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Contribution of Higher Order Terms in
the Reductive Perturbation Theory, II
- A Case of Strongly Dispersive Wave -

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Abstract

Contribution of hihger order terms in the reductive perturbation theory has been investigated for nonlinear propagation of strongly dispersive ion plasma wave. The basic set of fluid equation is reduced to a coupled set of the nonlinear Schrödinger equation for the first order perturbed potential and a linear inhomogeneous equation for the second order perturbed potential. A steady state solution of the coupled set of equations has been solved analytically in the asymptotic limit of small wave number.

§1. Introduction

In the previous paper[1], we have examined the higher order contributions in the reductive perturbation expansion applied for the case of weakly dispersive ion acoustic wave. The similar analysis has also been applied for the nonlinear shallow water wave in the long wave length limit[2]. We have shown that, when the analysis is extended to the second order terms, the reductive perturbation theroy reduces the basic set of nonlinear equations to a coupled set of the Korteweg-de Vries equation for the first order perturbed quantity and a linear inhomogeneous equation for the second order quantity. Dynamical effects of the second order contribution in the process of collisions of two solitons have been examined numerically by solving the linear inhomogeneous equation[3].

Here, turning our interest to a case of strongly dispersive wave, we develop similar analysis of the nonlinear wave modulation of strongly dispersive ion plasma wave in the collisionless plasma composed of cold ions and warm electrons. The main purpose of present analysis is to illustrate the higher order contributions of the reductive perturbation expansion in the strongly dispersive system, but not to provide theoretical interpretation on the nonlinear ion wave propagation taking place in the real world.

In the next section, we derive an equation for the second order perturbed potential, while the modulation of the first order potential is described by the nonlinear Schrödinger equation. In the third section, we examine the second order corrections by obtaining a steady state solution in the asymptotic limit of small wave number. We present concluding remarks in the last section.

§2. Contribution of the Second Order Terms to the Nonlinear Ion Plasma Wave

In order to examine the higher order contribution of the reductive perturbation theory for a case of the strongly dispersive wave, we examine a simple case of the ion plasma wave examined by Shimizu and Ichikawa[4]. Restricting our interest to the one dimensional motion of plasma composed of cold ions and isothermal electrons, we take the following set of equations

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial x} (nu) = 0 \ , \tag{1}$$

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = E \ , \tag{2}$$

$$\frac{\partial}{\partial x} n_e = - n_e E \ , \tag{3}$$

$$\frac{\partial}{\partial x} E = n - n_e \ , \tag{4}$$

where n and n_e are the densities of ions and electrons, u is the flow velocity of ions, and E the electric field. All these quantities, including the space-time variables (x,t), are normalized to be dimensionless as described in the ref.4. Physical quantities appeared in (1)-(4) are expanded into power series

$$U = U^{(0)} + \sum_{\alpha=1} \varepsilon^{\alpha} \sum_{\ell=-\infty}^{+\infty} U_{\ell}^{(\alpha)}(\xi,\tau)\exp\{i\ell(kx - \omega t)\} \quad , \tag{5}$$

where U represents a column vector,

$$U \equiv \begin{pmatrix} n \\ u \\ E \\ n_e \end{pmatrix} . \tag{6}$$

The space-time slow modulation of wave is described in terms of the stretched variables ξ and τ defined by

$$\xi = \varepsilon(x - \lambda t) \quad , \tag{7}$$

$$\tau = \varepsilon^2 t \quad , \tag{8}$$

where the parameter λ is determined through a compatibility condition for the components of $U_1^{(2)}$ as the group velocity

$$\lambda = \frac{\partial \omega}{\partial k} . \tag{9}$$

Substitution of the expansion (5) into the basic set of equations (1) - (4) yields the following equation for the α-th order terms,

$$W_{\ell} U_{\ell}^{(\alpha)} + M\frac{\partial}{\partial \xi} U_{\ell}^{(\alpha-1)} + A\frac{\partial}{\partial \tau} U_{\ell}^{(\alpha-2)} + N_{\ell}^{(\alpha)} = 0 \quad , \tag{10}$$

where the coefficients W_{ℓ}, M and A are given as

$$W_{\ell} = \begin{pmatrix} i\ell\omega, & i\ell k, & 0, & 0 \\ 0, & -i\ell\omega, & -1 , & 0 \\ 0, & 0 , & 1 , & i\ell k \\ -1, & 0 , & i\ell k, & 1 \end{pmatrix} \quad , \tag{11}$$

$$M = \begin{pmatrix} -\lambda, & 1, & 0, & 0 \\ 0, & -\lambda, & 0, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0 \end{pmatrix} , \tag{12}$$

