Variation Principle for Nonlinear Wave Propagation*

by

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ABSTRACT.: Variation principle is derived which determines stationary nonlinear propagation of electrostatic waves in the self-consistent density profile. Example is given for lower--hybrid waves and the relation to the variation principle for the Lagrangean density of electromagnetic fluids is discussed.

The present paper deals with a problem which is related to the heating of a plasma by a moderately strong electrostatic wave. When such a wave propagates through the plasma, the ponderomotive force associated with the oscillating field accelerates the particle and causes a modification of the plasma density profile, which in turn refracts the wave, resulting in, say, a self-focusing of the wave. Such nonlinear effects can strongly affect the penetration of the wave into plasma and hence the resulting plasma heating. Theoretically, the problem is to determine self-consistently the profiles of

* Work partially supported by the Scientific Research Fund of the Ministry of Education of Japan.

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both the electric field and the density. This problem usually requires a solution of highly complicated partial differential equations with appropriate boundary conditions.

In this paper, we show that the problem can be reduced to a simple variation principle under relatively general conditions. An advantage of the variation principle is that it contains only low-order differential operation, usually of second order, whereas the original differential equations often contain higher-order differentials. Therefore, the variation principle is suited for numerical analyses of the problem.

The basic assumptions and/or approximations can be stated as follows:

1) Nonlinear effects arise only through the local density modification of the plasma; the harmonic generation effect is negligible.
2) Thermal effects yield only small correction to the propagation characteristics of the wave and are negligible in the calculation of the nonlinear term. This assumption is justified when the characteristic wavelength of the wave is much greater than the Larmour radii, the Debye lengths and the excursion lengths due to the wave electric field.
3) The gradient scale length of the unperturbed density is much greater than the wavelength, so that the local dispersion relation can be used to describe the linear wave propagation.
4) The magnetic field can be treated uniform, which is justified when its gradient scale length is much greater than the scale length of the nonlinear density modification.

We first derive an expression for the density profile in terms of the wave electrostatic potential by using the static balance of force which acts on the fluid along the magnetic field, and then derive a variation principle from the Poisson equation for the wave potential.

Let us denote the fluid velocity, density and potential as

$$\vec{v} = \vec{v}(\omega) \exp[-i\omega t] + c.c$$

$$n = n_0(\vec{r}) + \{ n(\omega) \exp[-i\omega t] + c.c. \}$$

$$\varphi = \varphi_0(\vec{r}) + \{ \varphi(\omega) \exp[-i\omega t] + c.c. \}$$

Denoting the time average by angular bracket, we can write the static force-balance equation as

$$-q \nabla_z \varphi_0 - T \nabla_z \ln n_0 = m \langle \vec{v} \cdot \vec{\nabla} v_z \rangle \qquad (1)$$

where we chose the z-axis along the magnetic field and the rest of the notation is standard. Note that we ignored the nonlinear effects in the thermal correction term. One can assume charge neutrality for the static component, then by eliminating φ_0 from eq.(1) for the electron and the ion, one obtains

$$-\nabla_z \ln n_0 = \sum m \langle \vec{v} \cdot \vec{\nabla} v_z \rangle / (T_e + T_i) \qquad (2)$$

where the summation is over the species of the particle. To calculate the nonlinear term $\langle \vec{v} \cdot \vec{\nabla} v_z \rangle$, we neglect the thermal effect and use the linear approximation. Then in the Fourier representation in time, we have $\vec{v}(\omega) = -i\omega \vec{r}(\omega)$, where $d\vec{r}/dt = \vec{v}$, and $v_z(\omega)=q\nabla_z\varphi(\omega)/im\omega$. Using these relations, we obtain

$$\sum m\langle\vec{v}\cdot\vec{\nabla}v_z\rangle = \sum m\left\{\vec{v}(-\omega)\cdot\vec{\nabla}v_z(\omega)+\vec{v}(\omega)\cdot\vec{\nabla}v_z(-\omega)\right\}$$

$$= \vec{P}_0(-\omega)\cdot\vec{\nabla}\nabla_z\varphi(\omega)+\vec{P}_0(\omega)\cdot\vec{\nabla}\nabla_z\varphi(-\omega), \qquad (3)$$

where $\vec{P}_0(\omega)\equiv\sum q\vec{r}\,(\omega)$ is the polarization vector. It is related to the electric field by the linear relation

$$\vec{P}_0(\omega)=-\overleftrightarrow{\chi}_0(\omega)\cdot\vec{\nabla}\varphi(\omega), \quad \overleftrightarrow{\chi}_0(-\omega)=\overleftrightarrow{\chi}_0^{\,T}(\omega), \qquad (4)$$

where $\overleftrightarrow{\chi}_0$ is the polarizability (per particle) for the cold plasma model and is independent of the density. Equation (3) can then be written as

$$\sum m\langle\vec{v}\cdot\vec{\nabla}v_z\rangle = -\left[\overleftrightarrow{\chi}_0(-\omega)\cdot\vec{\nabla}\varphi(-\omega)\right]\cdot\nabla_z\vec{\nabla}\varphi(\omega)+c.c.$$

$$=-\frac{1}{2}\nabla_z\left\{\vec{P}_0(-\omega)\cdot\vec{\nabla}\varphi(\omega)+\vec{P}_0(\omega)\cdot\vec{\nabla}\varphi(-\omega)\right\}$$

