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FIBER BUNDLE GEOMETRY AND SPACE-TIME STRUCTURE*

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ABSTRACT. Within the framework of the geometric formulation of gauge theories in fiber bundles, the general relation between the bundle connection (gauge field) and the geometry of the base space is obtained. A possible gauge theory for gravitation is presented.

RESUMO. Considerando a formulação geométrica das teorias de Gauge em espaços fibrados encontra-se a relação geral entre o campo de Gauge e a geometria do espaço base. Uma teoria de Gauge para o campo gravitacional é apresentada dentro deste contexto.

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1. INTRODUCTION

A geometric unified formulation of local gauge theories was proposed recently.[1,2,3] This approach consists in treating a gauge theory as a theory of connections of a superspace, known in mathematics as a principal fiber bundle.[4] The geometry of fiber bundles allows us to describe not only individual gauge fields, but also a collection of such fields introducing over each space-time point many fibers. In particular, the supersymmetric theories can also be included in this scheme.[1,2,5]

Some difficulties arise when one tries to construct a gauge theory for the gravitational field in a superspace. In fact, as far as a geometrical theory for gravitation does already exist, the relation between the space-time geometry and that of a bundle manifold must be clearly specified.[6]

In this paper the influence of a gauge field on the base space geometry is investigated in the most general case. The base space is assumed to be a four dimensional metric manifold. Different geometries are introduced in this space changing the structural group.

In section 2 the properties of a bundle manifold and its relation to gauge theories are briefly discussed.

In section 3 we consider the projection of the fiber into the base space in order to establish the general relation between the connection of the base space and the bundle connection (gauge field). It turns out that in the most general case the base space geometry "induced" by the gauge field is a geometry with non-metric and nonsymmetric transport, according to Schouten's classification.[7]

Finally, a possible gauge theory for gravitation with Lorentz structure group is presented in section 4.

2. THE GEOMETRY OF THE SUPERSPACE

We shall summarize the properties of a bundle manifold and its relation to gauge theories.[2] Locally, the superspace of interest to us is the direct product of a 4-dimensional base manifold and an n-parameter group manifold. In order to define a basis for the superspace let us introduce a set of n+4 vector fields $\{\vec{h}_i\}$, $i = 0,1,\ldots,n+3$, with the following antisymmetric properties:

$$
\begin{aligned}
&[\vec{h}_\mu, \vec{h}_\nu] = 0 \\
&[\vec{h}_a, \vec{h}_b] = f^c_{\ ab}\vec{h}_c \\
&[\vec{h}_a, \vec{h}_\mu] = 0 , \qquad \mu,\nu = 0,1,2,3 \\
&\qquad\qquad\qquad\qquad a,b = 1,\ldots,n
\end{aligned}
\tag{2.1}
$$

It is easy to see that the commutation relations (2.1) correspond to the choice of a coordinate induced basis for the horizontal tangent space and an invariant basis for the vertical part of the tangent space. The symmetric properties are given through the superspace metric

$$G_{ij} = \vec{h}_i \cdot \vec{h}_j \tag{2.2}$$

Although very simple, the basis $\{\vec{h}_i\}$ is not convenient for physical interpretation because the relevant geometrical objects such as the metric tensor, the connection coefficients, the curvature tensor, etc, cannot be written in such a basis in a gauge covariant (or invariant) form. For this reason it is more advantageous to work in the "gauge-covariant" basis $\{\vec{E}_i\}$ where the metric (2.2) takes a block-diagonal form:

$$
\begin{aligned}
G_{\mu\nu} &= \vec{E}_\mu \cdot \vec{E}_\nu = g_{\mu\nu} \\
G_{ab} &= \vec{E}_a \cdot \vec{E}_b = \bar{g}_{ab} \\
G_{a\mu} &= G_{\mu a} = 0 \ ,
\end{aligned}
\tag{2.3}
$$

where $g_{\mu\nu}$ is the base manifold metric tensor and $\bar{g}_{ab}$ the group metric. Except when abelian groups or subgroups are involved, we take $\bar{g}_{ab}$ to be the Killing metric.