$$A = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix} . \tag{13}$$

The vector $N_\ell^{(\alpha)}$ is defined as

$$N_\ell^{(\alpha)} = \sum_{\alpha'} \sum_{\ell'} \begin{pmatrix} \Big[i\ell' k \, n_{\ell-\ell'}^{(\alpha-\alpha')} \, u_{\ell'}^{(\alpha')} + i\ell' k \, u_{\ell-\ell'}^{(\alpha-\alpha')} \, n_{\ell'}^{(\alpha')} \\ + n_{\ell-\ell'}^{(\alpha-\alpha'-1)} \frac{\partial}{\partial\xi} U_{\ell'}^{(\alpha')} + U_{\ell-\ell'}^{(\alpha-\alpha'-1)} \frac{\partial}{\partial\xi} n_{\ell'}^{(\alpha')} \Big] \\ i\ell' k \, U_{\ell-\ell'}^{(\alpha-\alpha')} \, U_{\ell'}^{(\alpha')} + U_{\ell-\ell'}^{(\alpha-\alpha'-1)} \frac{\partial}{\partial\xi} u_{\ell'}^{(\alpha')} \\ n_{e,\ell-\ell'}^{(\alpha-\alpha')} \, E_{\ell'}^{(\alpha')} \\ 0 \end{pmatrix} . \tag{14}$$

In the following, we summarize the previous result obtained in the ref.4. The ion wave dispersion relation is reduced from the compatibility condition for the components of $U_1^{(1)}$.

$$\det|W_1| = 0 , \tag{15}$$

as to given by

$$\omega^2 = \frac{k^2}{1 + k^2} \tag{16}$$

Assuming the $\ell = 0$ component of the first order density perturbation $n_0^{(1)}$ is equal to zero, we obtain

$$U_\ell^{(1)} = 0 \qquad \text{for } \ell \neq 1 \;, \tag{17-a}$$

$$U_{\pm1}^{(1)} = R_{\pm1}^{(1)}\; n_{\pm1}^{(1)} \;, \tag{17-b}$$

$$U_2^{(2)} = R_2^{(2)}\; n_1^{(1)}\; n_1^{(1)} \;, \tag{17-c}$$

$$U_1^{(2)} = R_1^{(1)}\; n_1^{(2)} + S_1^{(1)} \frac{\partial}{\partial \xi}\, n_1^{(1)} \;, \tag{17-d}$$

$$U_0^{(2)} = R_0^{(2)} (|n_1^{(1)}|^2 - C) \;, \tag{17-c}$$

$$U_1^{(3)} = R_1^{(1)} n_1^{(3)} + S_1^{(1)} \frac{\partial}{\partial \xi}\, n_1^{(2)} + T\, \frac{\partial}{\partial \tau}\, n_1^{(1)} + S_1^{(2)} \frac{\partial}{\partial \xi^2}\, n_1^{(1)} + (Q_0 + Q_2)\, |n_1^{(1)}|^2\, n_1^{(1)} - Q_0 C n_1^{(1)} \;, \tag{17-f}$$

with the definitions of various column vectors

$$R_{\pm1}^{(1)} = \begin{pmatrix} 1 \\ \omega/k \\ \mp i\omega^2/k \\ \omega/k^2 \end{pmatrix}, \qquad R_2^{(2)} = \begin{pmatrix} 2 + \frac{1}{3k^2} + \frac{\omega^2}{6k^2} \\ \frac{\omega}{k}\left(1 + \frac{1}{3k^2} + \frac{\omega^2}{6k^2}\right) \\ -i\frac{\omega^2}{k}\left(1 + \frac{2}{3k^2} + \frac{\omega^2}{3k^2}\right) \\ \frac{\omega^2}{k^2}\left(\frac{1}{2} + \frac{1}{3k^2} + \frac{2\omega^2}{3k^2}\right) \end{pmatrix}$$

$$S_1^{(1)} = \begin{pmatrix} 0 \\ i\,\frac{\omega^3}{k^2} \\ \frac{\omega^2}{k^2} - 2\,\frac{\omega}{k}\lambda \\ i2\,\frac{\omega^4}{k^3} \end{pmatrix}, \qquad R_0^{(2)} = -\frac{1}{1-\lambda^2} \begin{pmatrix} \frac{\omega^2}{k^2}(1 + \frac{\omega^2}{k^2}) \\ \frac{\omega}{k}(2 + \frac{\omega^4}{k^4} - \frac{\omega^6}{k^6}) \\ 0 \\ \frac{\omega^2}{k^2}(1 + \frac{\omega^2}{k^2}) \end{pmatrix},$$

$$T = \begin{pmatrix} 0 \\ i/k \\ 2\omega/k \\ i2\omega/k^2 \end{pmatrix}, \qquad S_1^{(2)} = \begin{pmatrix} 0 \\ -\omega^3/k^3 \\ i\lambda\omega^3 \\ (\omega^6/k^6) - 3\omega^4/k^4 \end{pmatrix},$$

$$Q_0 = \frac{\omega}{k}\,\frac{1}{1-\lambda^2} \begin{pmatrix} 0 \\ 2 + \frac{\omega^2}{k^2} + 2\,\frac{\omega^4}{k^4} - \frac{\omega^6}{k^6} \\ -i\omega(4 + \frac{\omega^2}{k^2} + 3\,\frac{\omega^4}{k^4} - 2\,\frac{\omega^6}{k^6}) \\ \frac{\omega}{k}\,(4 + 2\,\frac{\omega^4}{k^4} - 2\,\frac{\omega^6}{k^6}) \end{pmatrix},$$

$$Q_2 = \frac{\omega}{k} \begin{pmatrix} 0 \\ -(3 + \frac{\omega^2}{3k^2} + \frac{2}{3k^2}) \\ i\omega(4 + \frac{\omega^2}{2k^2} + \frac{1}{k^2}) \\ -\frac{\omega}{k}(4 + \frac{1}{k^2} + \frac{\omega^4}{3k^4} - \frac{\omega^2}{3k^4}) \end{pmatrix}. \tag{18}$$