Substituting this relation into eq.(2) and integrating with respect to z give

$$n_0(\vec{r})=N(x,y)\exp\left\{\left[\vec{P}_0(-\omega)\cdot\vec{\nabla}\varphi(\omega)+\vec{P}_0(\omega)\cdot\vec{\nabla}\varphi(-\omega)\right]/2(T_e+T_i)\right\} \qquad (5)$$

where N(x, y) denotes the unperturbed density. Noting the relation (4), one can see that n_0/N depends on the position only through the potential $\varphi(\pm\omega)$.

We now consider the Poisson equation which yields

$$\nabla^2\varphi(\omega) = -4\pi\vec{\nabla}\cdot n_0\vec{P}(\omega) \qquad (6)$$

where $\vec{P}$ now stands for the polarization including the thermal effect and we again used the charge neutrality for n_0. Now from eqs.(4) and (5), we have

$$\frac{\partial n_o}{\partial \vec{\nabla}\varphi(-\omega)} = \frac{n_o}{2(T_e+T_i)}\left\{ \vec{P}_o(\omega) - \vec{\nabla}\varphi(\omega)\cdot \overleftrightarrow{\chi}_o(-\omega)\right\}$$

$$= \{ n_o/(T_e+T_i)\}\ \vec{P}_o(\omega),$$

so that eq.(6) becomes

$$\nabla^2\varphi(\omega) = -4\pi(T_e+T_i)\vec{\nabla}\cdot\frac{\partial}{\partial\vec{\nabla}\varphi(-\omega)}n_o - 4\pi\vec{\nabla}\cdot n_o\Delta\vec{P}(\omega) \tag{7}$$

where $\Delta\vec{P}(\omega)=\vec{P}(\omega)-\vec{P}_o(\omega)$. If one neglects this term, one can immediately derive a variation principle by multiplying both sides of eq.(7) by $\delta\varphi(-\omega)$, integrating over the space and taking its real part. The result is

$$\delta\int d^3\vec{r}\left\{ |\vec{\nabla}\varphi(\omega)|^2 + 4\pi n_o(T_e+T_i)\right\} = 0, \tag{8}$$

where we ignored the contribution from the boundaries, assuming either a periodic boundary condition or the vanishing of the potential at the boundaries. This equation is closed in the electric field and does not contain its derivatives. Therefore, for an isotropic plasma, eq.(8) simply yields an algebraic equation for $|\vec{\nabla}\varphi(\omega)|^2$. The solution then depends only on the local density N(x, y) and is independent of the density of the neighbouring points. This situation is, however, drastically changed by including thermal corrections, i.e. the last term of eq.(7).

We calculate this term using the linear theory. We use the local dispersion relation, which is equivalent to using the relation

$$\Delta\vec{P}(\omega) = -\int\frac{d^3\vec{k}}{(2\pi)^3}\,\Delta\overleftrightarrow{\chi}(\vec{k},\omega)\cdot i\vec{k}\,\varphi(\vec{k},\omega)\exp[i\vec{k}\cdot\vec{r}]$$

$$= -\,\Delta\overleftrightarrow{\chi}(-i\vec{\nabla},\omega)\cdot\vec{\nabla}\varphi(\omega), \qquad (9)$$

where $\Delta\overleftrightarrow{\chi}(\vec{k},\omega)$ denotes the thermal correction to the polarizability. To lowest order, its matrix element may be written in the form

$$\Delta\chi_{\alpha\beta}(\vec{k},\omega) = A_{\alpha\beta}(\omega)\,k_\alpha k_\beta \qquad (\alpha,\beta = x,y,z). \ (10)$$

Substituting eqs.(9) and (10) into eq.(7) and carrying out the same procedure as we did in deriving eq.(8), we obtain the correction term, which is to be added to the left-hand side of eq.(8), as

$$-\,\delta\int d^3\vec{r}\left\{4\pi N\sum_{\alpha,\beta}A_{\alpha\beta}(\omega)\,|\nabla_\alpha\nabla_\beta\varphi(\omega)|^2\right\}. \qquad (11)$$

Thus the thermal correction contains the derivative of the electric field and hence produces a coupling of the field at different spatial positions. The resulting variation principle has a structure similar to that for quantum mechanics. For an isotropic plasma, the term (11) corresponds to the contribution of the kinetic energy, while the term (8) to that of the potential energy. Note that n_0 is given by eqs.(5) with (4).

