THE GROWTH OF MODULATIONAL INSTABILITY FOR RANDOMIZED INITIAL CONDITIONS

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RESEARCH AND TRAINING PROGRAMME ON CONTROLLED THERMONUCLEAR FUSION AND PLASMA PHYSICS (EUR-NE)

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Introduction

Recently the creation of density depletions due to purely
growing modes in turbulent plasmas has been studied
intensely [1-16]. This problem is of great interest for the
laser-plasma interaction at the critical density
[9-13] as well as for beam-plasma interaction and the
general understanding of the development of plasma
turbulence [1-8]. For the laser plasma interaction, the
laser radiation acts as a pump wave for the instability
while for an isolated plasma the decay of plasma waves
into plasma waves with smaller k and ion acoustic waves
(plasmon condensation) yields a pumping of plasmons at
small k values. As it turns out, the plasmons with very
small k values for which decay is forbidden, are unstable
against modulation of the amplitude, associated with
modulation of plasma density. For the laser-plasma case
the corresponding instability for constant pump wave is
to: the oscillating two stream instability [14]. The
threshold for the instability for the case where the
presence of ion-acoustic waves can be disregarded may be
estimatted to be

$$\frac{c_0 |E|^2}{N_0 M c_s^2} \geq 12 \frac{k_{eff}^2}{k_D^2}$$

where $k_{eff}$ is a typical wave number of the electron wave,
$k_D$ is the Debye radius, $c_s$ is the ion acoustic velocity,
$N_0$ is the unperturbed plasma density and $M$ is the ion mass.

The present investigation is concerned with the self-consistant
situation where there is no pumping. The situation is moreover assumed to be turbulent in the sense that the phases
of different k modes are chosen in a random way. This
problem has also been briefly considered in [17] where,
besides, extensive references to the whole field are given.
Basic Equations

The self-consistent system of equations governing the modulational instability, derived by Zakharov [2] may be written

\[ 2 i \omega_{\text{pe}} \frac{\partial E}{\partial t} + 3 \nu^2 \frac{\partial^2 E}{\partial x^2} = \omega^2 \frac{n}{\text{pe}} E \quad (1a) \]

\[ \frac{\partial^2 n}{\partial t^2} - \frac{c_s^2}{\partial x^2} \frac{\partial^2 n}{\partial x^2} = \frac{\epsilon_0}{4M} \frac{\partial^2 |E|^2}{\partial x^2} \quad (1b) \]

where standard notations were used. By introducing the renormalizations

\[ x = \frac{3}{2} \frac{v_{\text{te}}}{\omega_{\text{pe}} c_s}, \quad t = \frac{3}{2} \frac{v_{\text{te}}}{\omega_{\text{pe}} c_s} t, \quad n = \frac{4c_s^2 n_0}{3v_{\text{te}}} n \]

\[ |E|^2 \rightarrow \frac{16MN_0}{3c_0 v_{\text{te}}} \frac{c_s^4}{v_{\text{te}}^2} |E|^2 \]

the system (1) is transformed to the dimensionless form

\[ i \frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} = nE \quad (2a) \]

\[ \frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} = \frac{\partial^2 |E|^2}{\partial x^2} \quad (2b) \]

This system of equations has the constants of motion (see for instance [15])

\[ I_1 = \int_{-\infty}^{\infty} |E|^2 dx \]

\[ I_2 = \int_{-\infty}^{\infty} \left[ \frac{i}{2} \left( E \frac{\partial E^*}{\partial x} - E^* \frac{\partial E}{\partial x} \right) + nV \right] dx \]

\[ I_3 = \int_{-\infty}^{\infty} \left[ |\frac{\partial E}{\partial x}|^2 + n|E|^2 + \frac{1}{2} n^2 + \frac{1}{2} V^2 \right] dx \]
where $V$ is defined through

$$\frac{\partial n}{\partial t} + \frac{\partial V}{\partial x} = 0 \quad (3)$$

The constants of motion $I_1 - I_3$ correspond to conservation of number of quanta, momentum and energy respectively. The quantity $V$ corresponds to a hydrodynamic flux.

The system (2) has stationary solutions of the form

$$n = n(x - vt) \quad (4a)$$

$$E = \left[ -(1-v^2)n(x - vt) \right]^{1/2} e^{i(kx - \Delta \omega t)} \quad (4b)$$

where the shifts in wave-number and frequency are due to Doppler shift and nonlinearity. These stationary solutions are due to a balance between nonlinearity and dispersion and are called solitons. By substitution of (4a) into (2b) and integration with respect to $x$ we obtain

$$n = - \frac{|E|^2}{1 - v^2}$$

By introducing this into (2a) we obtain the nonlinear Schrödinger equation

$$i \frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} + \frac{1}{1 - v^2} |E|^2 E = 0 \quad (5)$$

It has been shown that this equation has an infinite number of constants of motion [16]. For the type of solution (4a), (3) may be integrated to give

$$V = v \cdot n$$
and thus the constants of motion $I_1 - I_3$ may be expressed only in $E$ and $n$ and derivatives of $E$. In the quasistatic case we take the limit $v \approx 0$ and then the terms proportional to $V$ in $I_2$ and $I_3$ are zero.

