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ON CLEBSH-GORDAN EXPANSION
FOR $O(2h=1,1)$

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**ON CLEBSH-GORDAN EXPANSION
FOR $O(2h=1,1)$**

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1. INTRODUCTION

A. Physical motivation

The problem of the direct product decomposition of two spin 0 unitary representations of the supplementary series of $SO(2h+1,1)$ (h - positive integer) arose in the study of a conformal covariant model of a self interacting (quantized) scalar field (see [M 1]). The model originates from the Green function formulation of a (renormalizable) ϕ^3 theory in six-dimensions given by Symanzik [S 1] some 13 years ago; the discussion of conformal covariance was facilitated by a modification of the equation for the propagator involving the stress energy tensor [M 3]. It is the desire to incorporate this 6-dimensional model along with the physical 4-dimensional case that led us to work from the outset with the general case of $2h$ -space-time dimensions. This generality has also the advantage that we are able to check our formulas for $h=1$ with the known case of $SO(3,1)$ (see [N 1]). We are concerned throughout with the Euclidean formulation of the theory (see, e.g., [S 2]) in which the Lorentz group is replaced by $SO(2h)$ and the covariance under infinitesimal conformal transformations of $SO(2h,2)/\mathbb{Z}_2$ is extended to global $O^\uparrow(2h+1,1)$ covariance (the arrow \uparrow indicates that we do not consider transformations which change the sign of the $2h + 2$ axis).

It is shown in [M 1] that the conformal expansion of the Euclidean Green functions allows one to diagonalize and solve the infinite set of integral equations for these functions. As a result one obtains conformal covariant operator product expansions which have been

suggested in a number of papers (see [F1], [M5] and references therein). It is also related to the problem of duality considered in [F1] .

For background and further references on the conformal group and its applications to quantum field theory the reader may consult the recent reviews [M2],[F1] and [R1] . Some preliminary results of this paper are quoted in Appendix A of [T2].

B. A two-parameter family of infinite dimensional representations of the Euclidean conformal group

In order to fix notation and terminology we start with a brief description of a two-parameter family of (irreducible) representations $\chi = [\ell, c]$ of $O^+(2h+1, 1)$ ($\ell = 0, 1, 2, \dots$; c is an arbitrary complex number). (A complete classification of the unitary irreducible representations of $SO(n, 1)$ is given in [H 1] [0 2] . In the case of $n = 2h+1$ these representations are labelled by one continuous and h discrete parameters.)

We shall introduce a space C_χ of infinitely differentiable (symmetric, traceless) tensor-valued functions

$$f(x) = f_{\mu_1 \dots \mu_\ell} (x_1, \dots, x_{2h}) \quad (\mu_i = 1, \dots, 2h)$$

on \mathbb{R}^{2h} , whose behaviour at infinity is dictated by conformal covariance. In order to reveal the meaning of the latter statement we shall first display the action of the representation χ on C_χ .

The Euclidean conformal group $O^+(2h+1, 1)$ acts transitively on the compactification S^{2h} of \mathbb{R}^{2h} . Here S^{2h} is the unit

sphere in $2h+1$ dimensions related to \mathbb{R}^{2h} through the stereographic projection

$$\begin{aligned} \hat{\xi}_\mu &= \frac{2x_\mu}{1+x^2}, \quad \mu = 1, \dots, 2h, \quad x^2 = x_1^2 + \dots + x_{2h}^2, \\ \hat{\xi}_{2h+1} &= \frac{1-x^2}{1+x^2} \quad \left(\sum_{\mu=1}^{2h} \hat{\xi}_\mu^2 + \hat{\xi}_{2h+1}^2 = 1 \right). \end{aligned} \quad (I.1)$$

Its action is generated by the following transformations in \mathbb{R}^{2h} :

- a) translations and Euclidean rotations: $x' = a + \Lambda x$
 $a = (a_1, \dots, a_{2h}), \quad \Lambda \in O(2h);$
 b) dilatations $x = \varrho x, \quad \varrho > 0;$
 c) conformal inversion

$$Rx = -\frac{x}{x^2}. \quad (I.2)$$

The special conformal transformations are given by

$$x' = RT_\theta Rx = \frac{x - x^2 \theta}{1 - 2\theta x + \theta^2 x^2}, \quad (I.3)$$

where T_θ is a translation: $T_\theta x = x + \theta$.

We shall define the representation $\mathcal{U} = [\ell, c]$ of the generating transformations a), b), c) of $O(2h+1, 1)$ in \mathcal{L}_2 in the following way (cf. [T1]):

$$[\mathcal{U}(a, \Lambda) f](x) = \Lambda^{\otimes \ell} f(\Lambda^{-1}(x-a)) \quad (I.4)$$

$$[\mathcal{U}(\varrho) f](x) = \varrho^{-h-c} f\left(\frac{x}{\varrho}\right) \quad (I.5)$$

$$[U(R)f](x) = \frac{\tau(x)^{\otimes \ell}}{(x^2)^{c+h}} f(Rx), \quad (I.6)$$

where

$$\tau(x)_{\mu\nu} = -\delta_{\mu\nu} + 2 \frac{x_\mu x_\nu}{x^2}, \quad \tau^2 = 1 \quad (I.7)$$

and

$$[B^{\otimes \ell} f(x)]_{\mu_1 \dots \mu_\ell} = B_{\mu_1 \nu_1} \dots B_{\mu_\ell \nu_\ell} f_{\nu_1 \dots \nu_\ell}(x) \quad (I.8)$$

($B = \Lambda, Z$). [The exponent $h+c$ in (I.5) is often denoted by d and called (scale) dimension of f .]

Now we are in a position to determine the behaviour at infinity of the tensor functions $f(x) \in C_X$. Conformal covariance implies that if $f \in C_X$ then also $U(R)f \in C_X$. Using (I.6) and the involution property (I.7) of $\tau(x)$, we find that

$$(1+x^2)^{h+c} \tau(x)^{\otimes \ell} [U(R)f](x) \xrightarrow{x \rightarrow \infty} f(0).$$

Accordingly, we shall postulate that for any $f(x) \in C_X$ there exists a finite (tensor-valued) limit

$$\lim_{x \rightarrow \infty} (1+x^2)^{h+c} \tau(x)^{\otimes \ell} f(x) = A^f. \quad (I.9)$$

We shall see in Sec. II that the representations \mathcal{R} , so defined, can be extended (by an appropriate completion of C_X) to unitary (irreducible) representations of $O^\uparrow(2h+1, 1)$ for the following values of c and ℓ :

$$c - \text{pure imaginary } (c = i\sigma), \quad \ell - \text{arbitrary (principal series)} \quad (I.10a)$$

$$\left. \begin{array}{l} -h < c < h \quad \ell = 0 \quad (h \geq 1) \text{ and} \\ 1-h < c < h-1 \quad \ell = 1, 2, \dots \quad (h \geq 2) \\ c \neq 0 \end{array} \right\} \text{(supplementary series) (I.10b)}$$

(Note that our terminology follows the analogy with the Lorentz group $O^+(3,1)$ and differs from the terminology adopted in [K1] [02]).

C. Outline of results

We consider the problem of decomposition of the direct product of two unitary representations of the supplementary series

$$\chi_{0\ell} \otimes \chi_{0\ell}, \quad \chi_{0a} = [0, c_a], \quad a = 1, 2 \quad (\text{I.11})$$

into irreducible unitary representations.

In other words we would like to expand each

$$f(x_1, x_2) \in C_{\chi_{0\ell}} \otimes C_{\chi_{0\ell}} \quad (\text{I.12})$$

in functions $F_\ell(x)$ transforming according to the unitary representation χ of $O^+(2h+1,1)$.

For $|c_1| + |c_2| \leq h$ the direct product (I.11) is expanded in representations $\chi = [\ell, c = i\sigma]$ of the principal series (cf. [N1] and Appendix A to [M1]):

$$f(x_1, x_2) = \int dV \int dx V(x, c_1, x_2, c_2; x \tilde{\chi}) F_\ell(x). \quad (\text{I.13})$$

Here $\tilde{\chi}$ is the representation dual to $\chi = [\ell, c]$. x)

x) For $h = 1$ the representation $[\ell, 0]$ ($\ell > 0$) should be identified with the irreducible representation $[\ell, c] \oplus [\ell, -c]$ of the full Lorentz group.

$$\nu = [l, c] \Rightarrow \tilde{\nu} = [l, -c] \quad ; \quad (I.14)$$

$$\oint d\nu = \sum_{l=0}^{\infty} \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \rho_l(\sigma) = \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} \rho_l(\sigma), \quad (I.15)$$

where

$$\rho_l(\sigma) = \frac{2(l+h-1)!}{(2\pi)^k l!} \left| \frac{\Gamma(h-1+i\sigma)}{2\Gamma(i\sigma)} \right|^k [\sigma^2 + (h+l-1)^2] \quad (I.16).$$

is the Plancherel measure (of $[R^2]$); finally, $V(x_1, c_1, x_2, c_2; x \tilde{\nu})$ are the Clebsch-Gordan kernels.

We start in Sec. II by defining an invariant bi-linear form of the type

$$(f, g)_2 = \int dx_1 \int dx_2 f(x_1) G_{\tilde{Z}}(x_1 - x_2) g(x_2) \quad (I.17)$$

on $C_x \times C_x$.