As an example, consider the propagation of a lower-hybrid wave. For this wave, one can treat ions as unmagnetized, while for electrons only their motion parallel to the magnetic field is to be taken into account in the calculation of the nonlinear term. The cold plasma polarizability then becomes

$$\chi_{ozz} = -\frac{e^2}{m\omega^2}, \quad \chi_{oxx} = \chi_{oyy} = \frac{e^2}{m\Omega_e^2} - \frac{Z^2e^2}{M\omega^2}$$

and the other components vanish. Here m and M are the mass of the electron and the ion, respectively, and Ω_e is the electron cyclotron frequency. Thermal correction can be calculated by including both the fluid effect and the finite Larmour radius effect for the electron; we get

$$A_{zz} = 3\,\frac{e^2 T_e}{m^2\omega^4}, \qquad A_{xx} = A_{yy} = 3\left[\frac{e^2 T_e}{4m^2\Omega_e^4} + \frac{Z^2e^2 T_i}{m^2\omega^4}\right],$$

$$A_{xz} = A_{yz} = \frac{1}{2}\,\frac{e^2 T_e}{m^2\Omega_e^2\omega^2}, \qquad A_{xy} = 0.$$

Numerical analysis of the variation principle using these coefficients is now underway and some of the results will be reported at the Conference.

Let us finally discuss the relation of this variation principle to that for the Lagrangean density for the electromagnetic fluid. The Lagrangean density can be written as[1])

$$L = \sum \int d^3\vec{a}\,\left\{ \frac{m}{2} n_a \dot{\vec{r}}^{\,2} + \frac{n_a q}{c}\,\dot{\vec{r}}\cdot\vec{A} - n_a q\varphi - n_a H(n) + \frac{n_a}{8\pi n}|\vec{\nabla}\varphi|^2 \right\}. \tag{12}$$

where n_a is the density as a function of the initial coordinate $\vec{a}$, $\vec{A}$ is the vector potential and $H = \int dnp/n^2$ with $p = p(n)$ being the pressure which satisfies the equation of state. Note that we added the contribution of the electric field to the expression given in ref.1). We write $\vec{r} = \vec{r}_0 + \{\vec{r}(\omega)\exp[-i\omega t] + c.c.\}$ and express the density as a function of $\vec{r}_o$ and t using the relation $n(\vec{r}_c, t) = n_a[\partial^3\vec{r}_o/\partial^3\vec{a}]^{-1}$. Taking only the slowly-varying part, we then obtain

$$L[\vec{r}_o, \varphi(\pm\omega)] = \sum \int d^3\vec{a} \Big\{ \frac{m}{2} n_a \dot{\vec{r}}_o^{\,2} + \frac{q}{c} n_a \dot{\vec{r}}_o \cdot \vec{A} - n_a H(n) \tag{13}$$
$$- \Big[q n_a \varphi_o + q n_a \langle \vec{r}(-\omega)\cdot\vec{\nabla}\varphi(\omega) + \vec{r}(\omega)\cdot\vec{\nabla}\varphi(-\omega) \rangle \Big] + \frac{n_a}{4\pi n} |\vec{\nabla}\varphi(\omega)|^2 \Big\}.$$

Note that φ_o is an explicit function of $\vec{r}_o$ and $\varphi(\pm\omega)$ because of the charge neutrality condition. One can derive the equations of motion for given $\varphi(\pm\omega)$ from this Lagrangean. The force-balance along the magnetic field then yields the relation

$$- q n_a \langle \vec{r}(-\omega)\cdot\vec{\nabla}\varphi(\omega) + \vec{r}(\omega)\cdot\vec{\nabla}\varphi(-\omega) \rangle - q n_a \varphi_o - n_a H(n)$$
$$= [\text{ function of } rA_\theta], \tag{14}$$

where we assumed the axial symmetry. Variation of L with respect to $\varphi(\pm\omega)$ operates only on the second line of eq.(13). Using eq.(14) for the term in the square bracket one obtains

$$\delta_\varphi L = \sum \int d^3\vec{a} \Big\{ \frac{1}{4\pi} \frac{n_a}{n} \delta |\vec{\nabla}\varphi(\omega)|^2 + n_a \delta H(n) \Big\}$$
$$= \sum \int d^3\vec{r}_o \Big\{ \frac{1}{4\pi} \delta |\vec{\nabla}\varphi(\omega)|^2 + P(n)\delta n/n \Big\}$$
$$= \delta \sum \int d^3\vec{r}_o \Big\{ |\vec{\nabla}\varphi(\omega)|^2/4\pi + \int p(n)\, dn/n \Big\}.$$

For the isothermal process, the relation $\delta_\varphi L = 0$ gives eq.(8).

1) N. G. Van Kampen and B. U. Felderhof, Theoretical Method in Plasma Physics (North-Holland Pub., Amsterdam, 1967) p.71.