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DU AND UD-INVARIANTS OF UNITARY GROUPS*

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I - INTRODUCTION

Invariants of unitary groups can be constructed in a number of ways. For example, they can be constructed by generalizing the concept of Casimir operators [1] through an extensive use of the estructure constants of the associated Lie algebras. In this paper, we shall consider two classes of invariants constructed in an easier way. The first of them has been used by physicists in connection with problems like those related to the labeling of states characterized by a given irreducible representation (irrep) of some unitary group. The other class of invariants we shall discuss here was introduced by the present authors in dealing with the problem of finding relationships between invariants of complementary unitary groups [2].

For reasons to be clear in the next section, we shall denote the invariants belonging to the first class by UD-invariants while the second-class ones will be named DU-invariants. There is no fundamental distinction between DU- and UD-invariants. Any of them can be used equally well in the applications. However, the DU-invariants lead to somewhat slightly simpler expressions for the eigenvalues of the invariants of U(n) and SU(n) as will be seen later on.

Although each class is formed by independent invariants, the classes themselves are not mutually independent. There is, however, no linear relation between UD- and DU-invariants. It is not yet known how to express an n-order DU-invariant is terms of UD-invariants, or vice-versa. Surely, this problem is soluble, but only for low order invariants the solution is explicitly known.

Next section is devoted to introduce the notation to be used throughout this paper and to define the objects we shall deal with. Some properties of the elementary symmetric functions are listed. In Section III, we present recursive relations that allow us to obtain the eigenvalues of DU-invariants of unitary groups in any irreducible representation. These relations will also serve to prove the validity of two closed formulas, presented in Section V, for getting those eigenvalues. A clear advantage of this approach is that no explicit reference to the basis supporting the considered irrep is needed.

Section IV is dedicated to a brief discussion of the $D^{(n)}$-function associated with unitary groups. Three closed formulas for obtaining the eigenvalues of invariants (DU and UD types) are discussed in Section V. Finally, for the sake of completeness, the connection between the invariants of U(n) and SU(n) is presented in Section VI.

II. DEFINITIONS AND NOTATION

In the following, we shall introduce some standard definitions and notation that will be necessary for our present purposes.

The objects η_i and ξ^i, $i=1,2,\ldots,n$, are generally named boson vector operators if their components satisfy the following commutation relations

$$[\eta_{is},\xi^{jt}] \equiv \eta_{is}\xi^{jt} - \xi^{jt}\eta_{is} = \delta_i^j\,\delta_s^t \,, \tag{2.1a}$$

and
$$[\eta_{is},\eta_{jt}] = [\xi^{is},\xi^{jt}] = 0 \,, \tag{2.1b}$$

for $i,j = 1,2,\ldots,n$ and $s,t = 1,2,\ldots,d$.

If, instead, they satisfy similar anti-commutation relations, namely ,

$$\{\eta_{is},\xi^{jt}\} \equiv \eta_{is}\xi^{jt} + \xi^{jt}\eta_{is} = \delta_i^j\,\delta_s^t \,, \tag{2.2a}$$

and
$$\{\eta_{is},\eta_{jt}\} = \{\xi^{is},\xi^{jt}\} = 0 \,, \tag{2.2b}$$

for $i,j = 1,2,\ldots,n$ *and* $s,t = 1,2,\ldots,d$,then they are named fermion vector operators.

Consider now the n^2 operators defined by

$$A_i^j \equiv \eta_i \cdot \xi^j = \sum_{s=1}^{d} \eta_{is}\,\xi^{js} \,, \quad i,j = 1,2,\ldots,n, \tag{2.3}$$

where η_{is} and ξ^{js} are the components of either boson or fermion vector operators. In any case, it is easy to verify that the commutation rules for the A's are given by

$$[A_i^j, A_k^l] = \delta_k^j A_i^l - \delta_i^l A_k^j \,. \tag{2.4}$$

The main property of the operators (2.3) we are interest in here is that they generate the group U(n), i.e., the group of all unitary transformations in the complex vector space of dimension n. So, any operator that commutes with all the A's will be invariant under unitary transformations. In other words, it will be an invariant of U(n).

We can form a k-order invariant of U(n) by doing the following contractions

$$C_k^{(n)} \equiv A_{i_1}^{\;i_2} A_{i_2}^{\;i_3} \cdots A_{i_k}^{\;i_{k-1}} \quad , \quad k = 1, 2, \ldots \qquad (2.5)$$

From now on, we shall adopt the usual convention of summing repeated indices over all the values they can assume.

Let us refer to the class of invariants (2.5) as DU-invariants of U(n) and maintain the convention that $C_0^{(n)} = n$.