The constant C in (17-e) and (17-f) is the boundary value of $|n_1^{(1)}|^2$ at $\xi = -\infty$. A compatibility condition for the components of $U_1^{(3)}$ is reduced to the nonlinear Schrödinger equation for the first order density perturbation $n_1^{(1)}(\xi,\tau)$ as following,

$$i\frac{\partial}{\partial\tau}n_1^{(1)} + p\frac{\partial^2}{\partial\xi^2}n_1^{(1)} + q|n_1^{(1)}|^2 n_1^{(1)} + rCn_1^{(1)} = 0 , \qquad (19)$$

with

$$p = -\frac{3}{2}\frac{\omega^5}{k^4} , \qquad (20\text{-a})$$

$$q = \frac{\omega}{2}\Big[-\frac{1}{k^2} - \frac{k^2}{2\omega^2} + \frac{\omega^4}{2k^4} + \frac{\omega^4}{3k^6} - \frac{\omega^6}{3k^6} + \frac{1}{1-\lambda^2}\frac{\omega^2}{k^2}\Big(1 + \frac{\omega^2}{k^2}\Big)^2\Big] , \qquad (20\text{-b})$$

$$r = -\frac{\omega}{2}\Big[4 + \frac{1}{1-\lambda^2}\frac{\omega^2}{k^2}\Big(1 + \frac{\omega^2}{k^2}\Big)^2\Big] , \qquad (20\text{-c})$$

in which some printing errors of the ref.3 have been corrected.

Now, turning our interest to determine explicitly effects of the second order density perturbation $n_1^{(2)}(\xi,\tau)$, we have to obtain the components $U_2^{(3)}$ and $U_0^{(3)}$. It is straightforward to get $U_2^{(3)}$ as

$$U_2^{(3)} = R_2^{(3)}\, n_1^{(1)}\, n_1^{(2)} + S_2 \frac{\partial}{\partial\xi}\big(n_1^{(1)}\, n_1^{(1)}\big) , \qquad (22)$$

with the definitions of

$$R_2^{(3)} = \begin{pmatrix} 4 + \frac{\omega^2}{k^2} + \frac{2}{3}\frac{\omega^2}{k^4} \\ \frac{\omega}{k}\big(2 + \frac{\omega^2}{k^2} + \frac{2}{3}\frac{\omega^2}{k^4}\big) \\ -i2\frac{\omega^2}{k}\big(1 + \frac{\omega^2}{k^2} + \frac{2}{3}\frac{\omega^2}{k^4}\big) \\ \frac{\omega^2}{k^2}\big(1 + 2\frac{\omega^2}{k^2} + \frac{2}{3}\frac{\omega^2}{k^4}\big) \end{pmatrix} \qquad (23)$$

$$S_2 = \begin{pmatrix} -\frac{i}{6k^3}(1 - 4\omega^2 - 2\frac{\omega^2}{k^2} - \frac{\omega^4}{k^4}) \\ i(\frac{\omega}{2k^2} + \frac{\omega^3}{4k^4} + \frac{\omega^3}{3k^6} - \frac{\omega^5}{4k^6}) \\ \frac{\omega^2}{k^2}(\frac{1}{2} + \frac{\omega^2}{2k^2} - \frac{\omega^4}{3k^4} + \frac{\omega^4}{3k^6}) \\ i\,\frac{\omega^2}{k^3}(\frac{1}{2} + \frac{3\omega^2}{2k^2} + \frac{\omega^2}{3k^4} - \frac{4\omega^4}{3k^4}) \end{pmatrix} . \qquad (24)$$

$E_0^{(3)}$ is derived from the $\ell = 0$ components of the third order terms of (10) as

$$E_0^{(3)} = \{ \frac{\omega^4}{k^4} + \frac{1}{1-\lambda^2}\frac{\omega^2}{k^2}(1+\frac{\omega^2}{k^2}) \}\frac{\partial}{\partial\xi}|n_1^{(1)}|^2 . \qquad (25)$$