The 2-point function $G_{\tilde{Z}}$ and the kernels V are determined uniquely (up to a constant factor) from conformal invariance. In writing down explicit expressions for the Green functions $G_{\tilde{Z}}$ and the kernels V it is convenient to use technique of homogeneous polynomials (see, e.g., [T3] [01] [21]) instead of multiple tensor indices. We write

$$\begin{aligned} V(x_1, c_1, x_2, c_2; x \tilde{\nu}) &= \frac{1}{\sqrt{l!}} V_{\mu_1 \dots \mu_l} (x_1, c_1, x_2, c_2, x \tilde{\nu}) z^{\mu_1} \dots z^{\mu_l} \\ &\equiv V(x_1, c_1, x_2, c_2; x \tilde{\nu}) \frac{z^{\otimes l}}{\sqrt{l!}}, \end{aligned} \quad (I.18)$$

$$\begin{aligned} G_{\tilde{Z}}(x_1 - x_2, z_1, z_2) &= \frac{1}{l!} z_1^{\otimes l} G_{\tilde{Z}}(x_1 - x_2) z_2^{\otimes l} \\ &= \frac{1}{l!} z_1^{\mu_1} \dots z_1^{\mu_l} G_{\tilde{Z}}(x_1 - x_2)_{\mu_1 \dots \mu_l \nu_1 \dots \nu_l} z_2^{\nu_1} \dots z_2^{\nu_l}, \end{aligned} \quad (I.19)$$

where $\tilde{z}, \tilde{z}_1, \tilde{z}_2$ are (complex) isotropic vectors:

$$\tilde{z} = \tilde{z}_1^2 + \dots + \tilde{z}_{2h}^2 = 0. \quad (1.20)$$

It is easily seen, for instance, that the homogeneous polynomial (I.18) is in one-to-one correspondence with the symmetric traceless tensors $V_{\mu_1 \mu_2}$. Indeed, each polynomial function $f(z)$ on the cone (I.20) can be extended in a unique way to a harmonic polynomial in $S \in \mathbb{C}^{2h}$ by setting

$$f(S) = \frac{z^{\ell} \rho^{\ell} \Gamma(2h + \ell - 2)}{\Gamma(2h + 2\ell - 2)} (S^{\dagger} \partial_z)^{\ell/2} P_{\ell}^{(h-\frac{1}{2}, h-\frac{1}{2})} \left(\frac{S \partial_z}{\sqrt{S^{\dagger} \partial_z^2}} \right) f(z), \quad (1.21)$$

$$(\Delta_S f(S) = 0)$$

where $P_{\ell}^{(\alpha, \beta)}(t)$ is the Jacobi polynomial satisfying the differential equation

$$[(1-t^2) \frac{d^2}{dt^2} + [\beta - \alpha - (\alpha + \beta + 2)t] \frac{d}{dt} + \ell(\ell + \alpha + \beta + 1)] P_{\ell}^{(\alpha, \beta)}(t) = 0$$

and the normalization condition

$$\frac{d^{\ell}}{dt^{\ell}} P_{\ell}^{(\alpha, \beta)}(t) = \frac{\Gamma(\alpha + \beta + 2\ell + 1)}{2^{\ell} \Gamma(\alpha + \beta + \ell + 1)}.$$

Then the inverse formula to (I.18) is

$$V_{\mu_1 \mu_2}(\dots, z, \tilde{z}) = \frac{1}{\sqrt{\ell!}} \frac{\partial}{\partial S_{\mu_1}} \dots \frac{\partial}{\partial S_{\mu_2}} V(\dots, z, \tilde{z}, S).$$

The contraction of two tensors \tilde{f} and \tilde{g} can be written in terms of the polynomials (I.21) as

$$f_{\mu_1 \mu_2} g_{\mu_1 \mu_2} = f(\partial_z) g(z) = g(\partial_z) f(z).$$

Further, in Sec. II we study the implications of the positivity condition G_2 and establish the unitarity of the representations of the supplementary series (I.10b). The method used—the $SO(2h-1)_p$ expansion of $\widetilde{G}_2(\rho)$ allows also to find restrictions on C for Minkowski space positivity condition of the corresponding Wightman functions. One has to use different normalization conventions for the two alternative interpretations of G_2 : first, when G_2 is regarded as the kernel of a $O(2h+1,1)$ invariant bi-linear form; second, when it is considered as analytic continuation to the Euclidean region of the τ -function of two tensor fields. In the first case, the adopted convention [see Eqs. (II.24) (II.25) below] implies

$$\int dy G_2(x, y) G_2(y, x_2) = \mathbb{1} \delta(x, x_2), \quad (\text{I.22})$$

where $\mathbb{1}$ stands for the unit operator in the space of symmetric traceless tensors of rank l . In the second case, the two-point function can be normalized in such a way that Wightman positivity is satisfied for

$$l=0, l \geq 1; \quad l=1, 2, \dots \quad C \geq l^2 h - 2. \quad (\text{I.23})$$

The choice of normalization [given by Eqs. (II.40), (II.41)] guarantees also the validity of a number of other desirable properties of the two-point Wightman function listed in Sec. II.D.

Sec.III is devoted to the evaluation of the normalization factor N_l of the invariant 3-point function V (III.4), which plays the role of a Clebsh-Gordan kernel. The normalization of V is determined by requiring a symmetry property with respect to "amputation of external legs" [Eqs. (III.7-8)] and from the Plancherel formula, which implies that Eq. (I.13) can be inverted in the form (III.12).

II. Invariant bi-linear forms. Supplementary series of unitary representations of $O^+(2h+1,1)$

A. Covariant two-point kernels and their Fourier transforms

According to (I.19) we are looking for a function $G(x, x_2; z_1, z_2)$ which is a homogeneous polynomial in each of the isotropic $2h$ -vectors z_1 and z_2 of degree l and transforms covariantly under the representation $\lambda = [l, c]$ of $O^+(2h+1,1)$. In other words we require that

$$G(\Lambda x, \Lambda x_2; \Lambda z_1, \Lambda z_2) = G(x, x_2; z_1, z_2) \quad (\text{II.1})$$

for $\Lambda \in O(2h)$, $a = (a_1, \dots, a_{2h})$;

$$p^{2(l+c)} G(px_1, px_2; z_1, z_2) = G(x_1, x_2; z_1, z_2) \text{ for } p > 0; \quad (\text{II.2})$$

$$\frac{1}{(x_1, x_2)^{l+c}} G(Rx_1, Rx_2; \tau(x_1)z_1, \tau(x_2)z_2) = G(x_1, x_2; z_1, z_2), \quad (\text{II.3})$$

where the conformal inversion R and its vector representation $\tau(x)$ are defined by (I.2) and (I.7).

The general form of G , satisfying the above condition is

$$G = G_2(x_{12}, z_1, z_2) = \frac{n(z)}{(2\pi)^h} \left(\frac{z}{z_1^2}\right)^{h-d} \frac{1}{z_1} (z_1 \tau(x_{12}) z_2)^2. \quad (\text{II.4})$$

where

$$x_{12} = x_1 - x_2, \quad z_1 \tau(x_{12}) z_2 = z_1 z_2 - \frac{z_1 x_{12} / (x_{12} z_2)}{x_1^2} \quad (\text{II.5})$$

and $n(z)$ is a normalization constant. The Euclidean and dilatation invariance of (II.4) are obvious. The verification of its R-invariance (i.e. of property (II.3)) is based on the identity

$$\tau(x_1) \tau(Rx_1 - Rx_2) \tau(x_2) = \tau(x_1 - x_2). \quad (\text{II.6})$$

The homogeneity property of G_2 with respect to x_2 is a consequence of dilatation invariance alone; the tensor structure of G_2 is fixed by R-invariance (cf. [II]).

Using the integral formula

$$\begin{aligned} \frac{\Gamma(d)}{(2\pi)^h} \int \left(\frac{z}{x^2}\right)^d e^{-ipx} dx &= \frac{1}{(2\pi)^h} \int_0^\infty d\alpha \alpha^{d-1} \int dx e^{-\frac{\alpha x^2}{2} - ipx} = \\ &= \int_0^\infty d\alpha \alpha^{d-h-1} e^{-\frac{1}{2\alpha} p^2} = \Gamma(h-d) \left(\frac{z}{p^2}\right)^{h-d} \end{aligned} \quad (\text{II.7})$$