Clearly, we can contract the indices of $A_i^{\;j}$ in a different way. Instead of contracting from "down" to "up" (DU) as we did in (2.5) , we can contract from "up" to "down" (UD) in the following way

$$\tilde{C}_k^{(n)} \equiv A_{i_k}^{\;i_1} A_{i_1}^{\;i_2} \cdots A_{i_{k-1}}^{\;i_k} \quad , \quad k = 1, 2, \ldots \qquad (2.6)$$

These new operators $\tilde{C}_k^{(n)}$ are also scalars with respect to U(n) since they also commute with all the generators (2.3). In analogy to the previous notation, they will be referred to as UD-invariants of U(n). It is also conventioned that $\tilde{C}_0^{(n)} = n$.

From the definitions (2.5) and (2.6), it is easy to see that $\tilde{C}_1^{(n)} = C_1^{(n)}$ and $\tilde{C}_2^{(n)} = C_2^{(n)}$. For $k>2$, however, the UD- and DU-invariants are connected through relations that get more and more complicated. For example, if $k=3$ then we have that

$$\tilde{C}_3^{(n)} = C_3^{(n)} + n\,C_2^{(n)} - \left[C_1^{(n)}\right]^2. \tag{2.7}$$

The general relation between UD- and DU-invariants is not yet known. From (2.7) it can be seen that that relation is non-linear.

Since there are at most n independent invariants of U(n) [3], each class of invariants discussed here has at most n independent elements. They can be taken as being the 1-,2-,..., n-order invariants as defined in (2.5) or (2.6).

The eigenvalues of the invariants, in a given irrep of U(n), depend only on the labels characterizing that representation. Such irreps can be characterized by a partition of an integer. The components of the partition are denoted h_{in}, $i=1,2,\dots,n$, and are such that

$$h_{1n} \geq h_{2n} \geq \cdots \geq h_{nn} \geq 0, \tag{2.8a}$$

and

$$h_{1n} + h_{2n} + \dots + h_{nn} = h = \text{integer}. \tag{2.8b}$$

The partition components h_{in} are defined by the following simultaneous operational equations

$$A_i^i = h_{in}, \quad i=1,2,\dots,n, \tag{2.9}$$

$$A_i^j = 0 \quad \text{for } j>i. \tag{2.10}$$

The irrep of U(n) characterized by the partition (2.8) is usually denoted by $[h] \equiv [h_{1n} h_{2n} \cdots h_{nn}]$. We shall indicate by $C_k^{(n)}(h_{1n}, h_{2n}, \ldots, h_{nn})$ and $\tilde{C}_k^{(n)}(h_{1n}, h_{2n}, \ldots, h_{nn})$, the corresponding eigenvalues of the k-order DU- and UD-invariants, respectively.

The theory of the symmetric group [4] associates with the partition (2.8) the concept of a hook. For our purpose here, we shall only need the partial hook defined [5] by

$$p_{ij} = h_{in} + j - i \quad , \quad i,j = 1,2,\ldots,n. \qquad (2.11)$$

We shall see, in Section V, that the eigenvalues of the DU- and UD-invariants can be given in terms of the particular partial hooks p_{i1} and p_{in} , respectively.

Finally, for further reference, we recall here the definition of the elementary symmetric functions [6] and some of their properties.

The elementary symmetric functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ of the ordered set $x = (x_1, x_2, \ldots, x_n)$ are defined by

$$\begin{aligned}
\varphi_1 &= \varphi_1(x) = \sum x_i = x_1 + x_2 + \ldots + x_n \; , \\
\varphi_2 &= \varphi_2(x) = \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n \; , \\
\varphi_3 &= \varphi_3(x) = \sum_{i<j<k} x_i x_j x_k \; , \\
&\vdots \\
\varphi_n &= \varphi_n(x) = \prod x_i = x_1 x_2 \cdots x_n \; .
\end{aligned} \qquad (2.12)$$

In terms of them, we can define the function

$$\beta_m(x) = \overline{\sum} \frac{(-1)^{m-\alpha}\,\alpha!}{\alpha_1!\alpha_2!\ldots\alpha_n!}\,\varphi_1^{\alpha_1}(x)\,\varphi_2^{\alpha_2}(x)\ldots\varphi_n^{\alpha_n}(x)\,, \qquad (2.13)$$

where the sum is taken over all α_i and the bar on the sum symbol is to indicate that the α_i are non-negative integers such that $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + n\alpha_n = m$. In addition, $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

The elementary symmetric functions can be used to express a Stevin product as

$$\prod_{i=1}^{n}(x_i + a) = \sum_{i=0}^{n} \varphi_i(x)\,a^{n-i}. \qquad (2.14)$$