Numerical treatment

In order to solve the system of equations (2) numerically it is convenient to make a Fourier transformation in space. This yields

\[ i \frac{\partial E_k}{\partial t} - k^2 E_k = \sum_{k'} n_{k',E_{k-k'}} \]  
\[ \frac{\partial^2 n_k}{\partial t^2} + k^2 n_k = -k^2 \sum_{k'} \frac{E_{k',E_{k-k'}}}{E_{k'}} \]  

Moreover, it is necessary to write the equations in real form. We then introduce

\[ E_k = E_{kR} + iE_{kI} \]

and

\[ n_k = n_{kR} + i n_{kI} \]

yielding the system

\[ \frac{\partial E_{kI}}{\partial t} + k^2 E_{kR} = -\sum_{k'} (n_{k'R}E_{k-k'R} - n_{k'I}E_{k-k'I}) \]  
\[ \frac{\partial E_{kR}}{\partial t} - k^2 E_{kI} = \sum_{k'} (n_{k'R}E_{k-k'R} + n_{k'I}E_{k-k'I}) \]  
\[ \frac{\partial^2 n_{kR}}{\partial t^2} + k^2 n_{kR} = -k^2 \sum_{k'} (E_{k'R}E_{k-k'R} + E_{k'I}E_{k'-k'I}) \]  
\[ \frac{\partial^2 n_{kI}}{\partial t^2} + k^2 n_{kI} = -k^2 \sum_{k'} (E_{k'I}E_{k-k'R} - E_{k'R}E_{k'-k'I}) \]
For the numerical treatment Eqs. (7c) and (7d) have to be split into two thus yielding a system of six equations for each Fourier component. In the present investigation the development in time of 31 Fourier components in space has been studied. The Fourier components have been chosen so as to make the space variation of \( n \) and \( E \) have the period \( L = 10 \), i.e. \( k = \pm m k_0 \) where \( k_0 = \frac{2\pi}{L} \) and \( m = 1, 2 \ldots \)

For comparison it is suitable to write the constants of motion \( I_1 - I_3 \) in terms of Fourier components, i.e. yielding the constants of motion of (6). The result is

\[
I_1 = \sum_k |E_k|^2
\]

\[
I_2 = -2 \sum_{k \neq 0} (k|E_k|^2 + \frac{1}{k} n_k \frac{\partial n_{-k}}{\partial t})
\]

and

\[
I_3 = \sum_{k \neq 0} (k^2|E_k|^2 + \frac{1}{2} |n_k|^2 + \frac{1}{2k^2} \frac{\partial n_k}{\partial t} \frac{\partial n_{-k}}{\partial t} + n_k \sum_{k_1} E_{k_1} E_{k+k_1}^*)
\]

where (3) was used i.e.

\[
V_k = -\frac{i}{k} \frac{\partial n_k}{\partial t}
\]

The terms corresponding to \( k = 0 \) are omitted since \( \frac{\partial^2 n_k}{\partial t^2} \sim k^2 \).

The initial conditions

As was pointed out earlier the intention is to make the phases of the Fourier components as random as possible. There are, however, certain restrictions on the choice of the initial values of the system (7). First the modulation of the density \( n \) which enters (2) is a real quantity. This follows from the derivation of (1) but also directly from the fact that \( n \) is the amplitude of a perturbation of zero frequency. This requirement also imposes restrictions on the derivatives of \( n_k \) which have to be given values
conserving the real character of \( n \). This yields the conditions

\[ n_{kR} = n_{-kR} \quad n_{kI} = -n_{-kI} \]  \hspace{1cm} (9a)

\[ \frac{\partial n_{kR}}{\partial t} = \frac{\partial n_{-kR}}{\partial t}, \quad \frac{\partial n_{kI}}{\partial t} = -\frac{\partial n_{-kI}}{\partial t} \]  \hspace{1cm} (9b)

These initial conditions are necessary for the conservation of \( I_1 - I_3 \).

In order to justify the limitation in the \( k \) values for the whole time development of (7) we moreover have to consider the mechanism of stabilization. The stabilizing term is the term \( k^2 E_k \) and this has in the quasistatic case to balance \( (|E|^2 E)_k \). In a one-dimensional description this will actually occur when the effective \( k \) value \( k_{\text{eff}} \) increases. By considering \( E_{k_{\text{eff}}} \) to be such an average \( E_k \) that it may be taken out of the sum in the r.h.s. of (6a) we obtain

\[ k_{\text{eff}}^2 \approx \sum_k |E_k|^2 = I_1 \]