Components of $U_0^{(3)}$ except $E_0^{(3)}$ are determined from the fourth order $\ell = 0$ components of (10) as

$$U_0^{(3)} = R_0^{(2)}(n_1^{(1)}n_{-1}^{(2)} + n_{-1}^{(1)}\ n_1^{(2)})$$
$$+ S_0\ \{(n_1^{(1)}\frac{\partial}{\partial\xi}n_{-1}^{(1)} - n_{-1}^{(1)}\frac{\partial}{\partial\xi}n_1^{(1)}) - \Gamma\} , \qquad (26)$$

in which $R_0^{(2)}$ is given in (18), and S_0 is defined as

$$S_0^{(2)} = \frac{i}{1-\lambda^2}\begin{pmatrix} \frac{\omega^6}{k^3} + \frac{3}{1-\lambda^2}\frac{\omega^6}{k^5}(1 + \frac{\omega^4}{k^4}) \\ \frac{\omega^3}{k^2}(1 + \frac{\omega^4}{k^4} - 2\frac{\omega^6}{k^6}) + \frac{3}{2}\frac{1}{1-\lambda^2}\frac{\omega^5}{k^4}(\frac{\omega^2}{k^2} + \frac{3\omega^4}{k^4} + \frac{\omega^8}{k^8} - \frac{\omega^{10}}{k^{10}}) \\ 0 \\ \frac{\omega^6}{k^3} + \frac{3}{1-\lambda^2}\frac{\omega^6}{k^5}(1 + \frac{\omega^4}{k^4}) \end{pmatrix}$$

(27)

The constant Γ of (26) is the boundary value of the flux $n_1^{(1)}(\partial n_{-1}^{(1)}/\partial\xi) - n_{-1}^{(1)}(\partial n_1^{(1)}/\partial\xi)$ at $\xi = -\infty$. We also take the boundary conditions,

$$n_{\pm 1}^{(2)} = 0 \qquad \text{at} \quad \xi = -\infty \, . \tag{28}$$

As the final step, we come to deal with the $\ell = 1$ components of the fourth order terms of (10)

$$W_1 U_1^{(4)} + M\frac{\partial}{\partial\xi}U_1^{(3)} + A\frac{\partial}{\partial\tau}U_1^{(2)} + N_1^{(4)} = 0 \, . \tag{29}$$

As the compatibility condition for the components of $U_1^{(3)}$ has been reduced to the nonlinear Schrödinger equation for the first order density perturbation $n_1^{(1)}(\xi,\tau)$, the compatibility condition for the components of $U_1^{(4)}$ given by (29) gives rise to the following equation for the second order density perturbation $n_1^{(2)}(\xi,\tau)$,

$$\begin{aligned} & i\frac{\partial}{\partial\tau}n_1^{(2)} + p\frac{\partial^2}{\partial\xi^2}n_1^{(2)} + rC\, n_1^{(2)} + a|n_1^{(1)}|^2\, n_1^{(2)} \\ & \quad + b\, n_1^{(1)}\, n_1^{(1)}\, n_{-1}^{(2)} = \frac{i}{3}\frac{dp}{dk}\frac{\partial^3}{\partial\xi^3}n_1^{(1)} + \\ & \quad ic\frac{\partial}{\partial\xi}\{|n_1^{(1)}|^2\, n_1^{(1)}\} + i\, d\, |n_1^{(1)}|^2 \frac{\partial}{\partial\xi}n_1^{(1)} \\ & \quad + ie\, n_1^{(1)} n_1^{(1)}\frac{\partial}{\partial\xi}n_{-1}^{(1)} + i\mu\, C\frac{\partial}{\partial\xi}n_1^{(1)} + i\nu\Gamma\, n_1^{(1)} \, , \end{aligned} \tag{30}$$

with the abbreviations of

$$a(k,\omega) = \frac{\omega}{1-\lambda^2}(1+\frac{\omega^2}{k^2} + 2\frac{\omega^4}{k^6}) - \omega(1+\frac{3}{2}\frac{\omega^2}{k^2} + \frac{\omega^2}{k^4} - \frac{1}{2}\frac{\omega^4}{k^4} - \frac{1}{3}\frac{\omega^6}{k^8}) \, , \tag{31-a}$$

$$b(k,\omega) = \frac{\omega}{1-\lambda^2}(\frac{1}{2} + \frac{1}{2}\frac{\omega^2}{k^2} + \frac{\omega^4}{k^4}) - \frac{\omega}{2}(1+\frac{1}{2}\frac{\omega^2}{k^2} - \frac{1}{2}\frac{\omega^4}{k^4} - \frac{1}{3}\frac{\omega^4}{k^6} + \frac{1}{3}\frac{\omega^6}{k^6}) \, , \tag{31-b}$$

$$c(k,\omega) = -\frac{1}{1-\lambda^2}\frac{\omega^5}{k^3} + \frac{1}{2}\frac{\omega}{k}\left(1 + \frac{1}{2}\frac{\omega^2}{k^2} - \frac{1}{3}\frac{\omega^4}{k^4} + \frac{1}{2}\cdot\frac{\omega^6}{k^6} - \frac{1}{3}\cdot\frac{\omega^8}{k^6}\right), \tag{31-c}$$