We close this section presenting the following further relations involving the elementary symmetric functions (2.12) and the function (2.13) [5]

$$\sum_{i=1}^{n} x_i^p \Big/ \prod_{k\neq i}^{n}(x_k - x_i) = (-1)^{n-1}\,\beta_{p-n+1}(x)\,, \qquad (2.15)$$

$$\sum_{i=0}^{m}(-1)^i\,\varphi_i(x)\,\beta_{m-i}(x) = 0\,, \qquad (2.16)$$

$$\beta_{m-n}(x) = 0 \quad \text{for} \quad m = 1,2,\ldots,n-1. \qquad (2.17)$$

III - RECURSIVE RELATIONS FOR THE DU-INVARIANTS

Throughout the rest of this paper, we shall apply extensively many of the ideas introduced in references 5,7 and 8, and prove many results used in reference 7.

To obtain the eigenvalues of the DU-invariants defined in (2.5), we can dispense with any explicit reference to the basis for the irrep considered by noting that those eigenvalues can be obtained by means of a constructive procedure using, alternative and conveniently, the following relations

$$C_1^{(n)}(h_{1n},h_{2n},\dots,h_{nn}) = h_{1n}+h_{2n}+ \dots + h_{nn} \,, \tag{3.1}$$

$$C_k^{(1)}(h_{11}) = h_{11}^k \,, \tag{3.2}$$

$$C_k^{(n)}(h_{1n},h_{2n},\dots,h_{nn}) = \sum_{i=0}^{k} \binom{k}{i} h_{nn}^{k-i} C_i^{(n)}(h_{1n}-h_{nn},\dots,h_{n-1n}-h_{nn},0), \tag{3.3}$$

$$C_k^{(n+1)}(h_{1n+1},h_{2n+1},\dots,h_{nn+1},0) = C_k^{(n)}(h_{1n+1},h_{2n+1},\dots,h_{nn+1}) + \sum_{i=1}^{k-1}(-n)^{k-i-1} C_i^{(n)}(h_{1n+1},h_{2n+1},\dots,h_{nn+1}) \,, \; k>1 . \tag{3.4}$$

The relation (3.2) is contained in (3.1). However, we consider it separately only for computational convenience. Both of them are immediately proven by taking into account the operational equation (2.8) and noting that, from the definition (2.5),

$$C_1^{(n)} = A_1^1 + A_2^2 + \dots + A_n^n \,, \tag{3.5}$$

and

$$C_k^{(1)} = (A_1^1)^k . \tag{3.6}$$

The relation (3.4) follows as a trivial particular case of a general relation between DU-invariants of unitary groups found by the present authors [2] . All we have to do is to consider the irreps $[h_{1n+1} \cdots h_{nn+1} 0]$ and $[h_{1n+1} \cdots h_{nn+1}]$ of $U(n+1)$ and $U(n)$, respectively, and take these groups as complementary in the chain

$$U(n(n+1)) \supset U(n) \times U(n+1). \qquad (3.7)$$

To prove (3.3), consider the identity

$$\sum_{\text{all } i=1}^{n} (A_{i_1}^{i_k} + \lambda \delta_{i_1}^{i_k})(A_{i_2}^{i_1} + \lambda \delta_{i_2}^{i_1}) \cdots (A_{i_k}^{i_{k-1}} + \lambda \delta_{i_k}^{i_{k-1}}) =$$

$$= \sum_{l=1}^{k} \binom{k}{l} \lambda^{k-l} C_l^{(n)}. \qquad (3.8)$$

Now, by repeated use of (2.10) and the commutation relation (2.4), we can reduce the LHS of (3.8) to the form $F_k^{(n)}(A_1^1+\lambda, A_2^2+\lambda, \ldots, A_k^k+\lambda)$. The same reducing steps lead us to the operational relation

$$F_k^{(n)}(A_1^1, A_2^2, \ldots, A_k^k) = C_k^{(n)}. \qquad (3.9)$$

Finally, relation (3.3) is obtained from (2.9) by replacing $h_{in} + \lambda$ by h_{in} and putting $\lambda = h_{nn}$.

The set of recursive relations (3.1) to (3.4) allows us to obtain the eigenvalues of the DU-invariants of U(n) in any given irreducible representation. They are also useful to prove the validity of some closed formulas for getting those eigenvalues as we shall see in the following.