(valid for $d < h$) we obtain the following expression for the Fourier transform of (II.4):

$$\begin{aligned}
\tilde{G}_2(p; z_1, z_2) &= \int G_2(x; z_1, z_2) e^{-ipx} dx \\
&= \frac{n(z)}{(2\pi)^h} \sum_{\kappa=0}^{\ell} \frac{(z_1, z_2)^{\ell-\kappa}}{(\ell-\kappa)!} \frac{[(z_1, \partial_p)(z_2, \partial_p)]^{\kappa}}{\kappa!} \int \left(\frac{z}{x^2}\right)^{h+c\kappa} e^{-ipx} dx \\
&= \frac{n(z)}{\Gamma(c+h+\ell)\Gamma(c+h-1)} \left(\frac{p^2}{2}\right)^c \sum_{\kappa=0}^{\ell} \frac{\Gamma(\ell-\kappa-c)\Gamma(h+\kappa+c-1)}{\kappa!(\ell-\kappa)!} \left[\frac{(p z_1)(p z_2)}{\frac{1}{2}p^2}\right]^{\ell-\kappa} (z_1, z_2)^{\kappa} \quad (II.8) \\
&= n(z) \frac{(-1)^{\ell} \Gamma(-c)}{\Gamma(c+h+\ell)} \left[\frac{(p z_1)(p z_2)}{\frac{1}{2}p^2}\right]^{\ell} \left(\frac{1}{2}p^2\right)^c P_{\ell}^{(c-\ell, h-2)}(\omega),
\end{aligned}$$

where

$$\omega = \cos \theta = 1 - \frac{p^2(z_1, z_2)}{(p z_1)(p z_2)} \quad (II.9)$$

and we have used the following expansion formula for the Jacobi polynomial

$$(-1)^{\ell} P_{\ell}^{(c-\ell, h-2)}(\omega) = \frac{1}{\Gamma(-c)\Gamma(c+h-1)} \sum_{\kappa=0}^{\ell} \frac{\Gamma(\ell-\kappa-c)\Gamma(h+\kappa+c-1)}{\kappa!(\ell-\kappa)!} \left(\frac{1-\omega}{2}\right)^{\kappa}.$$

[For comparing different representations of $P_{\ell}^{(\alpha, \beta)}$ the identity

$$\sin \pi x \Gamma(x)\Gamma(1-x) = \pi$$

is useful. It implies, in particular, that

$$(-1)^{\ell} \frac{\Gamma(\ell-\kappa-c)}{\Gamma(-c)} = (-1)^{\kappa} \frac{\Gamma(c+\ell)}{\Gamma(c+\kappa-\ell+1)} \quad] .$$

Note that θ in (II.9) is the angle between the vector \underline{z}_1 and \underline{z}_2 in the rest frame of p (in Minkowski space).

For real c we can assume that the space G_{χ} (see Sec. I.B) consists of real-valued functions. Then, the bi-linear form (I.17), defined by $G_{\tilde{F}}$ is real and symmetric. Its p-space picture

$$(f, g)_2 = \int \bar{f}(p) \tilde{G}_2(p) \tilde{g}(p) (dp), \quad (dp) = \frac{d^{2h}p}{(2\pi)^{2h}} \quad (\text{II.10})$$

could be regarded as a hermitian form on the set of Fourier transforms \tilde{C}_2 . To be sure, the reality of $f(x)$ implies that $\bar{f}(p) = \tilde{f}(-p)$, and $\tilde{C}_2 (= \mathcal{F}C_2)$ has to be considered as a vector space over the reals.

B. Expansion in projection operators. Positivity and normalization

The representation \mathcal{K} belongs to the supplementary series of unitary representations of $O^+(2h+1, 1)$ iff the Hermitian form (II.10) is positive-definite and thus defines an invariant scalar product. The operators U (I.4-6) would be unitary in the (real) Hilbert space \mathcal{H}_2 obtained from C_2 by completion with respect to the scalar product (I.17) [or (II.10)].

For fixed p the kernel $\tilde{G}_2(p)$ is an operator in the finite dimensional space \mathcal{T}_{2h}^p of $SO(2h)$ -irreducible (symmetric, traceless) tensors of rank l . A straightforward way to investigate the restrictions on \mathcal{K} imposed by positivity is to expand \tilde{G} in projection operators $\Pi^{ls}(p)$ defined as follows. $\Pi^{ls}(p)$ project onto the subspaces $\mathcal{T}_{2h-1}^s(p)$ of \mathcal{T}_{2h}^l irreducible with respect to the stability subgroup $SO(2h-1)_p \subset SO(2h)$ of the vector p . Note that the dimensions of the space \mathcal{T}_{2h}^l and of its subspaces \mathcal{T}_{2h-1}^s ($s = 0, 1, \dots, l$) are given by

$$\dim \mathcal{T}_{2h}^l = \frac{(2h+l-3)!}{l!(2h-2)!} (2h+2l-2) = \sum_{s=0}^l \dim \mathcal{T}_{2h-1}^s,$$

$$\dim \mathcal{T}_{2h-1}^s = \tau_l \Pi^{ls} = \frac{(2h+s-4)!}{(2h-3)! s!} (2h+2s-3). \quad (\text{II.11})$$

In 4-dimensional space-time (i.e., for $2h=4$) the number S plays the role of spin. In terms of such an expansion positivity of \tilde{G} is expressed as positivity of the scalar coefficients to $\Pi^{\ell s}$.

Let

$$S_{\mu\nu}^{(a)} = -i \left(z_{a\mu} \frac{\partial}{\partial z_{a\nu}} - z_{a\nu} \frac{\partial}{\partial z_{a\mu}} \right), \quad a = 1, 2 \quad (\text{II.12})$$

be the (hermitian) generators of the index part of $2h$ -rotations.

Then the functions

$$\Pi^{\ell s}(p, z_1, z_2) = \frac{1}{\ell!} z_1^{\otimes \ell} \Pi^{\ell s}(p) z_2^{\otimes \ell} \quad (\text{II.13})$$

can be found up to a normalization factor as solutions of the equation

$$\left[\frac{1}{2} S_{\mu\nu}^{(a)} S_{\mu\nu}^{(a)} - S_{\sigma\mu}^{(a)} S_{\sigma\nu}^{(a)} \frac{p_\mu p_\nu}{p^2} \right] \Pi^{\ell s} = s(s+2h-3) \Pi^{\ell s} \quad (\text{II.14})$$

(valid for both $a=1$ and $a=2$ - Cf. [T3]). The result is

$$\Pi^{\ell s}(p, z_1, z_2) = A_{\ell s} (-1)^s \left[\frac{(p z_1)(p z_2)}{\frac{1}{2} p^2} \right]^\ell P_s^{(h-2, h-2)}(\omega) \quad (\text{II.15})$$

The normalization constant $A_{\ell s}$ is determined from the condition that $\Pi^{\ell s}$ are (orthogonal) projectors

$$\Pi^{\ell s}(p) \Pi^{\ell' s'}(p) = \delta_{s s'} \Pi^{\ell s}(p) \quad (\text{II.16})$$

In order to evaluate $A_{\ell s}$ we use the completeness relation

$$\sum_{s=0}^{\ell} \Pi^{\ell s}(p) = \mathbb{1} \quad (\text{II.17})$$

According to (II.15) its z - picture expression is

$$\sum_{s=0}^{\ell} (-1)^s A_{\ell s} \rho_s^{(h-2, h-2)}(\omega) = \frac{1}{\ell!} \left(\frac{1-\omega}{2} \right)^\ell, \quad (\text{II.18})$$

where ω is given by (II.9). We multiply both sides by $(1-\omega)^{h-2}$ and integrate over ω in the interval $[-1, 1]$, using the orthonormalization property

$$\int_{-1}^1 (1-\omega)^{h-2} \rho_s^{(h-2, h-2)}(\omega) \rho_{s'}^{(h-2, h-2)}(\omega) d\omega = \frac{\sqrt{s!} 2^{2h-s} [(h+s-2)!]^2}{s! (2h+2s-3) (2h+s-4)!} \quad (\text{II.19})$$

and the integral formula (see, e.g., [G3] Eq. 7.391.4)

$$\frac{1}{2^\ell \ell!} \int_{-1}^1 (1-\omega)^{\ell+h-2} (1+\omega)^{h-2} \rho_s^{(h-2, h-2)}(\omega) d\omega = \frac{(-1)^s 2^{2h-s} (\ell+h-2)! (h+s-2)!}{s! (\ell-s)! (2h+\ell+s-3)!} \quad (\text{II.20})$$

The result is

$$A_{\ell s} = \frac{(2h+2s-3)(h+\ell-2)! (2h+s-4)!}{(\ell-s)! (h+s-2)! (2h+\ell+s-3)!} \quad (\text{II.21})$$

In order to expand the right-hand side of (II.8) in the projection kernels (II.15) we use the formula

$$\rho_\ell^{(c-\ell, h-2)}(\omega) = \frac{(\ell+h-2)!}{\Gamma(c+h-1)\Gamma(c-h-\ell+2)} \sum_{s=0}^{\ell} \frac{(2s+2h-3)(s+2h-4)! \Gamma(c+h+s-1) \Gamma(c-h-s+2)}{(\ell-s)! (s+\ell+2h-3)! (h-2+s)!} \rho_s^{(h-2, h-2)}(\omega) \quad (\text{II.22})$$

Combining (II.15), (II.21) with (II.22) we find

$$\left[\frac{(p_1^2)(p_2^2)}{-\frac{1}{2}p^4} \right]^l \rho_{\ell}^{(c-l, h-2)}(\omega)$$

$$= \sum_{s=0}^{\ell} (-1)^{\ell-s} \frac{\Gamma(c+h+s-1)\Gamma(c-h-s+2)}{\Gamma(c+h-1)\Gamma(c-h-\ell+2)} \prod^{\ell s} (p; z_1, z_2). \quad (\text{II.23})$$