In fact it turns out that when

\[ I_1 < \frac{1}{2} k_{\text{max}}^2 \]  \hspace{1cm} (10)

where \( k_{\text{max}} \) is the largest absolute \( k \) value of the Fourier components studied, the numerical solution is stabilized when \( k_{\text{eff}} < \frac{1}{2} k_{\text{max}} \) and the mode \( E_{k_{\text{max}}} \) is very weakly excited. This just means that we have modes with large enough \( k \) values present to obtain the stabilization by the dispersive term. This estimate however relies on the quasistatic approximation and for large initial values of \( \frac{\partial n}{\partial t} \) it may be violated.
For moderate and random initial values of $\frac{\partial n_k}{\partial t}$ the condition (10) however seems sufficient for the applicability of the description used. For such values of $\frac{\partial n_k}{\partial t}$ stable and stationary solitons are obtained as solutions of (7) (see Fig. 1). In order to make the solitons move it seems suitable to choose the initial conditions so as to describe two arbitrary but equal configurations moving in opposite directions i.e.

$$n = n_1(x-vt) + n_2(x+vt) \quad (n_1 = n_2)$$  \hspace{1cm} (11)

This means that

$$n = \sum_k n_{1k} e^{ik(x-vt)} + \sum_k n_{2k} e^{ik(x+vt)}$$

and we may write

$$n_{1,2k} = n'_{1,2k} e^{-ikvt}$$

yielding the initial values

$$\frac{\partial n_{1k}}{\partial t} = -ikvn_{1k} \hspace{1cm} (12a)$$

$$\frac{\partial n_{2k}}{\partial t} = ikvn_{2k} \hspace{1cm} (12b)$$

These initial values fulfill the conditions (9) for the total Fourier component $n_k$. In the simulations they have been chosen so that $|n_{1k}| = |n_{2k}|$ and so that they have a phase difference $k \Delta \phi$, where $\Delta \phi$ is common to all modes.

In order to start from a situation relevant for the nonlinear Schrödinger equation $n$ was initially chosen so that
For such a situation the condition (6) may not apply. If (9) continues to be valid during the growth of the instability we rather expect the condition (10) to take the form

\[ I_1 < \frac{1}{2} k_{\text{max}}^2 (1 - v^2) \]

Numerical solutions

In Fig. 1 we show a solution of (7) where we have chosen \( v = 0 \) i.e.

\[ n(0) = -|E(0)|^2 \]

and where the Fourier components have been summed up to give a solution in space. The solution is characterized by the values

\[ I_1 = 22.20, \quad I_2 = 0.00 \quad \text{and} \quad I_3 = -362.5 \]

The maximum variation of the constants of motion was 0.1\%. We also find that \( I_1 \) is well within the limit (10). To make the initial conditions to completely agree with the quasi-static case the initial derivatives of \( n \) were chosen as zero. This situation corresponds to \( V(0) = 0 \). It is in this case of interest to calculate the constants of motion \( I_1 - I_3 \) for \( V = 0 \) and compare them with the conserved quantities. For the initial conditions used here the solitons do not move. In Fig. 1a-d is shown the growth of solitons and in Fig. 1e and f is shown the variation of \( I_2 \) and \( I_3 \) when \( V \) is taken to be zero.
In Fig. 2 we show a solution of (7) where

\[ n(0) = -\frac{|E(0)|^2}{1-v^2} \]

and \( v = 0.7 \). This solution is characterized by \( I_1 = 16.39 \), \( I_2 = -14.65 \) and \( I_3 = 514.5 \). Here the maximum variation occurred in \( I_3 \) and was equal to 0.5%. For comparison with the variation 0.1% in the preceding solution it should be noted that the present one was studied during a longer time interval \((0 \leq t \leq 10)\). A comparison with the condition (14) shows that we should have \( I_1 < 20 \). This is obviously fulfilled. It should, however, be mentioned that another computer solution with the same \( v \) but with \( I_1 = 30 \) gave strongly excited \( E_{k_{\text{max}}} \) and was thus not stabilized within the validity of the model. This indicates that the condition (14) gives a good estimate for the maximum \( I_1 \) allowed for a given \( k_{\text{max}} \). Fig. 2a-h shows the solution for \( v = 0.7 \). The initial conditions were chosen according to (11) and this makes the solitons move with different velocities. Between \( t = 1.936 \) and \( t = 2.336 \) we can observe a collision between a large and a small soliton, where the small soliton is reflected.

Discussion

The solutions shown here (and in fact several others not shown here) indicate a strong tendency for creation of several solitons when the initial phases of the Fourier components in space of \( E \) are picked in a random manner. However, the limit of close packaging (compare Kingsep et al. [3]) is not reached. The comparison with the quasistatic approximation for the first solution shows large deviations of \( I_2 \) and \( I_3 \). However, the constraint (10) and its generalization (14) for the case of moving configurations show a good agreement with the numerical results. The structure of the solution given by (11) as initial condition is during the development of the instability maintained only to a limited extent. In order
to study more closely the interaction of solitons it thus seems necessary to abandon the condition of random phases.

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References


Fig. 1e. The variation of $I_2' = I_2(V = 0)$ when $I_2 = 0.00$
Fig. 1f. The variation of $I_3' = I_3(V = 0)$ when $I_3 = -362.5$
Fig. 2c. $t = 1.54$