IV - THE FUNCTION $D^{(n)}$

Consider the integers q_{in} defined by

$$q_{in} = h_{in} + 1 - i \quad , \quad i = 1, 2, \ldots, n, \tag{4.1}$$

which are partial hooks of the partition $[h_{1n} h_{2n} \ldots h_{nn}]$,of the type p_{i1} . Define the $D^{(n)}$-function by

$$D^{(n)}(h) = D^{(n)}(h_{1n}, h_{2n}, \ldots, h_{nn}) \equiv \frac{1}{(n-1)\natural} \prod_{i<j}^{n} (q_{in} - q_{jn}) , \tag{4.2}$$

where

$$m\natural = m!(m-1)!(m-2)! \ldots 2! \ . \tag{4.3}$$

Since $q_{in} - q_{jn} = h_{in} - h_{jn} + j - i$, we see that the $D^{(n)}$-function gives the dimension of the irrep $[h]$ of $U(n)$ when its arguments define the corresponding Young diagram.

In the following, we shall prove some properties of the $D^{(n)}$-function which are similar to, but different from those derived by Louck and Biedenharn [5] .

From the relation

$$\prod_{i<j}^{n+1} Q_{ij} = \prod_{\ell=1}^{n} Q_{\ell\, n+1} \prod_{i<j}^{n} Q_{ij} \ , \tag{4.4}$$

and the recursive character of $n\natural$, by taking $Q_{km} = q_{kn+1} - q_{mn+1}$, we can obtain a first property of the $D^{(n)}$-function, namely,

$$D^{(n+1)}(h_{1n+1}, h_{2n+1}, \ldots, h_{n+1\,n+1}) = \frac{1}{n\natural} D^{(n)}(h_{1n+1}, \ldots, h_{nn+1}) \prod_{k=1}^{n} (q_{kn+1} - q_{n+1\,n+1}) , \tag{4.5}$$

from which it follows easily that

$$D^{(n+1)}(h_{1n+1},\dots,h_{in+1}+1,\dots,h_{n+1n+1}) = \frac{1}{n!} D^{(n)}(h_{1n+1},\dots,h_{in+1}+1,\dots,h_{nn+1})$$
$$\times \left[1+\frac{1}{q_{in+1}-q_{n+1n+1}}\right] \prod_{k=1}^{n} (q_{kn+1}-q_{n+1n+1}),\ i=1,2,\dots,n. \tag{4.6}$$

After some simple manipulations of the product symbol $\prod$, we can also see that

$$D^{(n)}(h_{1n},\dots,h_{in}+1,\dots,h_{nn}) = D^{(n)}(h) \prod_{k\neq i}^{n} \left[1+\frac{1}{q_{in}-q_{kn}}\right]. \tag{4.7}$$

Finally, we note a clear translation invariance of the $D^{(n)}$-function, i.e.,

$$D^{(n)}(h+\lambda) = D^{(n)}(h_{1n}+\lambda, h_{2n}+\lambda,\dots,h_{nn}+\lambda) = D^{(n)}(h). \tag{4.8}$$

V - CLOSED FORMULAS FOR THE EIGENVALUES

In this section, we shall present three closed formulas for getting the eigenvalues of the DU and UD-invariants of U(n) in any given irrep.

Firstly, we shall prove that

$$C_k^{(n)}(h) = \sum_{i=1}^{n} q_{in}^{k}\ D^{(n)}(h_{1n},\dots,h_{in}+1,\dots,h_{nn})/D^{(n)}(h),\ k=1,2,\dots \tag{5.1}$$

It is sufficient to show that (5.1) satisfies the recursive relations (3.1) to (3.4), which, as we already saw, give the eigenvalue $C_k^{(n)}(h)$.

If $k=1$, the RHS of (5.1) reads

$$\sum_{i=1}^{n} q_{in} \prod_{j\neq i}^{n} \left[1+\frac{1}{q_{in}-q_{jn}}\right] = \tfrac{1}{2}n(n-1)+\sum_{i=1}^{n} q_{in}\,, \tag{5.2}$$

where use was made of (4.7). The equality in (5.2) follows from techniques discussed in the appendix.

Substituting the definition (4.1) of q_{in} in (5.2), we prove that (5.1) satisfies the first recursive relation (3.1).

If $n=1$, the RHS of (5.1) becomes

$$q_{11}^{k}\, D^{(1)}(h_{11}+1)/D^{(1)}(h_{11}) = h_{11}^{k}\,, \tag{5.3}$$

where we used (4.8). Thus, (5.1) satisfies (3.2) also.

Now, we notice that

$$C_k^{(n)}(h+\lambda) = \sum_{i=1}^{n} (q_{in}+\lambda)^{k} D^{(n)}(h_{in}+\lambda,\ldots,h_{in}+\lambda+1,\ldots,h_{nn}+\lambda)/D^{(n)}(h+\lambda)$$

$$= \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} C_j^{(n)}(h)\,, \tag{5.4}$$

where we used again (4.8). As discussed in section 3, the relation (5.4) is equivalent to (3.3).

The proof that (5.1) satisfies (3.4) is somewhat laborious.