The antisymmetric properties of the set $\{\vec{E}_i\}$ are given by

$$
\begin{aligned}
[\vec{E}_\mu, \vec{E}_\nu] &= -F^a_{\mu\nu} \vec{E}_a \\
[\vec{E}_a, \vec{E}_b] &= f^c_{ab} \vec{E}_c \\
[\vec{E}_\mu, \vec{E}_a] &= 0 \ .
\end{aligned}
\tag{2.4}
$$

To establish the relation between the two basis $\{\vec{E}_i\}$ and $\{\vec{h}_i\}$ we use the expansion

$$
\begin{aligned}
\vec{h}_j &= h^i_j \vec{E}_i \\
\vec{N}^j &= N^j_i \vec{\Omega}^i \ ,
\end{aligned}
\tag{2.5}
$$

where $\{\vec{N}^i\}$ and $\{\vec{\Omega}^i\}$ are dual to $\{\vec{h}_i\}$ and $\{\vec{E}_i\}$ respectively. The coefficients h^i_j, N^j_i of the expansion (2.5) are determined from the normalization conditions

$$
\begin{aligned}
\langle \vec{N}^i, \vec{h}_j \rangle &= \delta^i_j \\
\langle \vec{\Omega}^i, \vec{E}_j \rangle &= \delta^i_j
\end{aligned}
\tag{2.6}
$$

and from requirement that $\{\vec{h}_a\}$ satisfy the group algebra. They are:

$$h_a^b = N_a^b = \delta_a^b \; ; \quad h_a^\mu \equiv N_a^\mu = 0$$
$$h_\mu^\nu = N_\mu^\nu = \delta_\mu^\nu \; ; \quad h_\mu^a = - N_\mu^a \; . \tag{2.7}$$

It can be shown[2] that under local gauge transformations

$$\vec{E}_\mu \to e^{i\epsilon^a(x)\vec{h}_a} \vec{E}_\mu e^{-i\epsilon^a(x)\vec{h}_a} \tag{2.8}$$

the objects N_μ^a behave like the vector potentials in gauge theories

$$N_\mu^b \to N_\mu^b - i\epsilon^b_{,\mu} - if^b_{ac}N_\mu^a\epsilon^c \; . \tag{2.9}$$

Moreover, the "gauge fields" N_μ^a are related to the functions $F_{\mu\nu}^a$ in (2.4) as follows

$$F_{\mu\nu}^a = N^a_{\mu,\nu} - N^a_{\nu,\mu} + f^a_{bc}N^b_\mu N^c_\nu \quad . \tag{2.10}$$

The local gauge transformations are then interpreted as local rotations of the vectors $\{\vec{E}_\mu\}$ which leave the commutator $[\vec{E}_\mu,\vec{E}_\nu]$ invariant. Under this point of view the gauge fields N_μ^a are bundle connections. To make it more clear we may combine (2.5) with (2.7) to obtain

$$\vec{E}_\mu = \vec{h}_\mu + N_\mu^a\vec{E}_a \; . \tag{2.11}$$

Noting that $\vec{h}_\mu = \partial_\mu$ (coordinate induced basis), it is easy to see that $\vec{E}_\mu$ has the form of a covariant derivative operator with connection N_μ^a.

The geometry of the superspace provides a natural background for the description of all gauge theories. Its utilization in physics intrinsically lead us to admit the possibility that physical space, defined by the interactions, may have dimension larger than four, and even infinite. From this point of view the usual space-time description of microprocesses means a special projection of the objective geometry into one created by our macroscopic measuring apparatus. It would be interesting to know what we loose with this projection.

3. THE GEOMETRY OF THE BASE SPACE

We now turn to study the different geometries of the base space. From the structure of the superspace defined in the previous section we can expect that there exists a relation between the base space geometry and the bundle connection H^{i}_{μ}. In other words, we want to know what kind of geometries are induced in the base space by different gauge fields. This is a very important point if we want to justify what the structure of a fibre bundle has to do with the space-time of general relativity.