We shall fix the normalization constant $n(\lambda)$ in such a way that the coefficient to $\prod^{\ell 0}(p)$ in the spin-expansion of \tilde{G}_2 to be just $\left(\frac{p^2}{2}\right)^c$:

$$\tilde{G}_2(p) = \left[\prod^{\ell 0}(p) + \alpha_{\ell 1}(c) \prod^{\ell 1}(p) + \dots + \alpha_{\ell \ell}(c) \prod^{\ell \ell}(p) \right] \left(\frac{p^2}{2}\right)^c. \quad (\text{II.24})$$

This gives ^{x)}

$$n(\lambda) = \frac{(-1)^{\ell} \Gamma(c+h+\ell) \Gamma(c-h-\ell+2)}{\Gamma(c) \Gamma(c-h+2)} = \frac{\Gamma(c+h+2) \Gamma(h-c-1)}{\Gamma(c) \Gamma(\ell+h-c-1)}. \quad (\text{II.25})$$

We shall discuss the advantage (and peculiarities) of this choice in the next subsection. With such a normalization we obtain

$$\alpha_{\ell s}(c) \equiv \alpha_s(c) = (-1)^s \frac{\Gamma(c+h+s-1) \Gamma(c-h-s+2)}{\Gamma(c+h-1) \Gamma(c-h+2)} \quad (\text{II.26})$$

$$= \frac{\Gamma(c+h+s-1) \Gamma(h-1-c)}{\Gamma(h+s-1-c) \Gamma(h-1+c)} = \frac{(c+h-1) \dots (c+h+s-2)}{(h-c-1) \dots (h+s-c-2)}.$$

The scalar distribution $\left(\frac{p^2}{2}\right)^c$ is a positive measure on the space of fast decreasing functions of p for all $c > -h$. However, the scalar product $(f, g)_{\tilde{G}_2}$ [see (I.17)] can be

^{x)} We note that with this normalization the Plancherel measure (I.16) is given by

$$\rho_{\ell}(\sigma) = \frac{(\ell+h-1)!}{2(2\pi)^{h\ell}} n(\lambda) n(\tilde{\lambda}). \quad (\text{I.16}')$$

transformed to its p-space form (see (II.10) with $\lambda \rightarrow \tilde{\lambda}$) without recourse to analytic regularization only for $c < 0$ [since Eq. (I'.7) (with $d = h + c$) can be derived using ordinary convergent integrals only in that domain]. Combining this with (II.24) (II.26) we see that $G_2(x_1, x_2)$ is a positive distribution in $C_{\tilde{x}} \times C_{\tilde{x}}$ for

$$\begin{aligned} -h < c < 0 & \quad \text{if } \ell = 0 \\ -(h-1) < c < 0 & \quad \text{if } \ell \neq 1, h > 1 \end{aligned} \quad (\text{II.27})$$

We shall see in the next subsection, that the scalar product $(f, g)_{\tilde{x}}$ defined for $c > 0$ via analytic regularization (cf. [G1]) is positive also in the wider region (I.10b).

C. Equivalent representations and intertwining operators

Similarly to the special case of the Lorentz group obtained for $h=1$ (see [G2]), the representations

$$\chi = [\ell, c] \quad \text{and} \quad \tilde{\chi} = [\ell, -c] \quad (\text{II.28})$$

are equivalent. The intertwining operators for these representations are integral operators with kernels $G_2(x_1)$ and $G_{\tilde{2}}(x_1)$.

We have

$$U_2 G_2 = G_2 U_{\tilde{2}}, \quad G_{\tilde{2}} U_2 = U_{\tilde{2}} G_{\tilde{2}} \quad (\text{II.29})$$

$$G_2 G_{\tilde{2}} = \mathbb{1} \quad (\text{II.30})$$

or explicitly

$$\int G_2(x_1 - y; z_1, z_2) G_{\tilde{2}}(y - x_2, z_1, z_2) dy = \delta(x_1 - x_2) \frac{(z_1, z_2)^\ell}{\ell!} \quad (\text{II.31})$$

The last equation is obviously a consequence of (II.24) because of (II.16) and the property

$$\alpha_s(c)\alpha_s(-c) = 1, \quad (\text{II.32})$$

which follows from (II.26). (That is one reason for our choice of normalization.) We leave it to the reader to verify that if, for instance, $f_x(x) \in C_x$ then

$$\int G_x(x-y) f_x(y) dy \in C_x.$$

In the previous subsection we have established the unitarity of the representation $\tilde{\chi}$ for negative c , satisfying (I.10b). It follows from the equivalence of χ and $\tilde{\chi}$ that the representation χ is also unitary for such c . Hence, $\tilde{\chi}$ (or χ) is unitary for both positive and negative c in the domain (I.10b).

The coefficients $\alpha_s(c)$ (II.26) become zero or infinite for $s \geq 1$ and integer c such that $|c| \geq h-1$. We could have reversed the places of zeros and infinities by a different choice of normalization. With our choice G_x is well defined for all positive c and that is precisely what we need in the physical applications (cf. [M1] and [T2]).

The integer points with $|c| \geq h+l$ correspond to reducible, but non-decomposable representations of $O^+(2h+1,1)$. To see that, we consider first the case of a representation $\tilde{\chi}_{l,n} = [l, -c_n]$ with $c_n = h+l+rn$. In this case $C_{\tilde{\chi}_{l,n}}$ contains a finite dimensional invariant subspace: the space $E_{l,n}$ of all polynomials of degree $2(n+l) = 2(c_n-h)$ (or less). But $E_{l,n}$ does not have an invariant complement in $C_{\tilde{\chi}_{l,n}}$. The factor space $C_{\tilde{\chi}_{l,n}}/E_{l,n}$

is isomorphic to an infinite dimensional invariant subspace F_{2n} of C_{2n} that consists of all tensor functions $f(x) \in C_{2n}$ which satisfy the condition

$$\int f(x) P_k(x) dx = 0 \quad \text{for } k \leq 2(n+l), \quad (\text{II.33})$$

where $P_k(x)$ is an arbitrary polynomial of x of degree k .

According to (II.24) (II.25) the momentum space Green function $\tilde{G}_{2n}(p)$ is a homogeneous polynomial in p of degree $2(l+n+h)$. Therefore \tilde{G}_{2n} acts as a differential operator on C_{2n} which annihilates the finite dimensional invariant subspace E_{2n} . In this case the representations \mathcal{N}_{2n} and $\tilde{\mathcal{N}}_{2n}$ are not equivalent. The map $G_{2n}: C_{2n} \rightarrow C_{2n}$ only establishes equivalence between the irreducible representations realized in C_{2n}/E_{2n} and $F_{2n} \subset C_{2n}$.

D. Wightman positivity

Functions with the properties of G_c (for real c) arise not only in studying invariant bi-linear forms, but also in considering analytic continuation of Wightman functions

$$W(x_1, z_1; z_2, z_2) = \langle O(x_1, z_1) O(x_2, z_2) \rangle, \quad (\text{II.34})$$

(or \tilde{v} -functions) to Euclidean points (for which $x_0 = i x_{2h}$) in a conformal invariant quantum field theory (in the sense of [M4] [T1]). Here $O(x, z)$ is a (local) tensor field

$$O(x, z) = \frac{1}{\sqrt{l!}} O_{\mu_1 \dots \mu_l}(x) z^{\mu_1} \dots z^{\mu_l} \equiv O(x) \frac{z^{\otimes l}}{\sqrt{l!}}. \quad (\text{II.35})$$

Wightman positivity for the two-point function implies that

$$\tilde{W}(p; z, \bar{z}) \geq 0 \quad (\text{II.36})$$

in Minkowski space.

The Fourier transform of the Wightman function can be obtained from $\tilde{G}_Z(p)$ (II.8) by the following procedure. First of all, using (II.9) and (II.23) we find the following expression for the M-space τ -function

$$\tau_Z(p) = \frac{n_W(z) \Gamma(-c)}{\Gamma(c+h+l)} \left(\frac{1}{2} p^2 - i0\right)^{c-l} \sum_{s=0}^l \frac{\Gamma(c+h+s-1) \Gamma(c-h-s+2)}{\Gamma(c+h-1) \Gamma(c-h+l+2)} (-1)^{l-s} \left(\frac{p}{2}\right)^l \Gamma^{ls}(p) \quad (\text{II.37})$$

($p^2 = p^+ - p_0^2$) (We are writing $n_W(z)$ instead of $n(z)$, since we have to use a different normalization in the new interpretation of the 2-point function.) Then the p-space Wightman function is given by

$$\begin{aligned} \tilde{W}(p) &= -i \theta(p_0) [\tau_Z(p) - \bar{\tau}_Z(p)] \\ &= - \frac{2 \sin \pi(c-l) n_W(z) \Gamma(-c)}{\Gamma(c+h+l) \Gamma(c+h-1) \Gamma(c-h+l+2)} \theta(p_0) \left(\frac{1}{2} p^2\right)_+^{c-l} \sum_{s=0}^l \Gamma(c+h+s-1) \Gamma(c-h-s+2) (-1)^{l-s} \left(\frac{p}{2}\right)^l \Gamma^{ls}(p), \end{aligned} \quad (\text{II.38})$$

where $t_+^\lambda \equiv \theta(t) t^\lambda$ (of. [61]); in deriving the last equality we have used the identity

$$(Q+i0)^\lambda - (Q-i0)^\lambda = 2i \sin \pi \lambda (-Q)_+^\lambda. \quad (\text{II.39})$$