From (5.1) we have that

$$C_k^{(n+1)}(h_{1n+1},\dots,h_{nn+1},0)=\sum_{i=1}^{n+1} q_{in+1}^k \frac{D^{(n+1)}(h_{1n+1},\dots,h_{in+1}+1,\dots,h_{nn+1},0)}{D^{(n+1)}(h_{1n+1},\dots,h_{nn+1},0)} . \tag{5.5}$$

Taking apart the last term of the sum, noticing that $q_{n+1\,n+1}=-n$, since $h_{n+1\,n+1}=0$, and using the properties (4.5) and (4.6), the RHS of (5.5) becomes

$$C_k^{(n)}(h_{1n+1},\dots,h_{nn+1})+\sum_{j=0}^{k-1}(-n)^{k-j-1}C_j^{(n)}(h_{1n+1},\dots,h_{nn+1})+$$

$$(-n)^k\left[\frac{D^{(n+1)}(h_{1n+1},\dots,h_{nn+1},1)}{D^{(n+1)}(h_{1n+1},\dots,h_{nn+1},0)}+\sum_{i=1}^{n}\frac{1}{q_{in}+n}\frac{D^{(n)}(h_{1n+1},\dots,h_{in+1}+1,\dots,h_{nn+1})}{D^{(n)}(h_{1n+1},\dots,h_{nn+1})}\right], \tag{5.6}$$

where use was also made of the identity

$$\frac{x^k}{x+n}=\frac{(-n)^k}{x+n}+\sum_{j=0}^{k-1}(-n)^{k-j-1}x^j . \tag{5.7}$$

Finally, all we have to do now is to prove that the square brackets in (5.6) vanish. It is true and the demonstration envolve Stevin products and known properties of the elementary symmetric functions (2.12). Again, the reader is referred to the appendix for the details.

So, we have demonstrated that (5.1) gives the eigenvalues of the DU-invariants of U(n) in terms of the D-function (4.2) and the quantities q_{in} defined in (4.1). These eigenvalues can also be obtained in terms of the elementary symmetric functions (2.12) through the following closed formula

$$C_k^{(n)}(h_{1n},\dots,h_{nn}) = \sum_{l=0}^{k}(-)^l \varphi_l(x) \sum_{m=0}^{k-l} \binom{n-l}{k+1-l-m} \beta_m(x), \quad k=1,2,\dots, \tag{5.8}$$

where x stands for the ordered n-tuple $(q_{1n}, q_{2n},\dots,q_{nn})$ formed of the quantities (4.1), and $\beta_m(x)$ was defined in (2.13).

The expression (5.8) is similar to the corresponding one for UD-invariants derived by Louck and Biedenharn [5]. The present formula, connected with the DU character of the invariants, admits of factorization of φ_l facilitating the evaluation of the eigenvalue.

Let us now prove that (5.8) is true. Putting $x_i = q_{in}$ and using (4.7) we can write (5.1) as

$$C_k^{(n)}(h_{1n}, h_{2n},\dots,h_{nn}) = -\sum_{i=1}^{n} x_i^k \frac{\prod_{l=1}^{n}(x_l - x_i - 1)}{\prod_{j\neq i}^{n}(x_j - x_i)}. \tag{5.9}$$

For evaluating the sum above, we first develop the Stevin product (see Appendix) and then make use of the relation (2.15) and (2.17) to obtain

$$C_k^{(n)}(h_{1n},\dots,h_{nn}) = \sum_{l=0}^{k+1}(-)^l \varphi_l \sum_{m=0}^{k-l+1} \binom{n-l}{k-m-l+1} \beta_m. \tag{5.10}$$

Several terms of the sums above can be eliminated. Taking apart the term $l=k+1$ and then the contributions from $m=k-l+1$ we obtain

$$C_k^{(n)}(h_{1n},\dots,h_{nn})=\sum_{l=0}^{k}(-)^l\varphi_l\sum_{m=0}^{k-l}\binom{n-l}{k+1-l-m}\beta_m+\sum_{l=0}^{k+1}(-)^l\varphi_l\beta_{k+1-l}\,. \tag{5.11}$$

From (2.16), we see however that the last sum vanishes, completing the proof since the remaining expression coincides with (5.8).