$$d(k,\omega) = -\frac{3}{2}\frac{1}{(1-\lambda^2)^2}\frac{\omega^7}{k^5} + \frac{1}{2}\frac{1}{1-\lambda^2}\frac{\omega}{k}\left(1 + \frac{\omega^2}{k^2} + 3\frac{\omega^4}{k^4}\right) - $$
$$-\frac{1}{2}\frac{\omega}{k}\left(3 - \frac{13}{3}\frac{\omega^2}{k^2} + \frac{\omega^4}{k^4} - \frac{\omega^6}{k^6} + \frac{2}{3}\frac{\omega^8}{k^8}\right) - \frac{1}{6}\frac{\omega}{k^3}\left(7 - \frac{\omega^2}{k^2} - \frac{\omega^4}{k^4}\right), \tag{31-d}$$

$$e(k,\omega) = \frac{3}{2}\frac{1}{(1-\lambda^2)^2}\frac{\omega^7}{k^5} + \frac{3}{2}\frac{1}{1-\lambda^2}\frac{\omega^3}{k^3}\left(1 + \frac{2}{3}\frac{\omega^2}{k^2} + \frac{\omega^6}{k^4}\right)$$
$$-\frac{1}{4}\frac{\omega^3}{k^3}\left(3 - \frac{11}{3}\frac{\omega^2}{k^2} + \frac{\omega^4}{k^4} - \frac{4}{3}\frac{\omega^6}{k^6}\right) - \frac{1}{2}\frac{\omega}{k}\left(1 + \frac{1}{k^2} - \frac{\omega^4}{k^6} + \frac{1}{3}\frac{\omega^6}{k^8}\right), \tag{31-e}$$

$$\mu(k,\omega) = -\frac{1}{1-\lambda^2}\frac{\omega}{k}\left[2 + \frac{5}{2}\frac{\omega^4}{k^4} - \frac{\omega^6}{k^6} - \frac{3}{2}\frac{\omega^8}{k^8}\right] \tag{31-f}$$

$$\nu(k,\omega) = -\frac{3}{2}\frac{1}{(1-\lambda^2)^2}\frac{\omega^7}{k^5}\left[1 + 3\frac{\omega^2}{k^2} - \omega^2 - \frac{\omega^8}{k^4}\right]$$
$$-\frac{1}{1-\lambda^2}\frac{\omega^2}{k}\left(1 + \frac{\omega^4}{k^4} - 2\frac{\omega^6}{k^6} - \frac{1}{2}\frac{\omega^5}{k^2}\right] \tag{31-g}$$

As it was in the case of weakly dispersive wave, the second order equation (30) is now linear for the second order density perturbation $n_{\pm 1}^{(2)}(\xi,\tau)$ with the inhomogeneous source terms, of which the first term represents the higher order dispersion effect and the second, third and fourth terms describe the mode-mode interaction effects between the nonlinear first order waves. Concerning with the terms containing the constant C in (30), we notice that the term r C $n_1^{(2)}$ is elimitated together with the first order term rC $n_1^{(1)}$ of (19) in terms of the transformation of

$$n_1^{(1)}(\xi,\tau) = \tilde{n}^{(1)}(\xi,\tau)\ \exp(ir\,C\tau), \tag{32-a}$$

$$n_1^{(2)}(\xi,\tau) = \tilde{n}^{(2)}(\xi,\tau)\exp\ (ir\,C\tau). \tag{32-b}$$

The present analysis illustrates explicitly the characteristic structure of the reductive perturbation theory, which provides us a systematic approach to examine competitive process between the dispersion effect and the nonlinear effect to the higher order terms. We emphasize again the essential aspects of the nonlinear self-interaction of finite amplitude waves are taken into account fully in the nonlinear Schrödinger equation as in the Korteweg-de Vries equation for the first order perturbed quantities.

§3. A Steady State Solution in the Small Wave Number Region

Since the coefficients of the coupled set of equations (19) and (30) are too complicated, we examine a steady state solution of the asymptotic limit of small wave number $k^2 << 1$ of these equations. As has been discussed in the ref. 3, the asymptotic form of (19) is

$$i \frac{\partial}{\partial \tau} n_1^{(1)} - \frac{3}{2} k \frac{\partial^2}{\partial \xi^2} n_1^{(1)} - \frac{2}{3k} C n_1^{(1)} + \frac{1}{3k} |n_1^{(1)}|^2 n_1^{(1)} = 0 , \tag{33-a}$$

while the second order equation (30) is reduced to

$$i\frac{\partial}{\partial \tau} n_1^{(2)} - \frac{3}{2}k \frac{\partial^2}{\partial \xi^2} n_1^{(2)} - \frac{2}{3k} C n_1^{(2)} + \frac{2}{3k} |n_1^{(1)}|^2 n_1^{(2)}$$
$$+ \frac{1}{3k} n_1^{(1)} n_1^{(1)} n_{-1}^{(2)} = - \frac{i}{2} \frac{\partial^3}{\partial \xi^3} n_1^{(1)} + i \frac{2}{3k^2} |n_1^{(1)}|^2 \frac{\partial}{\partial \xi} n_1^{(1)}$$
$$+ i \frac{1}{k^2} n_1^{(1)} n_1^{(1)} \frac{\partial}{\partial \xi} n_{-1}^{(1)} - i \frac{2}{3k^2} C \frac{\partial}{\partial \xi} n_1^{(1)} - i\frac{2}{3k^2} \Gamma n_1^{(1)} . \tag{33-b}$$

Although (33-a) admits a solution moving in the frame of ξ-coordinate with a velocity proportional to its amplitude, we seek the following type of solutions,

$$n_1^{(1)}(\xi,\tau) = f(\xi)\ \exp(\ -i(\frac{2}{3k}C - \alpha)\tau)\quad , \tag{34-a}$$

$$n_1^{(2)}(\xi,\tau) = i\ g(\xi)\exp(-i(\frac{2}{3k}C - \alpha)\tau)\quad . \tag{34-b}$$

Hence for the type of solution (34-a), we have $\Gamma=0$ in the second order equation (33-b).