In this case we shall use the normalization

$$\begin{aligned} n_W(z) &\equiv 2^c (c+h+l-1) \frac{\Gamma(c+h-1) \Gamma(c-h+l+2)}{\Gamma(c-h-l+2)} \\ &= 2^c \frac{\Gamma(c+h-1) \Gamma(-c) \Gamma(c-h+2) \Gamma(c-h+l+2)}{\Gamma(c+h+l-1) \Gamma^2(c-h-l+2)} n_2(z) (-1)^l. \end{aligned} \quad (\text{II.40})$$

which gives

$$\tilde{w}(p) = \frac{2\pi\theta(p_0)}{\Gamma(c+1)} (-p^2)_+^{c-\ell} \sum_{s=0}^{\ell} \frac{\Gamma(c+h+s-1)\Gamma(c-h-s+2)}{\Gamma(c+h+\ell-1)\Gamma(c-h-\ell+2)} \dots (-1)^s (p^2)^{\ell} |p^2|^{-\ell} \quad (\text{II.41})$$

In order to establish when the right-hand side of (II.41) is positive, we notice that the operator

$$(-1)^s (p^2)^{\ell} |p^2|^{-\ell}$$

is positive, since, according to (II.19) (II.21)

$$(p^2)^{\ell} (-1)^s |p^2|^{-\ell} (p, z, \bar{z}) = A_{\ell s} [2|p z|^2]^{\ell} |p^2|^{-\ell} \omega^{(h-\ell, h-\ell)}(\omega) \neq 0 \quad (\text{II.42})$$

for $\omega = 1 - \frac{p^2 z \bar{z}}{|p z|^2} \neq 1$. The last inequality ($\omega \neq 1$) is fulfilled because, for $z^2 = 0$ we have $z \bar{z} = \underline{z} \bar{z} - \underline{z} \bar{z} \neq 0$. Therefore, $\tilde{w}(p)$ is positive for

$$c \geq -1 \text{ if } \ell = 0; \quad c \geq h + \ell - 2 \text{ for } \ell = 1, 2, \dots$$

This result was obtained by different methods also in [R2] and [F2].

Our choice of normalization (II.40) ensures the following additional properties of $\tilde{w}(p)$:

(1) For $\ell = 0$, $c = -1$ (II.41) goes into the conventional expression for the two-point function of a free zero-mass field

(ii) For canonical dimensions

$$c = h + \ell - 2 \quad (\ell > 0) \quad (\text{II.44})$$

we recover the two-point functions of conserved (tensor) currents (while the expression (II.26) for $\mathcal{L}_{\ell}(c)$ is going to infinity for such a c).

III. Direct product expansions and Clebsh-Gordan kernels

1. General form of the expansion. Normalization conditions

We consider now the direct product space $C_{\lambda_0} \otimes C_{\lambda_0}$ ($\lambda_0 = [c, c_a]$) of infinitely smooth functions $f(x_1, x_2)$ satisfying the asymptotic conditions

$$\lim_{x_i \rightarrow \infty} (x_i^c)^{h+\nu_i} f(x_1, x_2) = f_a(x_c) \in C_{\lambda_0} \quad (\text{III.1})$$

where (a, b) stands for $(1, 2)$ or $(2, 1)$. For c_a in range (I.10b) we can expand $f(x_1, x_2)$ in irreducible (unitary) representations of $O^+(2h+1, 1)$ as follows

$$f(x_1, x_2) = \int d\lambda \int dx V(x, c_1, x_2, c_2, x, \lambda) F_\lambda(x) + D.T. \quad (\text{III.2})$$

where D.T. indicates (possible) discrete terms and the summation and integration is spread over the principal series of unitary representations (see (I.15) (I.16)).

The conformal "Fourier transform" $F_\lambda(x)$ satisfies the symmetry condition

$$F_\lambda(x) = \int G_\lambda(x-y) F_\lambda(y) dy. \quad (\text{III.3})$$

Conformal invariance implies that the Clebsh-Gordan kernel is given by

$$V(x, c_1, x_2, c_2; x_3, \lambda_0, z) = \frac{N_\lambda(c_1, c_2, \delta) (1z)^c}{(2\pi)^h \left(\frac{1}{2}x_{12}\right)^{h+c_1-\delta+\frac{1}{2}} \left(\frac{1}{2}x_{13}\right)^{\delta+c_2-\frac{1}{2}} \left(\frac{1}{2}x_{23}\right)^{\delta+c_2-\frac{1}{2}} \sqrt{|\theta|}} \quad (\text{III.4})$$

(cf. [T1] Sec. IV.2C) . Here we have used the definition (I.20) and the following notation:

$$c_{\pm} = \frac{1}{2}(c_1 \pm c_2) , \quad \lambda_{\pm} = [l, 2\delta - h] \quad (\text{III.5})$$

$$\lambda_{\mu} = 2 \frac{(x_{13})_{\mu}}{x_{13}^2} - 2 \frac{(x_{23})_{\mu}}{x_{23}^2} . \quad (\text{III.6})$$

The normalization constant N_{ℓ} will be fixed by the following conditions:

$$\int dx_1' V(x_1' c_1, x_2 c_2; x_3 \lambda) G_{-c_1}(x_1' - x_1) = V(x_1 - c_1, x_2 c_2; x_3 \lambda); \quad (\text{III.7a})$$

$$\int dx_2' V(x_1 c_1, x_2' c_2; x_3 \lambda) G_{-c_2}(x_2' - x_2) = V(x_1 c_1, x_2 - c_2; x_3 \lambda); \quad (\text{III.7b})$$

(G_c is a shorthand for $G_{[0,c]}$);

$$\int dx_3' V(x_1 c_1, x_2 c_2; x_3' \lambda) G_{\tilde{\lambda}}(x_3' - x_3) = V(x_1 c_1, x_2 c_2; x_3 \tilde{\lambda}); \quad (\text{III.8})$$

$$\begin{aligned} & \frac{1}{2} \int dx_1 \int dx_2 V(x_1 - c_1, x_2 - c_2; x_3 \lambda) \otimes V(x_1 c_1, x_2 c_2; x_3 \tilde{\lambda}') \\ & = \frac{1}{2} [\mathbb{1} \delta(\lambda, \lambda') + G_{\lambda} \delta(\lambda, \tilde{\lambda}')] , \end{aligned} \quad (\text{III.9})$$

where

$$\delta(\lambda, \lambda') = \frac{\delta_{\text{ee}'}}{\rho_{\ell}(\sigma)} \delta(\sigma - \sigma') \varepsilon_{\tilde{\pi}} \quad (\text{III.10})$$

($f_2(\sigma)$ is the Plancherel measure (I.16)) and the unit operator is defined in the (x, z) -picture as follows

$$\frac{1}{z'} z^{\alpha'} \int z^{\alpha'} = \delta(x_j - x_j') \frac{(z z')^{\alpha'}}{z'} . \quad (\text{III.11})$$

Eq. (III.9) along with the symmetry property (III.3) implies that the expansion (III.2) can be inverted and the conformal Fourier transform of $f(x_1, x_2)$ is given by

$$F_z(x) = \int dx_1 \int dx_2 V(x_1 - c_1, x_2 - c_2; x z) f(x_1, x_2) . \quad (\text{III.12})$$

We shall see in what follows that conditions (III.7) and (III.9) are sufficient to determine the normalization constant N'_2 . Eq. (III.8) then can be derived as a consequence.

B. Amputation of scalar lines

We start with the exploitation of the symmetry property (III.7).

The calculation is based on the integral formula

$$\begin{aligned} [(x_1, \alpha_1^{\sigma_1}, x_2, \alpha_2^{\sigma_2}, x_3, \alpha_3^{\sigma_3})] &= \frac{1}{(2\pi)^h} \int \frac{\Gamma(\delta_1)}{[\frac{1}{z}(x_1 - z)]^{\delta_1}} \frac{\Gamma(\delta_2)}{[\frac{1}{z}(x_2 - z)]^{\delta_2}} \frac{\Gamma(\delta_3)}{[\frac{1}{z}(x_3 - z)]^{\delta_3}} dz \\ &= \int_{\alpha_1}^{\infty} d\alpha_1 \int_0^{\infty} d\alpha_2 \int_0^{\infty} d\alpha_3 \frac{\alpha_1^{\delta_1} \alpha_2^{\delta_2} \alpha_3^{\delta_3}}{(k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3)^h} \exp \left\{ - \frac{\alpha_1 \alpha_2 z_1^2 + \alpha_1 \alpha_3 z_3^2 + \alpha_2 \alpha_3 z_2^2}{2(k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3)} \right\} \\ &\quad (k_i \geq 0, \sum k_i > 0) \end{aligned}$$

Changing the order of summation and using the sum rule

$$\sum_{k=j}^{\ell} (-1)^k \binom{\ell}{k} \binom{k}{j} \frac{\Gamma(\alpha+k)}{\Gamma(\beta+k)} = (-1)^j \binom{\ell}{j} \frac{\Gamma(\beta-\alpha+\ell-j)\Gamma(\alpha+j)}{\Gamma(\beta+\ell)\Gamma(\beta-\alpha)} \quad (\text{III.15})$$