Another representation of the eigenvalues of $C_k^{(n)}$ can be obtained by means of a matrix, an idea developed by Perelomov and Popov [8] . In their paper, these authors use the UD convention. For the DU-invariants, let us introduce the $n\times n$ matrix $a=a^{(n)}(h_{1n},h_{2n},\dots,h_{nn})$ whose elements are given by

$$a_{ij} = q_{in}\delta_{ij}+t_{ij}\,,\quad i,j=1,2,\dots,n, \tag{5.12}$$

with

$$t_{ij} = \begin{cases} 1 & \text{if } i>j \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.13}$$

In these conditions, we have simply that

$$C_k^{(n)}(h_{1n},h_{2n},\dots,h_{nn}) = \sum_{i,j=1}^{n}(a^k)_{ij}\,,\quad k=1,2,\dots, \tag{5.14}$$

i.e., the eigenvalue of $C_k^{(n)}$ in the irrep $[h]$ of U(n) is given by the sum of all matrix elements of a^k . For the proof, we shall follow the same steps as we did to prove the formula (5.1), i.e., we shall show that the RHS of (5.14) satisfies the recursive relations (3.1)-(3.4).

First of all, it is easy to see that

$$\sum_{i=1}^{n} a_{ij} = h_{jn} + n + 1 - 2j \quad , \quad j = 1,2,\dots,n. \tag{5.15}$$

and

$$\sum_{j=1}^{n} a_{ij} = h_{in} \quad , \quad i = 1,2,\dots,n. \tag{5.16}$$

Hence, summing (5.15) on the index j , or (5.16) on i , we get

$$\sum_{i,j=1}^{n} a_{ij} = \sum_{i=1}^{n} h_{in} = h = C_1^{(n)}(h_{1n}, h_{2n}, \dots, h_{nn}) , \tag{5.17}$$

and the relation (3.1) is verified.

From the definition, we have that $a_{11}^{(1)} = q_{11} = h_{11}$, so that $C_k^{(1)}(h_{11}) = h_{11}^k$ which is relation (3.2) .

It is also easy to see that

$$a^{(n)}(h_{1n}+\lambda, h_{2n}+\lambda, \dots, h_{nn}+\lambda) = a^{(n)}(h_{1n}, h_{2n}, \dots, h_{nn}) + \lambda I , \tag{5.18}$$

where I is the $n \times n$ identity. As a scalar matrix commutes with any matrix (of same dimensions, obviously), we have from (5.18) that

$$[a^{(n)}(h_{1n}+\lambda, h_{2n}+\lambda, \dots, h_{nn}+\lambda)]^k = \sum_{l=0}^{k} \binom{k}{l} \lambda^{k-l} [a^{(n)}(h_{1n}, \dots, h_{nn})]^l , \tag{5.19}$$

or, equally well,

$$[a^{(n)}(h_{1n}, \dots, h_{nn})]^k = \sum_{l=0}^{k} \binom{k}{l} h_{nn}^{k-l} [a^{(n)}(h_{1n} - h_{nn}, \dots, h_{n-1n} - h_{nn}, 0)]^l , \tag{5.20}$$

and summing up all matrix elements of both sides of (5.20), in view of (5.14) we obtain (3.3).

As before, to verify (3.4) we have to work a bit more. Let us consider an irrep of $U(n+1)$ with $h_{n+1\,n+1} = 0$. In this case, we have explicitly that

$$a^{(n+1)}(h_{1n+1},\ldots,h_{nn+1},0) = \begin{pmatrix} q_{1n+1} & 0 & 0 & \ldots & 0 & 0 \\ 1 & q_{2n+1} & 0 & \ldots & 0 & 0 \\ 1 & 1 & q_{3n+1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & q_{nn+1} & 0 \\ 1 & 1 & 1 & \ldots & 1 & -n \end{pmatrix} \tag{5.21}$$

and we see that this matrix can be conveniently decomposed as a sum of two matrices in the following way

$$a^{(n+1)} = b + c \tag{5.22}$$

where b is the direct sum of $a^{(n)}$ with the nullmatrix 1×1 ,i.e.,

$$b = a^{(n)}(h_{1n+1},\ldots,h_{nn+1}) \oplus 0\,, \tag{5.23}$$

and c is also an $(n+1)\times(n+1)$ matrix whose only non-vanishing row is the last one. Explicitly, we have then that

$$a^{(n+1)}(h_{1n+1},\dots,h_{nn+1},0)=\left(\begin{array}{c:c} a^{(n)}(h_{m+1},\dots,h_{nn+1}) & 0 \\ \hdashline 0 & 0 \end{array}\right)+\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 1 & -n \end{pmatrix}. \tag{5.24}$$

Now, we notice that the matrices b and c are such that

$$bc=0 \tag{5.25}$$

$$c^k=(-n)^{k-1}c \quad , \quad k=1,2,\dots, \tag{5.26}$$

$$b^k=\left[a^{(n)}\right]^k \oplus 0\,, \quad k=1,2,\dots, \tag{5.27}$$

$$(c\,b^\ell)_{ij}=\begin{cases} \sum\limits_{m=1}^{n}\left[(a^{(n)})^\ell\right]_{mj}\delta_{in} \;, \quad j\leq n\,, \\ 0\,, \quad j=n+1\,. \end{cases} \tag{5.28}$$