Substituting (34-a) into (33-a), we obtain a steady solution

$$n_1^{(1)}(\xi,\tau) = A\ \tanh(\frac{A}{3k}\xi)\ \exp\ (-i\ \frac{A^2}{3k}\ \tau)\ , \tag{35}$$

where the amplitude A is defined through a relation

$$A^2 = 3k\ \alpha \tag{36}$$

and $C = A^2$ by the definition. The factor $A^2/3k$ is nothing but the nonlinear frequency shift of the carrier wave.

On the other hand, substitution of (34-b) and (35) with (36) into (33-b) yields a differential equation for $g(\xi)$,

$$\frac{d^2}{d\xi^2}\ g(\xi) + 2(\frac{A}{3k})^2\ \{1 - 3k\ \tanh\ (\frac{A}{3k}\xi)\}\ g(\xi)$$

$$= \frac{2A^2}{3k^2}\ \{-\ \frac{7}{3} + 4\ \mathrm{sech}^2\ (\frac{A}{3k}\xi)\}\ \mathrm{sech}^2(\frac{A}{3k}\xi)\ . \tag{37}$$

Transforming the variable ξ into a new variable

$$\mu = \tanh\ (\frac{A}{3k}\ \xi)\quad , \tag{38}$$

we can reduce (37) to a standard form of the second order differential equation,

$$\frac{d}{d\mu}[(1-\mu^2)\frac{d}{d\mu}g(\mu)] + 2g(\mu)$$

$$= \frac{2}{3}\frac{A^2}{k^2}\{4(1-\mu^2) - \frac{7}{3}\} \quad . \tag{39}$$

Since the two independent solutions of the associated homogeneous equation of (39) are given by the Legendre functions,

$$P_1(\mu) = \mu \ , \tag{40-a}$$

$$Q_1(\mu) = \frac{1}{2}\mu\log\frac{1+\mu}{1-\mu} - 1 \ , \tag{40-b}$$

the solution of (39) can be obtained by the method of variation of undetermined coefficients setting as

$$g(\mu) = u_1(\mu)P_1(\mu) + u_2(\mu)Q_1(\mu) \quad . \tag{41}$$

The coefficients $u_1(\mu)$ and $u_2(\mu)$ are specified by solving

$$\frac{d}{d\mu}u_1 = -\frac{2A^2}{3k^2}\{4(1-\mu^2) - \frac{7}{3}\}Q_1 \ , \tag{42-a}$$

$$\frac{d}{d\mu}u_2 = \frac{2A^2}{3k^2}\{4(1-\mu^2) - \frac{7}{3}\}P_1 \quad . \tag{42-b}$$

Referring to our boundary conditions on $n_1^{(2)}(\xi=-\infty) = 0$, we can determine u_1 and u_2 as

$$u_1 = \frac{2A^2}{3k^2}[-\mu^3 + \frac{11}{6}\mu + \frac{5}{6}$$

$$-\frac{1}{2}(1-\mu^2)\{\frac{7}{6} - (1-\mu^2)\}\log\frac{1+\mu}{1-\mu}] \ , \tag{43-a}$$

$$u_2 = \frac{2A^2}{3k^2}(1-\mu^2)\{\frac{7}{6} - (1-\mu^2)\} \quad . \tag{43-b}$$

Thus, finally we obtain a steady solution of the second order density perturbation as

$$n_1^{(2)}(\xi,\tau) = i(\frac{A}{3k})^2 5 \{1 + \tanh(\frac{A}{3k}\xi) - \frac{6}{5}\operatorname{sech}^2(\frac{A}{3k}\xi)\}$$

$$\exp(-i\frac{A^2}{3k}\tau) . \qquad (44)$$

Therefore, the steady solution of the self-modulation of ion plasma wave is determined up to the second order terms as follows

$$n(\xi,\tau) = W(\xi)\exp\{i(\frac{A^2}{3k}\tau + \Theta(\xi))\} , \qquad (45)$$

with the definitions of

$$W(\xi)^2 = A^2\tanh^2(\frac{A}{3k}\xi) + 25(\frac{A}{3k})^2\{1 + \tanh(\frac{A}{3k}\xi)$$

$$- \frac{6}{5}\operatorname{sech}^2(\frac{A}{3k}\xi)\}^2 , \qquad (46\text{-a})$$

$$\tan\Theta = \frac{5A}{9k^2}\cdot\tanh(\frac{A}{3k}\xi)^{-1}\{1 + \tanh(\frac{A}{3k}\xi) - \frac{6}{5}\operatorname{sech}^2(\frac{A}{3k}\xi)\} . \qquad (46\text{-b})$$