(see Appendix A) for $\alpha = c_2 - j$, $\beta = \delta - c_2 - \frac{\ell}{2}$
we obtain

$$\begin{aligned} & \int dx_2 V(x_1, c_1, x_2, c_2; x_3, z, z) G_{-c_2}(x_2' - x_2) \\ &= \frac{N_2(c_1, c_2, \delta) \Gamma(\delta - c_2 + \frac{\ell}{2}) \Gamma(h - \delta + c_2 + \frac{\ell}{2})}{\sqrt{\ell!} (2\pi)^h \Gamma(h + c_2 - \delta + \frac{\ell}{2}) \Gamma(\delta - c_2 + \frac{\ell}{2})} \frac{(\lambda z)^\ell}{(\frac{1}{2} x_{12})^{h - \delta + c_2 + \frac{\ell}{2}} (\frac{1}{2} x_{13})^{\delta - c_2 - \frac{\ell}{2}} (\frac{1}{2} x_{23})^{\delta - c_2 - \frac{\ell}{2}}} \\ &= \frac{N_2(c_1, c_2, \delta) \Gamma(-c_2 + \delta + \frac{\ell}{2}) \Gamma(h - \delta + c_2 + \frac{\ell}{2})}{N_2(c_1, c_2, \delta) \Gamma(h + c_2 - \delta + \frac{\ell}{2}) \Gamma(\delta - c_2 + \frac{\ell}{2})} V(x_1, c_1, x_2, c_2; x_3, z, z). \end{aligned} \quad (\text{III.16})$$

Applying the obvious symmetry property

$$V(x_1, c_1, x_2, c_2, x_3, z, z) = (-1)^\ell \frac{N_2(c_1, c_2, \delta)}{N_2(c_2, -c_1, \delta)} V(x_2, c_2, x_1, c_1; x_3, z, z) \quad (\text{III.17})$$

we can derive from (III.16) another relation of that type, involving integration over the first argument of V (say x_1').

Combining these two equations and comparing with (III.7) we obtain

$$\frac{N_2(c_2, c_1, \delta)}{N_2(-c_2, -c_1, \delta)} = \frac{\Gamma(h + c_2 - \delta + \frac{\ell}{2}) \Gamma(\delta + c_2 + \frac{\ell}{2})}{\Gamma(h - c_2 - \delta + \frac{\ell}{2}) \Gamma(\delta - c_2 + \frac{\ell}{2})} \quad (\text{III.18})$$

C. The Plancherel formula

In order to give a precise meaning of the singular equation (III.9) we start with the following regularization of the left-hand side:

$$I_{\varepsilon}(x_3, \ell \delta z, x_3' \ell' \delta' z') = \frac{\sqrt{\ell'} \sqrt{\ell'}}{N_{\ell}(-c_+, -c_-, \delta') N_{\ell'}(c_+, c_-, h-\delta')} \quad (III.19)$$

$$\times \int dx_1 \int dx_2 V(x_1 - i, x_2 - i; x_3, \ell z) V(x_1, c_1 - 2\varepsilon, x_2, c_2; x_3', \ell' z')$$

and shall go to the limit $\varepsilon \rightarrow r0$ only after smearing with an analytic test function of δ' .

Setting

$$x_i - x_3' = x_{i,3'} \quad i = 1, 2, 3 \quad (III.20)$$

and performing the integration in (III.19) over x_2 we obtain:

$$I_{\varepsilon} = \frac{1}{(2\pi)^h} \sum_{k=0}^{\ell} \sum_{k'=0}^{\ell'} (-1)^{k+k'} \binom{\ell}{k} \binom{\ell'}{k'} \frac{\Gamma(\delta-\delta'-\frac{\ell+\ell'}{2}+c) \Gamma(h-\delta-c-\frac{\ell}{2}) \Gamma(\delta'+c-\frac{\ell'}{2}-\varepsilon)}{\Gamma(h-\delta+\delta'+\frac{\ell+\ell'}{2}-\varepsilon) \Gamma(\delta+c-\frac{\ell}{2}+k) \Gamma(h-\delta'-c_+-k'-\frac{\ell'}{2}+c)}$$

$$\int dx_1 \frac{(z x_{1,3})^{\ell-k} (z' x_{1,3}')^{\ell'-k'}}{(\frac{1}{2} x_{1,3})^{\delta-c-\frac{\ell}{2}-k} (\frac{1}{2} x_{1,3}')^{h-\delta'+c_+-\frac{\ell'}{2}-k'-\varepsilon}} (z \theta_3)^k (z' \theta_3')^{k'} \left[\left(\frac{z}{x_{3,3}} \right)^{\delta-\delta'-\frac{\ell+\ell'}{2}+\varepsilon} \left(\frac{z'}{x_{3,3}'} \right)^{h-\delta-c-\frac{\ell}{2}} \left(\frac{z}{x_{1,3}} \right)^{\delta'+c_+-\frac{\ell'}{2}-\varepsilon} \right]$$

Here we have used again (III.14) and (III.13). Further, we apply the binomial formula:

$$\frac{(z x_{1,3})^{\ell-k} (z' x_{1,3}')^{\ell'-k'} (z \theta_3)^k (z' \theta_3')^{k'}}{(\frac{1}{2} x_{1,3})^{\delta-c-\frac{\ell}{2}-k} (\frac{1}{2} x_{1,3}')^{h-\delta'+c_+-\frac{\ell'}{2}-k'-\varepsilon}} \left[\left(\frac{z}{x_{3,3}} \right)^{\delta-\delta'-\frac{\ell+\ell'}{2}+\varepsilon} \left(\frac{z'}{x_{3,3}'} \right)^{h-\delta-c-\frac{\ell}{2}} \left(\frac{z}{x_{1,3}} \right)^{\delta'+c_+-\frac{\ell'}{2}-\varepsilon} \right] =$$

$$= \sum_{j=0}^k \sum_{j'=0}^{k'} \binom{k}{j} \binom{k'}{j'} \frac{\Gamma(h-\delta-c_+ + \frac{\ell}{2} + k-j) \Gamma(\delta'+c_- + \frac{\ell'}{2} + k-j-\varepsilon)}{\Gamma(h-\delta-c_+ + \frac{\ell}{2}) \Gamma(\delta'+c_- + \frac{\ell'}{2} - \varepsilon)}$$

$$\times \frac{(z_1 x_{1j})^{\ell j} (z_1' x_{1j}')^{\ell' j'}}{(z_1 x_{1j})^{\delta \ell + \frac{\ell+\ell'}{2} j - \varepsilon} (z_1' x_{1j}')^{2h-\delta-\delta'+\frac{\ell+\ell'}{2} j' - \varepsilon}} (z_2)^\delta (z_2')^{\delta'} \left(\frac{z}{x_{33}}\right)^{\delta-\delta'+\frac{\ell+\ell'}{2} + \varepsilon}$$

then change the order of summation in k and j , and in k' and j' , and use twice the sum rule (III.15) with $\alpha = \delta' + c_- + \frac{\ell'}{2} - j' - \varepsilon$
 $\beta = \delta + c_+ - \frac{\ell}{2}$ (and $\alpha' = h - \delta' - c_- + \frac{\ell}{2} - j'$, $\beta' = h - \delta' - c_- - \frac{\ell'}{2} + \varepsilon$)

The result is

$$\Gamma_\varepsilon(x, \ell \sqrt{x}, x_3, \ell' \sqrt{x_3}) = (2\pi)^h \sum_{j=0}^{\ell} \sum_{j'=0}^{\ell'} (-1)^{j+j'} \binom{\ell}{j} \binom{\ell'}{j'} \frac{\Gamma(\delta-\delta'+\frac{\ell-\ell'+\varepsilon}{2}) \Gamma(\delta-\delta'+\frac{\ell-\ell'+\varepsilon}{2} + \varepsilon)}{\Gamma(\delta-\delta'+\frac{\ell-\ell'+\varepsilon}{2} + j + \varepsilon) \Gamma(\delta-\delta'+\frac{\ell-\ell'+\varepsilon}{2} + j' + \varepsilon)}$$

$$\times \frac{\Gamma(\delta-\delta'+\frac{\ell+\ell'+\varepsilon}{2}) \Gamma(\delta'+\frac{\ell'}{2}+c_--\varepsilon) \Gamma(h-\delta+\frac{\ell}{2}-c_+)}{\Gamma(h-\delta+\delta'+\frac{\ell+\ell'+\varepsilon}{2}) \Gamma(\delta'+\frac{\ell'}{2}+c_+)} \left[(z_2)^\delta (z_2')^{\delta'} \left(\frac{z}{x_{33}}\right)^{\delta-\delta'+\frac{\ell+\ell'}{2} + \varepsilon} \right]$$

$$\times \frac{\Gamma(\delta+\delta'+\frac{\ell'-\ell}{2}-h+\varepsilon) \Gamma(h-\delta-\delta'+\frac{\ell'-\ell}{2}+\varepsilon)}{\Gamma(\delta+\delta'+\frac{\ell'+\ell}{2}-j-\varepsilon) \Gamma(2h-\delta-\delta'+\frac{\ell+\ell'}{2}-j'-\varepsilon)} (z_2)^\delta (z_2')^{\delta'} \delta_\varepsilon(x_{33}),$$