Let us make use of the projection into the space tangent to the base space. Namely, consider a vector ϕ^{i} in the tangent space and its covariant derivative. The projection can be specified by the following relations

$$\phi^i = h^i_\mu \phi^\mu$$
$$\phi^i_{,\mu} - N^a_\mu (La)^i_k \phi^k = h^i_\rho (\phi^\rho_{,\mu} + \Gamma^\rho_{\mu\nu}\phi^\nu) \quad , \tag{3.1}$$

where $(La)^i_k$ is the generator of the 4-dimensional representation of Gn. From (3.1) it is easy to derive the general relation between the connection of the base space and the bundle connection:

$$\Gamma^\alpha_{\mu\nu} = h^\alpha_i h^i_{\nu,\mu} - h^\alpha_i N^a_\mu (La)^i_k h^k_\nu \ . \tag{3.2}$$

It is important to notice that (3.2) is equivalent to requirement that the "projectors" h^i_μ (h^μ_i) are constant with respect to the total covariant derivative

$$\nabla_\mu h^i_\nu = h^i_{\nu,\mu} - N^a_\mu (La)^i_k h^k_\nu - \Gamma^\lambda_{\mu\nu} h^i_\lambda = 0 \tag{3.3}$$

$$\nabla_\mu h^\lambda_i = h^\lambda_{i,\mu} + h^\lambda_k N^a_\mu (La)^k_i + \nabla^\lambda_{\mu\nu} h^\nu_i = 0 \ . \tag{3.4}$$

From (3.3) and (3.4) we see that the local basis of the representation of Gn transforms into itself under parallel transport in the base space. To ensure the existence of the objects $\{h^\lambda_i\}$ satisfying (3.4) we write the integrability condition

$$\nabla_{[\mu\nu]} h^\lambda_i = R_{\mu\nu\sigma}{}^\lambda h^\sigma_i + F^a_{\mu\nu}(La)^k_i h^\lambda_k + S^\sigma_{\mu\nu}\nabla_\sigma h^\eta_i \ , \tag{3.5}$$

where $F^a_{\mu\nu}$ is the field tensor (2.10), $S^\sigma_{\mu\nu}$ the torsion tensor and $R_{\mu\nu\sigma}{}^\lambda$ the curvature tensor of the base space. As a consequence of (3.3), (3.4) and (3.5) one obtains the relation between the field tensor and the curvature of the base space:

$$R_{\mu\nu\sigma}{}^{\lambda} = h^{\lambda}_{i}(La)^{i}_{k}h^{k}_{\sigma}F^{a}_{\nu\mu} \ . \qquad (3.6)$$

For a complete specification of the base space geometry we need also to compute the covariant derivative of the metric tensor $g_{\mu\nu} = h^{i}_{\mu}h_{i\nu}$. Multiplying (3.2) by $g_{\alpha\lambda}$ and performing a symmetrization in μ,ν, we have

$$g_{\mu\nu;\lambda} = N^{a}_{\lambda}[h_{i\mu}(La)^{i}_{k}h^{k}_{\nu} + h_{i\nu}(La)^{i}_{k}h^{k}_{\mu}] \ . \qquad (3.7)$$

Thus, in the most general case the gauge field induces in the base space a geometry with torsion and with non-zero covariant derivative of the metric tensor. If the representation of Gn is such that the matrices $(La)^{i}_{k}$ are real and antisymmetric, then $g_{\mu\nu;\lambda} = 0$, and we have a geometry with metric transport.[7]

For the subsequent calculations with concrete structure groups it is important to observe that the matrix elements $(La)^{i}_{k}$ with one group and one base index vanish. To see this put (2.7) into (3.4) to obtain

$$\delta^{\lambda}_{\sigma}N^{\sigma}_{\mu}(La)^{\sigma}_{b} = 0 \ ,$$

which implies $(La)^{\sigma}_{b} = 0$.