The property (5.25) allows us to write

$$\left[a^{(n+1)}(h_{1n+1},\dots,h_{nn+1},0)\right]^k=\sum_{l=0}^{k}c^{k-\ell}b^\ell=c^k+b^k+\sum_{\ell=1}^{k-1}c^{k-\ell}b^\ell$$

$$=(-n)^{k-1}c+b^k+\sum_{\ell=1}^{k-1}(-n)^{k-\ell-1}c\,b^\ell\,, \tag{5.29}$$

where (5.26) was used in the last step.

Summing up all matrix elements of both sides of (5.29) it results that

$$\sum_{ij=1}^{n+1}\left[\left\{a^{(n+1)}(h_{1n+1},\dots,h_{nn+1},0)\right\}^{k}\right]_{ij} =$$

$$\sum_{ij=1}^{n}\left[\left\{a^{(n)}(h_{1n+1},\dots,h_{nn+1})\right\}^{k}\right]_{ij} + \sum_{l=1}^{k-1}(-n)^{k-l-1}\sum_{ljm=1}^{n}\left[\left\{a^{(n)}(h_{1n+1},\dots,h_{nn+1})\right\}^{l}\right]_{mj}\delta_{in}, \tag{5.30}$$

where we have isolated the last terms with $i=j=n+1$, and used (5.27) and (5.28). The delta symbol allows an easy sum over leading us, finally, to the relation (3.4), completing the proof of the validity of (5.14).

We have thus obtained three closed formulas for getting the eigenvalues of the DU-invariants in any irrep of U(n) in terms of the partial hooks (4.1). In the case of UD-invariants, similar formulas were derived by Louck and Biedenharn [5] , and Perelomov and Popov [8] . We give in the following their results.

For the UD-invariants we must consider the partial hooks p_{in} , i.e.,

$$p_{in} = h_{in} + n - i \tag{5.31}$$

and the formula corresponding to (5.1) reads [5]

$$\tilde{C}_k^{(n)}(h) = \sum_{i=1}^{n} p_{in}^{k}\, D^{(n)}(h_{1n},\dots,h_{in}-1,\dots,h_{nn})/D^{(n)}(h), \tag{5.32}$$

where $D^{(n)}$ was defined in (4.2) [note that $p_{in}-p_{jn}=q_{in}-q_{jn}$].

The UD-eigenvalues can also be obtained in terms of the elementary symmetric functions (2.12), and (2.13) [5] , i.e.,

$$\tilde{C}_k^{(n)}(h) = \sum_{l=0}^{k} \sum_{m=0}^{l} (-)^{k-l+m} \binom{n-m}{k-l+1} \varphi_m(x) \beta_{l-m}(x) \ , \quad k = 1,2,\dots, \tag{5.33}$$

where now $x = (p_{1n}, p_{2n}, \dots, p_{nn})$.

Finally, the matricial procedure to obtain the UD-eigenvalues is the same as before, i.e., [8]

$$\tilde{C}_k^{(n)}(h) = \sum_{ij=1}^{n} (\tilde{a}^k)_{ij} \quad , \ k = 1,2,\dots, \tag{5.34}$$

where

$$\tilde{a}_{ij} = p_{in} \delta_{ij} - \tilde{t}_{ij} \quad , \tag{5.35}$$

and

$$\tilde{t}_{ij} = \begin{cases} 1 & \text{if } i<j \ , \\ 0 & \text{otherwise}. \end{cases} \tag{5.36}$$

VI - INVARIANTS OF UNIMODULAR UNITARY GROUPS

The unimodular unitary group SU(n) plays an important role in modern physics. The eigenvalues of the invariants $\mathcal{C}_k^{(n)}$ of that group can be obtained from those of U(n) in the irrep characterized by the same Young diagram of SU(n) through the known relation

$$\mathcal{C}_k^{(n)} = \sum_{l=0}^{k} \binom{k}{l} \left(-\frac{h}{n}\right)^{k-l} C_l^{(n)} \ , \quad k \geq 0 . \tag{6.1}$$

This relation holds irrespectively of the DU or UD criterion considered. It however preserves the character of the invariants. So, for instance, DU invariants of U(n) provide DU invariants of SU(n). It also gives $\mathcal{C}_1^{(n)} \equiv 0$, i.e., the first order invariant of SU(n) —— DU or UD —— is identically zero[its eigenvalue vanishes in any irrep of SU(n).]

As an example, we give explicitly the eigenvalues of the first-, second- and third order invariants of U(3) and SU(3) in the irrep $[h_1 h_2 0]$.