(III.21)

where

$$\delta_\varepsilon(x_{33}) = \frac{1}{(2\pi)^{2h}} \frac{\Gamma(\delta+\delta'+\frac{\ell'-\ell}{2}-\varepsilon) \Gamma(2h-\delta-\delta'+\frac{\ell'-\ell}{2}-\varepsilon)}{\Gamma(h-\delta-\delta'+\frac{\ell'-\ell}{2}+\varepsilon) \Gamma(\delta+\delta'+\frac{\ell'-\ell}{2}-h+\varepsilon)} \int \frac{dx_3}{\left(\frac{x_3}{z}\right)^{\delta+\delta'+\frac{\ell'-\ell}{2}-\varepsilon} \left(\frac{x_3}{z}\right)^{2h-\delta-\delta'+\frac{\ell'-\ell}{2}-\varepsilon}}$$

$$= \int \frac{dp}{(2\pi)^{2h}} \left(\frac{z}{p}\right)^{2\varepsilon} e^{ipx_{33}} = \frac{\Gamma(h-2\varepsilon)}{(2\pi)^h \Gamma(2\varepsilon)} \left(\frac{z_{33}}{z}\right)^{h-2\varepsilon} \left(\frac{\rightarrow}{\varepsilon \rightarrow 0} \delta(x)\right). \quad \text{(III.22)}$$

Because of the distribution character of the limit $\varepsilon \rightarrow 0$ [as is already suggested by Eq. (III.9)] we shall first smear the right-hand side of (III.21) by a suitable test function of the representation label

$$c' = 2\sigma' - h. \quad (\text{III.23})$$

Let

$$2\sigma' = h + c, \quad c = i\sigma \quad (\sigma - \text{real}), \quad (\text{III.24})$$

and let $f(c')$ be an analytic function in some finite strip

$$0 \leq \text{Re } c' < a, \quad (\text{III.25})$$

fast decreasing at infinity inside the strip. We shall evaluate the integral

$$I_\varepsilon^f(x_{33}, \sigma, l) = \int_{-\infty}^{+\infty} \frac{dc'}{4\pi i} \beta_0(-i\sigma) I_\varepsilon(x_3, l, \frac{h+c'}{2}, x_3, l, \frac{h+c'}{2}) f(c') \quad (\text{III.26})$$

in the limit $\varepsilon \rightarrow 0$ by closing the contour of integration in the strip (III.25). (In order to simplify the calculation we have set $l' = l$.) The result is

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon^f = -\frac{1}{2} \beta_0(\sigma) \lim_{\varepsilon \rightarrow 0} \left[f(i\sigma) \text{Res}_{c'=i\sigma} I_\varepsilon + f(-i\sigma) \text{Res}_{c'=-i\sigma} I_\varepsilon \right]. \quad (\text{III.27})$$

For the first residue we obtain from (III.21) (for $l' = l$)

$$\begin{aligned} -\lim_{\varepsilon \rightarrow 0} \text{Res}_{c'=i\sigma} I_\varepsilon &= 2(2\pi)^h \sum_{j=0}^l (-1)^{l-j} \frac{\rho^j c' \Gamma(i\sigma) \Gamma(-i\sigma)}{j! j! (h+l-1)! \Gamma(h+l-j+i\sigma) \Gamma(h+l-j-i\sigma)} \left[(z_0)_j^2 (z'_0)_j \left(\frac{2x_3}{z} \right)^l \right] \delta^l(x_{33}) \\ &= 2(2\pi)^h \frac{\rho^l}{(h+l-1)!} \left| \sum_{j=0}^l \binom{l}{j} (-1)^{l-j} \frac{(l-j)! \Gamma(i\sigma)}{\Gamma(h+l-j+i\sigma)} \right|^2 \delta^l(x_{33}) (z')^l. \end{aligned} \quad (\text{III.28})$$

In writing down the first equality (III.28) we used that for any test function $u(x_j)$:

$$\begin{aligned} & \int [(\mathbb{Z}'\partial_j)^{\ell-j} (\mathbb{Z}'\partial_j)^{\ell-j} \delta(x_{3j})] u(x_j) [(\mathbb{Z}'\partial_j)^j (\mathbb{Z}'\partial_j)^j] \left(\frac{x_{3j}^2}{2}\right)^\ell dx_j \\ &= (-1)^{\ell-j} \int \delta(x_{3j}) (\mathbb{Z}'\partial_j)^{\ell-j} (\mathbb{Z}'\partial_j)^{\ell-j} \{ u(x_j) (\mathbb{Z}'\partial_j)^j (\mathbb{Z}'\partial_j)^j \} \left(\frac{x_{3j}^2}{2}\right)^\ell dx_j \\ &= (-1)^{j-j} \int \delta(x_{3j}) u(x_j) (\mathbb{Z}'\partial_j)^\ell (\mathbb{Z}'\partial_j)^\ell \left(\frac{x_{3j}^2}{2}\right)^\ell dx_j \end{aligned}$$

(since the terms containing derivatives of $u(x_j)$ vanish); we also applied the identity

$$\frac{\Gamma(\ell-\ell)\Gamma(\ell)}{\Gamma(\ell+j-\ell)\Gamma(\ell+j-\ell)} = (-1)^{j+j} \frac{\Gamma(1+\ell-j-\ell)\Gamma(1+\ell-j-\ell)}{\Gamma(1+\ell-\ell)\Gamma(1-\ell)}$$

Next we take into account that

$$\sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \frac{(\ell-j)!}{\Gamma(h+\ell-j+i\sigma)} = \frac{1}{(i\sigma+h+\ell-1)\Gamma(h+i\sigma-1)} \quad (\text{III.29})$$

(see Appendix A). Inserting it in (III.28) and (III.27) and recalling the notation (III.20) and the expression for the Plancherel measure (I.16) we obtain:

$$-p_{\ell}(\sigma) R_{2S} I_{\ell=0} = \delta(x_{3j}) (\mathbb{Z}'\partial_j)^\ell. \quad (\text{III.30})$$

The orthogonality relation for $l \neq l'$ implied by (III.9), (III.10) can also be verified by a slightly more complicated calculation along the same lines.

Now we proceed to the evaluation of the second residue term in (III.27). Setting $z' = z$ ($l' = l$) for δ' given by (III.24) and using (II.4) (II.25) we find

$$\begin{aligned}
 & -\lim_{\epsilon \rightarrow 0} \text{Res}_{z' = -\delta + 2\delta} I_{\epsilon}(x_3, l, \delta, z, x_3, l', \frac{h+c}{2}, z) \\
 &= 2^{1/2} (h-1)! \sum_{j, j'=0}^{\ell} (-1)^{j+j'} \binom{\ell}{j} \binom{\ell}{j'} \frac{[\Gamma(2\delta-h)]^2 \Gamma(2\delta-h-l) \Gamma(h-\delta+c+\frac{\ell}{2}) \Gamma(h-\delta-c+\frac{\ell}{2})}{\Gamma(2h-2\delta+l) \Gamma(\delta+c+\frac{\ell}{2}) \Gamma(\delta-c+\frac{\ell}{2}) \Gamma(2\delta-h-l+j) \Gamma(2\delta-h-l+j')} \\
 & \times \frac{1}{\Gamma(h+l-j) \Gamma(h-l-j')} \left[(\partial_3 z)^{\ell-j} (z \partial_3)^{\ell-j} \left(\frac{\partial}{\partial z} \right)^h \right] (2z_3)^j (z \partial_3)^{j'} \left(\frac{\partial}{\partial z} \right)^{2\delta-h-l} \\
 &= (2\pi)^h \sum_{j, j'=0}^{\ell} (-1)^{j+j'} \binom{\ell}{j} \binom{\ell}{j'} \frac{[\Gamma(2\delta-h)]^2 \Gamma(2\delta-h-l+j+j') \Gamma(2\ell+h-j-j')}{\Gamma(2\delta-l-h+j) \Gamma(2\delta-l-h+j') (l-j+h-1)! (l-j'+h-1)!} \\
 & \times \frac{\Gamma(h-\delta-c+\frac{\ell}{2}) \Gamma(h-\delta+c+\frac{\ell}{2}) \Gamma(h-2\delta) \Gamma(2h-2\delta+l-1)}{\Gamma(\delta-c+\frac{\ell}{2}) \Gamma(\delta+c+\frac{\ell}{2}) \Gamma(2h-2\delta+l) \Gamma(2\delta+l) \Gamma(2h-2\delta-1)} \\
 & \times G_{\mathcal{N}}(x_{33}; z, z).
 \end{aligned}
 \tag{III.31}$$

On the other hand, as is shown in Appendix B,

$$\begin{aligned}
 & \sum_{j, j'=0}^{\ell} (-1)^{j+j'} \binom{\ell}{j} \binom{\ell}{j'} \frac{\Gamma(i\sigma-l+j+j') (2\ell-j-j'+h-1)!}{\Gamma(i\sigma+j-l) \Gamma(i\sigma+j'-l) (l-j+h-1)! (l-j'+h-1)!} \\
 &= \frac{\ell! \Gamma(i\sigma+h+l-1)}{(h+l-1)! \Gamma(i\sigma) \Gamma(i\sigma+h-1)}.
 \end{aligned}
 \tag{III.32}$$