We observe that the second order contributions modify not only the depth of amplitude modulation, but also give rise to the local phase modulation as shown by (46-b). In Fig.1, we illustrate the envelop profile $W(\xi)$ together with the functions $f(\xi)$ and $g(\xi)$ for arbitrary chosen parameters $k = 1/3$ and $A=0.08$. Fig.2 shows the local modulation $\Theta(\xi)$ for the same parameters. The remarkable effect of the second order contribution is to fill the central portion of modulation depth with a finite amount of amplitude.

§4. Concluding Remarks

Applying the reductive perturbation theory to a model system describing the ion plasma wave, we have examined the second order contribution in some details. Since the effects of resonant ions with the group velocity[5] modify the basic structure of the nonlinear Schrödinger equation for the amplitude modulation of the ion plasma wave, we do not expect that the present analysis provides us new information observable in the laboratory experiments, but the effects could be detected in numerical experiments carried out on the system described by the set of equations (1) - (4).

Both of our analysis carried out in the preceeding sections and in the previous paper demonstrate the characteristic structure of the reductive perturbation theory, in which the nonlinear self-interaction effect is fully accounted for the nonlinear fundamental equation derived in the lowest order and the higher order equations describes effects of interaction among the nonlinear waves treating the competing process between the higher order dispersion effect and the nonlinear effect as weak perturbation. Basing on the analysis on steady state solutions of the coupled set of equations, we can conclude that the coupled set of equations such as (19) and (30) describes dressed envelop soliton, of which stable core is given as a solution of the lowest order nonlinear Schrödinger equation (19), and clouds arround the core is determined from the second order equations such as (30). Thus, we can investigate dynamical behaviour of the dressed solitons or of the dressed envelop solitons in systematic ways on the basis of the reductive perturbation theory. It remains for us to investigate theoretical inter-relationship between our approach and the perturbation theory of Oikawa and Yajima[6] presented for the studies of interacting nonlinear waves.