3.1. Abelian Group

As a first example let us consider the group of transformations $g'_{\mu\nu} = \lambda(x)g_{\mu\nu}$ which leave the interval ds^2 invariant. The corresponding transformation of the gauge field N_{μ} is

$$N'_{\mu} = N_{\mu} - \frac{i}{2e}\,[\ln\lambda(x)]_{,\mu} \ .$$

The projection of the 4-dimensional representation of this abelian group gives

$$\Gamma^{\alpha}_{\mu\nu} = h^{\alpha}_{i} h^{i}_{\nu,\mu} - ieN_{\mu}\delta^{\alpha}_{\nu} ,$$

$$S^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{[\mu\nu]} = - ieN_{[\mu}\delta^{\alpha}_{\nu]} \tag{3.8}$$

$$g_{\lambda\nu;\mu} = 2ieg_{\lambda\nu}N_{\mu} .$$

Thus, the gauge field associated with this group induces in the base space a geometry with semimetric transport. This is the Weyl-Cartan geometry.

3.2. GL(4)

Inserting the generator of the 4-dimensional representation of this group into (3.2) and (3.7) we obtain respectively

$$\Gamma^{\alpha}_{\mu\nu} = h^{\alpha}_{i} h^{i}_{\nu,\mu} - \bar{g}_{\lambda\nu}N^{\lambda\alpha}_{\mu}$$

$$g_{\lambda\nu;\mu} = N^{\alpha\beta}_{\mu}[g_{\lambda\beta}\bar{g}_{\alpha\nu} + g_{\nu\beta}\bar{g}_{\alpha\lambda}] , \tag{3.9}$$

where $\bar{g}_{\lambda\nu}$ is the GL(4) metric. From (3.9) we see that the gauge field $N^{\alpha\beta}_{\mu}$ induces in the base space a geometry with non-metric and non-symmetric transport.

4. GAUGE THEORY FOR GRAVITATION

We present now a possible gauge theory of gravitation with Lorentz structure group. As a starting point we write the action[2]

$$S = \int d^4x \, dVg \, \sqrt{-G} \, R = \int d^4r \, dVg \, \sqrt{-G} \, [R_{s.t.} + R_G - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a] \quad (4.1)$$

where $R_{s.t.}$ = scalar curvature of the base space and R_G = scalar curvature of the group manifold.

This is the only action satisfying the following requirements:

a) In the limit of flat base-space the equations of motion for $F^a_{\mu\nu}$ are the same as that for non-abelian gauge theories

b) For $F^a_{\mu\nu} = 0$ the theory reduces to Einstein's.

The dependence on R_G can be eliminated by the replacement:

$$L \rightarrow L' = \sqrt{-G} \, [R - R_G] \; . \quad (4.2)$$

After integration over the group parameters (4.1) becomes

$$S = \int d^4x \, \sqrt{-g} \, [R_{s.t.} - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a] \; . \quad (4.3)$$

From (4.3) the Einstein-Yang-Mills equations can be easily obtained:

$$\begin{aligned} & F_a{}^{\mu\nu}{}_{;\nu} = 0 \\ & R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \lambda^2 (g_{\alpha\nu} F^a_{\mu\sigma} F^{\sigma\alpha}_a - \frac{1}{4} g_{\mu\nu} F^a_{\alpha\beta} F^{\alpha\beta}_a) \quad , \end{aligned} \quad (4.4)$$

where we have introduced a constant λ with dimension of length. Projecting (4.4) in the base space with the aid of (3.6), we have

$$R_{\lambda\ \ ;\mu}^{\mu\nu\sigma} = 0$$

$$R_{\mu\nu} = \frac{1}{4}\lambda^2 C_{\mu\lambda\sigma\nu} R^{\lambda\sigma} , \qquad (4.5)$$

where $C_{\mu\lambda\sigma\nu}$ is the conformal tensor

$$C_{\mu\lambda\sigma\nu} = R_{\mu\lambda\sigma\nu} + \frac{1}{2}(R_{\mu\sigma}g_{\lambda\nu} + R_{\lambda\nu}g_{\mu\sigma} - R_{\mu\nu}g_{\lambda\sigma} - R_{\lambda\sigma}g_{\mu\nu})$$
$$+ \frac{R}{6}(g_{\lambda\sigma}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\sigma}) .$$

The first equation in (4.5) is analogous to Yang's equation for gravitation.[8] The second equation in (4.5) can be important to eliminate the non-physical conformally flat solutions of Yang's equation.[9,10,11]

In concluding, on could say that the utilization of fiber bundle geometry shows that 4-dimensional Riemannian geometry is only a particular case among possible dynamical geometries.

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