Inserting (III.32) in (III.31) we obtain

$$\begin{aligned}
 & - \gamma_2(\sigma) \lim_{\epsilon \rightarrow 0} \operatorname{Res}_{\nu = -i\sigma + \epsilon} I_\epsilon(x_3, \ell \frac{hrc}{2} z, x_3', \ell \frac{hrc'}{2} z) \\
 & = 2! \frac{\Gamma(h-c_+ - \delta + \frac{\ell}{2}) \Gamma(hrc_+ - \delta + \frac{\ell}{2})}{\Gamma(\delta - c_+ + \frac{\ell}{2}) \Gamma(\delta + c_+ + \frac{\ell}{2})} G_2(x_{33'}, z, z)
 \end{aligned} \tag{III.33}$$

Comparing (III.27) (III.30) (III.33) with the normalization condition (III.9-11) we find that for imaginary $2\delta = h$

$$N_\ell(-c_+, -c_-, \delta) N_\ell(c_+, c_-, h-\delta) = 1, \tag{III.34}$$

$$N_\ell(-c_+, -c_-, \delta) N_\ell(c_+, c_-, \delta) = \frac{\Gamma(\delta - c_+ + \frac{\ell}{2}) \Gamma(\delta + c_+ + \frac{\ell}{2})}{\Gamma(h - \delta - c_+ + \frac{\ell}{2}) \Gamma(h - \delta + c_+ + \frac{\ell}{2})} \tag{III.35}$$

Multiplying Eqs. (III.18) with (III.35) (side by side) we obtain

$$N_\ell(c_+, c_-, \delta) = \left\{ \frac{\Gamma(h+c_+ - \delta + \frac{\ell}{2}) \Gamma(c_+ + \delta + \frac{\ell}{2}) \Gamma(\delta - c_+ + \frac{\ell}{2}) \Gamma(\delta + c_+ + \frac{\ell}{2})}{\Gamma(h-c_+ - \delta + \frac{\ell}{2}) \Gamma(-c_+ + \delta + \frac{\ell}{2}) \Gamma(h - \delta - c_+ + \frac{\ell}{2}) \Gamma(h - \delta + c_+ + \frac{\ell}{2})} \right\}^{1/2}. \tag{III.36}$$

The sign of the square root can be fixed, requiring that for real dimensions and $-1 \leq c_+ \leq 0$, $|c_-| < \delta < hc_+$ the factors N_ℓ be positive. Notice that Eq. (III.34) (which was not used in the derivation of (III.36)) is satisfied automatically by this expression.

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APPENDIX A

A summation formula involving ratios of Γ -functions

Eqs. (III.15) and (III.29) used in Sec. III can both be derived from the following known formula for the value of the hypergeometric function $F = {}_2F_1$, at the point $x=1$:

$$F(a, b; c; 1) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m) \Gamma(b+m) \Gamma(c)}{\Gamma(a) \Gamma(b) \Gamma(c+m) m!} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad (\text{A.1})$$

(see e.g. [8] Eq. 9.122.1).

In order to reduce Eq. (III.15) to the form (A.1) we set $k-j = m$, $l-j = n$ and continue to non-integer n , writing (III.22) in the form

$$\sum_{m=0}^{\infty} \frac{\Gamma(m-n) \Gamma(d+j+m)}{\Gamma(-n) m! \Gamma(\beta+j+m)} = \frac{\Gamma(\beta-d+n) \Gamma(d+j)}{\Gamma(\beta+j+n) \Gamma(\beta-d)} \quad (\text{A.2})$$

Here, we have used the identity

$$(-1)^m \frac{\Gamma(n+1)}{\Gamma(n-m+1)} = \frac{\Gamma(m-n)}{\Gamma(-n)} \quad (\text{A.3})$$

Eq. (III.29) is established in a similar way.

There exists also a direct elementary proof of Eqs. (III.15) and (III.29) which exploits their similarity to the Newton binomial formula.

In the above notation, Eq. (III.15) assumes the form

$$f_n(a, b) = \sum_{m=0}^n (-1)^m \binom{n}{m} (a)_m (b+m)_{n-m} = (b-a)_n \quad (\text{A.4})$$

where $a = d+j$, $b = \beta+j$ ($n = l-j$) and

$$x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)\dots(x+k-1) \quad (\text{A.5})$$

is the finite-difference counter part of the power x^k .

In order to prove (A.4), we evaluate the finite difference

$f_n(a, b) - f_n(a, b-1)$ using

$$(x)_k - (x-1)_k = [x+k-1-(x-1)](x)_{k-1} = k(x)_{k-1}. \quad (\text{A.6})$$

Thus, we find the recurrence relation

$$f_n(a, b) - f_n(a, b-1) = n f_{n-1}(a, b) \quad (\text{A.7})$$

with the initial condition

$$f_1(a, b) = b - a. \quad (\text{A.8})$$

In order to fix $f_n(a, b)$ uniquely we have to evaluate it for a particular value of b . For $b = a$ we have

$$f_n(a, a) = (a)_n \sum_{m=0}^n (-1)^m \binom{n}{m} = (a)_n (1-1)^n = 0. \quad (\text{A.9})$$

It is easily seen that the only polynomial solution of (A.7-9) is given by the right-hand side of (A.4).

Eq. (III.29) can also be reduced to the form (A.4). Indeed multiplying both sides by $\Gamma(h+i\sigma+l)$ and substituting the summation index $j \rightarrow k = l-j$ we obtain

$$\sum_{k=0}^l (-1)^k \binom{l}{k} (1)_k (h+i\sigma+k)_{l-k} = (h+i\sigma-1)_l \quad (\text{A.10})$$

which is a special case of (A.4) (with $a=1$, $b=h+i\sigma$).

Appendix B. Evaluation of a double sum involving Γ - functions

We wish to evaluate the expression

$$S \equiv \sum_{j,j'=0}^{\ell} (-1)^{j+j'} \binom{\ell}{j} \binom{\ell}{j'} \frac{\Gamma(i\sigma - \ell + j + j') (2\ell - j - j' + h - 1)!}{\Gamma(i\sigma + j - \ell) \Gamma(i\sigma + j' - \ell) (\ell - j + h - 1)! (\ell - j' + h - 1)!} \quad (\text{B.1})$$

First we make a change of summation variables to $z = \ell - j, z' = \ell - j'$.

Using the familiar identity $\Gamma(x)\Gamma(1-x) = \pi / \sin \pi x$

and the definition of the Euler Beta-function B we may then

rewrite (B.1) as

$$S = \pi^{-2} \sin^2 i\pi\sigma \frac{\Gamma(i\sigma + h + \ell)}{\Gamma(i\sigma + h - 1)^2} S', \quad \text{with} \quad (\text{B.2})$$

$$S' = \sum_{z,z'=0}^{\ell} \binom{\ell}{z} \binom{\ell}{z'} B(1-i\sigma+z, i\sigma+h-1) B(1-i\sigma+z', i\sigma+h-1) B(i\sigma+\ell-z-i, h+i\sigma) \cdot$$

We now insert the standard integral representation of the

B-functions,

$$B(a, b) = \int_0^1 dx x^{a-1} (1-x)^{b-1}$$

The result is

$$S' = \sum_{z,z'=0}^{\ell} \binom{\ell}{z} \binom{\ell}{z'} \int_0^1 \int_0^1 \int_0^1 dx dy dz \left\{ \frac{z(1-x)}{1-z} \right\}^z \left\{ \frac{z'(1-y)}{1-z'} \right\}^{z'} \left\{ (1-x)(1-y) \right\}^{-i\sigma} \cdot (xy)^{i\sigma+h-2} z^{h-1} (1-z)^{i\sigma+\ell-1}$$

Both summations can now be performed with the help of the binomial theorem, this gives

$$S' = \int_0^1 \int_0^1 \int_0^1 dx dy dz (xy)^{i\sigma+h-2} \left\{ (1-x)(1-y) \right\}^{-i\sigma} (1-z)^{\ell} (1-zy)^{\ell} z^{h-1} (1-z)^{i\sigma+\ell-1}$$

The x and y integrations can be performed, each of them produces a Jacobi-polynomial op. [A3], formulae 8.962.1 and 9.111.

This gives

$$S' = \left\{ \frac{\ell! \Gamma(i\sigma) \Gamma(i\sigma + h - 1)}{\Gamma(h + \ell)} \right\}^2 \int_{-1}^1 dt \left(\frac{1-t}{2} \right)^{i\sigma - \ell - 1} \left(\frac{1+t}{2} \right)^{h-1} \left\{ P_{\ell}^{(i\sigma - \ell - 1, h - 1)}(t) \right\}^2.$$

We have introduced $t = 2z - 1$ as a new variable of integration. Finally, the t -integration can also be performed with the help of the standard orthonormality relation of Jacobi-polynomials, op. [G3], formula 7.391.1. One obtains

$$S' = \ell! \frac{\Gamma(i\sigma) \Gamma(i\sigma + h - 1)}{\Gamma(h + \ell)} \frac{\Gamma(1 - i\sigma)^2}{i\sigma + h + \ell - 1}.$$

This has to be inserted into Eq. (B2). Splitting the \sin^2 into Γ -functions again, we obtain after some cancellations the final result

$$S = \frac{\ell!}{(h + \ell - 1)!} \frac{\Gamma(i\sigma + h + \ell - 1)}{\Gamma(i\sigma) \Gamma(i\sigma + h - 1)}. \quad (\text{B.3})$$

We remark that result and derivation are equally valid when h is not an integer. The factorials $(h + \ell - 1)!$, etc., have to be read as Γ -functions in this case.

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