DU criterion

$$C_1^{(3)}(h_1,h_2,0) = h_1+h_2 ,$$
$$C_2^{(3)}(h_1,h_2,0) = h_1^2+h_2^2+2h_1 ,$$
$$C_3^{(3)}(h_1,h_2,0) = h_1^3+h_2^3+2h_1^2-h_2^2+h_1h_2-2h_1-2h_2 ,$$
$$\mathcal{C}_2^{(3)}(h_1,h_2) = \tfrac{2}{3}\left[h_1^2+h_2^2-h_1h_2+3h_1\right] ,$$
$$\mathcal{C}_3^{(3)}(h_1,h_2) = \tfrac{1}{9}\left[2h_1^3+2h_2^3-3h_1^2h_2-3h_1h_2^2\right]-h_2^2-h_1h_2-2h_1-2h_2 .$$

UD criterion

$$\tilde{C}_1^{(3)}(h_1,h_2,0) = C_1^{(3)}(h_1,h_2,0),$$
$$\tilde{C}_2^{(3)}(h_1,h_2,0) = C_2^{(3)}(h_1,h_2,0),$$
$$\tilde{C}_3^{(3)}(h_1,h_2,0) = h_1^3+h_2^3+4h_1^2+h_2^2-h_1h_2+4h_1-2h_2 ,$$
$$\tilde{\mathcal{C}}_2^{(3)}(h_1,h_2) = \mathcal{C}_2^{(3)}(h_1,h_2),$$
$$\tilde{\mathcal{C}}_3^{(3)}(h_1,h_2) = \tfrac{1}{9}\left[2h_1^3+2h_2^3-3h_1^2h_2-3h_1h_2^2\right]+2h_1^2+h_2^2-3h_1h_2+4h_1-2h_2 .$$

Notice the more symmetrical expressions for the DU eigenvalues.

Appendix

Let us prove that the square brackets in (5.6) vanishes.

Let $x_i = q_{i,n+1} + n$. From the definition (4.2) and the usual convention that $C_0^{(n)} = n$, in any irrep, we see that we have to prove that

$$\prod_{l=1}^{n} \frac{x_l - 1}{x_l} + \sum_{i=1}^{n} \frac{1}{x_i} \prod_{l \neq i}^{n} \frac{x_l - x_i - 1}{x_l - x_i} = 1 . \tag{A.1}$$

where use was made of (4.7). Multiplying both sides of (A.1) by $\varphi_n(x) = \prod x_l$, we are left with

$$\prod_{l=1}^{n} (x_l - 1) + \sum_{i=1}^{n} \prod_{l \neq i}^{n} x_l \left[1 - \frac{1}{x_l - x_i}\right] = \varphi_n(x) . \tag{A.2}$$

The first term in the LHS of (A.2) is a Stevin product (2.14) with $a = -1$ so that it reads

$$\prod_{l=1}^{n} (x_l - 1) = \sum_{l=0}^{n} (-)^{n-l} \varphi_l(x). \tag{A.3}$$

The second term can be treated in the following way

$$P_i(x) \equiv \prod_{l \neq i}^{n} x_l \left[1 - \frac{1}{x_l - x_i}\right] = \frac{\varphi_n(x)}{x_i} \left[-\prod_{j=1}^{n} (x_j - x_i - 1)\right] \Big/ \prod_{k \neq i}^{n} (x_k - x_i) . \tag{A.4}$$

The square brackets contain a Stevin product (2.14) with $a = -1 - x_i$, so that

$$P_i(x) = \varphi_n(x) \sum_{l=0}^{n} (-)^{n-l+1} \varphi_l(x) \sum_{j=0}^{n-l} \binom{n-l}{j} x_i^{j-1} \Big/ \prod_{k \neq i}^{n} (x_k - x_i) . \tag{A.5}$$

Separating the term $l = n$ and taking into account that

$$\sum_{i=1}^{n} x_i^{j-1} \Big/ \prod_{k\neq i}^{n} (x_k - x_i) = \begin{cases} 1/\varphi_n(x) & \text{if } j=0\,, \\ 0 & \text{if } j=1,2,\dots,n-1\,, \\ (-1)^{n-1} & \text{if } j=n\,, \end{cases} \qquad \text{(A.6)}$$

it is easy to see that

$$\sum_{i=1}^{n} P_i(x) = \sum_{l=0}^{n-1} (-)^{n-l+1} \varphi_l(x)\,, \qquad \text{(A.7)}$$

which summed to (A.3) gives exactly (A.2), and the proof that (A.1) is true is completed.

The same technique can be used, with $x_i = q_{in}$, to help in the derivation of (5.2) and (5.10).

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