structure and replace $\tilde{G}$ by an 11-dimensional extension P of the bundle of Galilei frames over V. We interpret ω^{A} and ω^{AB}, now given by (6.2), as components of the connection form corresponding to the Newtonian connection on V . But the generalization of γ to curved space is somewhat less evident. In (6.3) ω^{*} can be considered given by (6.2) for some $A_{\alpha}(x)$ but the only way to write the other two terms "covariantly" is at the expense of introducing a unit timelike vector field u^{α} and replacing γ by

$$\gamma = \omega^{*} - \frac{1}{2} \overset{u}{\gamma}_{\alpha\beta} e^{\alpha}_{0} e^{\beta}_{0} \theta^{0} - \overset{u}{\gamma}_{\alpha\beta} e^{\alpha}_{0} e^{\beta}_{A} \theta^{A} \tag{6.8}$$

To compute the exterior derivative of Θ we first note that from (6.2) we obtain, using the fact that $\{ e^{\alpha}_{a} \}$ is a Galilei frame, that

$$de^{\alpha}_{0} = e^{\alpha}_{A} \omega^{A} - \Gamma^{\alpha}_{\lambda\mu} e^{\lambda}_{r} e^{\mu}_{0} \theta^{r} , \quad de^{\alpha}_{A} = e^{\alpha}_{K} \omega^{K}_{\ \ A} - \Gamma^{\alpha}_{\lambda\mu} e^{\lambda}_{r} e^{\mu}_{A} \theta^{r}$$

and then

$$d\gamma = -\frac{1}{2} F_{rs} \theta^r \wedge \theta^s + \left(u_{A/0} + u_B u^B{}_{/A} \right) \theta^0 \wedge \theta^A$$

$$- u_{A/B} \theta^A \wedge \theta^B + \delta_{AB} \theta^A \wedge \omega^B$$

where $u^a := \theta^a_\alpha u^\alpha$, $F_{rs} = 2 e^\alpha_r e^\beta_s \partial_{[\alpha} A_{\beta]}$ and $u^a{}_{/b} = \theta^a_\alpha e^\beta_b \nabla_\beta u^\alpha$. (To derive this the first Cartan equations

$$d\theta^0 = 0 \quad , \quad d\theta^A = \theta^0 \wedge \omega^A - \omega^A_{\cdot B} \wedge \theta^B$$

have been used). Then using also the second Cartan equations

$$d\omega^{AB} = \frac{1}{2} R^{AB}_{\cdot\cdot rs} \theta^r \wedge \theta^s - \omega^A_{\cdot C} \wedge \omega^{CB}$$

$$\omega \quad = \frac{1}{2} R^A_{\cdot 0 rs} \theta^r \wedge \theta^s - \omega^A_{\cdot B} \wedge \omega^B$$

where $R_{bcd} = \theta^a_\alpha e^\beta_b e^\gamma_c e^\delta_d R^\alpha_{\cdot\beta\gamma\delta}$, we find

$$\begin{aligned} \sigma &:= d\theta \\ &= \frac{1}{2} \left(-m F_{rs} + \frac{1}{2} S_{AB} R^{AB}_{\cdot\cdot rs} \right) \theta^r \wedge \theta^s \\ &+ m \left(u_{A/0} + u_B u^B{}_{/A} \right) \theta^0 \wedge \theta^A - m u_{A/B} \theta^A \wedge \theta^B \\ &+ m \delta_{AB} \theta^A \wedge \omega^B - \frac{1}{2} S_{AB} \omega^{AC} \wedge \omega_C^{\cdot B} \end{aligned} \tag{6.9}$$

For a curve $\tau \mapsto \left(x^\alpha(\tau) , e^\alpha_a(\tau) , \chi(\tau) \right)$ to be tangent to $\ker \sigma$ we find again by the method of [11]

$$v^\alpha := \dot{x}^\alpha = v^0 e^\alpha_0$$

$$\dot{e}^\alpha_0 = \gamma^{\alpha\rho} \left(F_{\rho\sigma} + 2 \overset{u}{\partial}_{\lambda[\rho} \nabla_{\sigma]} u^\lambda \right) v^\sigma - (2m)^{-1} R^\alpha_{\cdot\lambda\mu\nu} v^\lambda S^{\mu\nu} \tag{6.10}$$

$$\dot{e}^\alpha_A = \lambda e^\alpha_K S^K_{\cdot A} \quad , \quad \dot{\chi} = \zeta \tag{6.11}$$

where $\dot{} := (dx^\alpha/d\tau)\nabla_\alpha$, $S^{\alpha\beta} := e^\alpha_A e^\beta_B S^{AB}$ and v^0, λ and ζ are arbitrary. The phase space has thus dimension 8. Choosing τ to be the worldtime we have $v^0 = 1$ and hence $v^\alpha = e^\alpha_0$.
The space-time components $S^{\alpha\beta}$ of the spin satisfy (5.8) and (6.11) implies

$$\dot{S}^{\alpha\beta} = 0$$

Now we note that we get exactly the same equations of motion as (5.9) provided the first term on the right hand side of (6.10) vanishes. But this is the case by (3.9) if the so far undetermined "gauge fields" A_α and u^α satisfy (3.2), i.e. define the same connection $\Gamma^\gamma_{\alpha\beta}$ that was already given in terms of ω^A and ω^{AB}.

This is perhaps not so surprising if compared with the relativistic model in [11]. There the Cartan 1-form ϑ can be formulated covariantly in terms of $g_{\alpha\beta}$ and $\Gamma^\gamma_{\alpha\beta}$ and it is also understood that the $\Gamma^\gamma_{\alpha\beta}$ represent the Levi-Cività connection.

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