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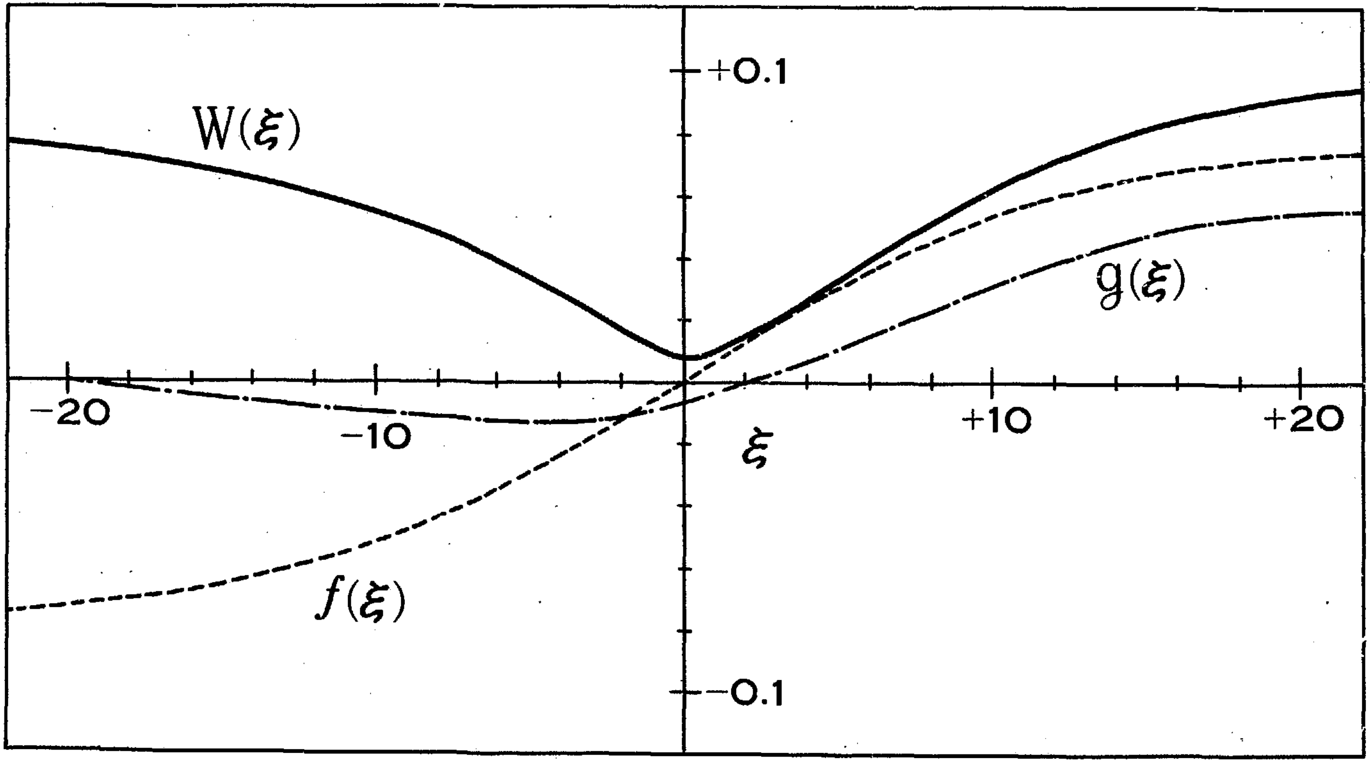
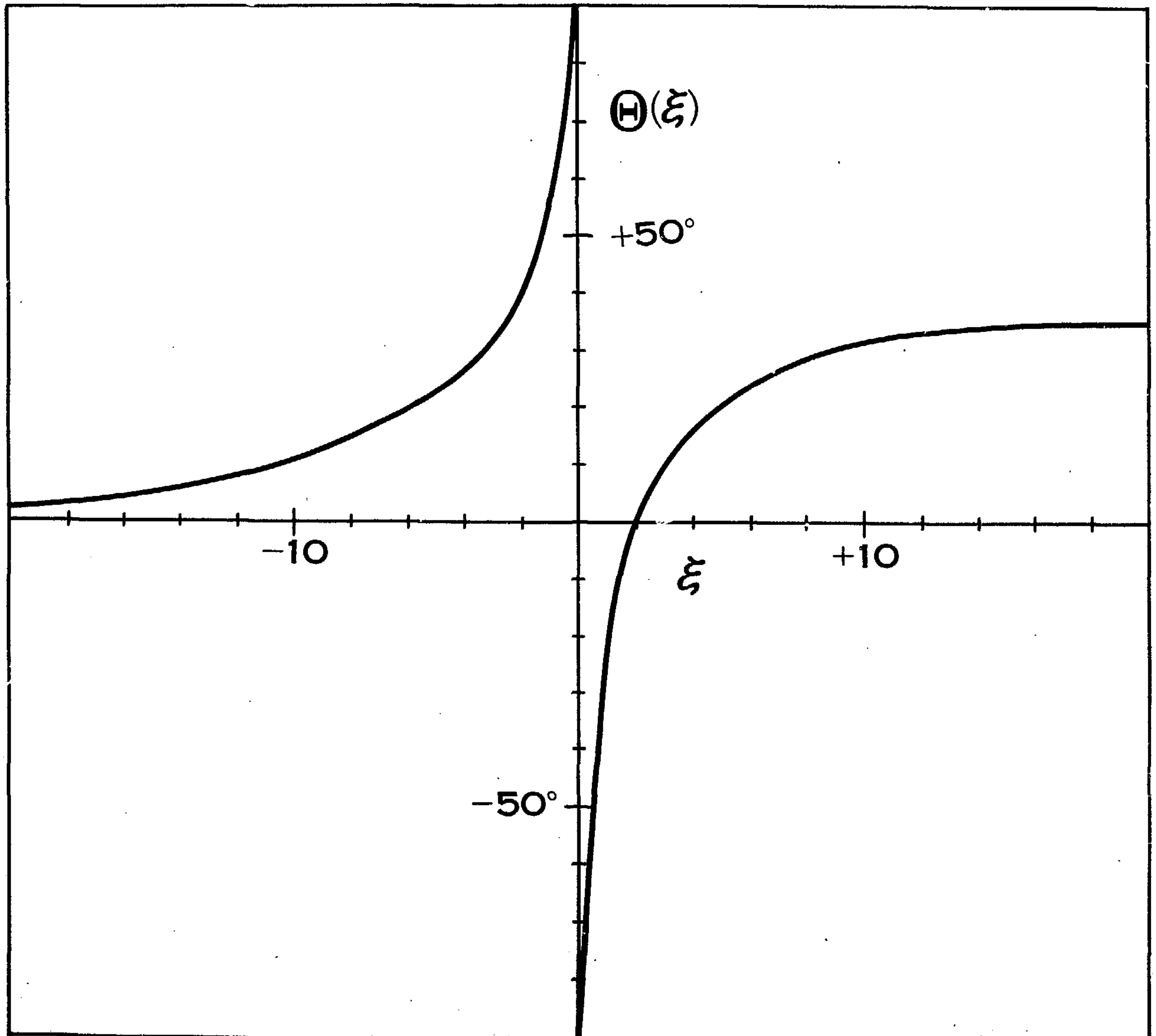
Captions of Figures

Fig. 1 Modulated amplitude $W(\xi)$ with the first order solution $f(\xi)$ and the second order solution $g(\xi)$ for values of $A = 0.08$ and $k = 1/3$.

Fig. 2 Modulated phase angle $\Theta(\xi)$ for values of $A = 0.08$ and $k = 1/3$.



W(ξ)
g(ξ)
f(ξ)
ξ
+0.1
−0.1
−20
−10
+10
+20


Θ(ξ)
+50°
−10
ξ
+10
−50°