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in an RF-Plugged Inhomogeneous Plasma Slab

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Abstract

A theory based on the fluid and perturbation theories is developed to analytically study a nonlinear electrostatic ion cyclotron wave excited in an rf-plugged inhomogeneous plasma slab by applying a pair of external potentials $\phi_{ext}(x,z) = \pm\Phi_o \cos\omega_o t \cdot \exp(-z^2/2h^2)$ at its boundaries $x = \pm L$. Here, $\mathbf{B}_o$ is applied along the z-axis. The potential forms of the fundamental and the nonlinear second harmonic are found as functions of x, z and t provided the field-free densities vary as $\exp(-x^2/2d^2)(d^2/h^2 \ll 1)$. In the core region of the slab, the electric field (perpendicular to $\mathbf{B}_o$) created by the fundamental potential can approximately be regarded as a dipole field, provided that $|1-\mu| \ll 3$ with $\mu=(\omega_o^2-\omega_{ci}^2)m_i d^2/(\gamma_i T_i+Z\gamma_e T_e)$. Under the stricter condition $\mu \approx 1$, a dipole-like electric field can also be excited in the entire region for the case of high density and weak nonlinearity. It is shown that the assumption $\omega_o \ll h^{-1}\sqrt{\gamma_e T_e/m_e}$ can lead to the Boltzmann relation for the electron fluid even in inhomogeneous plasmas. Moreover, the density depletion δN_i obtained here contains a new considerable term proportional to $|\phi|^2$, in addition to the usual term proportional to $-|\partial\phi/\partial x|^2$ which originates from the ponderomotive force.

§1. Introduction

In order to study the rf-plugging of plasma particles which are magnetically confined in open-ended systems, both experimental and theoretical efforts have been made continuously, in particular, intensively in these several years.[1] Among these works dealing with the rf-plugging, there are some theoretical works which consider along which branch of normal modes a pair of externally applied potentials oscillating near the ion cyclotron frequency ω_{ci} should penetrate, producing an rf-field mainly along the x-direction perpendicular to the static magnetic field $\mathbf{B}_0=\mathbf{e}_z B_0(z)$ ($\mathbf{e}_z$ is the unit vector along the z-axis), into the cusp regions of an open-ended system.[2-4] Here it is known that the excited normal mode can be an electrostatic ion cyclotron wave with an electric field nearly parallel to the x-direction.[5-6] And this wave creates the ponderomotive forces to repel plasma particles (especially ions) escaping through the cusp regions, back to the main region of the system,[5-6] simultaneously heating ions through a resonant absorption of the rf-power.[7] These theoretical works, however, seems not to consider what function of x the potential of the externally excited normal mode should take as penetrating and simultaneously rf-plugging an inhomogeneous sheet or slab plasma. Here, for convenience' sake, we denote by a "sheet" plasma a bounded thin layer whose half thickness L is comparable to the ion Larmor radius ρ_i, and by a "slab" plasma a bounded one with L being considerably greater than ρ_i.

In this paper, on the basis of the warm fluid description for inhomogeneous plasmas, we will therefore focus attention on the derivation of the wave form of a normal mode which will be expected to penetrate an inhomogeneous slab plasma as a nonlinear electrostatic ion cyclotron wave. The normal mode is supposed to be excited by a pair of externally applied potentials oscillating near but above ω_{ci} and having a weak spatial inhomogeneity parallel to the magnetic field $\mathbf{B}_o = \mathbf{e}_z B_o$ where B_o is hereafter constant in space and time. This parallel inhomogeneity of the applied potentials naturally produces a small but finite parallel component of the electric field of the electrostatic ion cyclotron wave excited within the slab plasma. The main part of the electric field is, of course, in the x-direction along which the inhomogeneity of the plasma density is chosen. In order to make it possible to analytically find the required wave form as a function of x, we will make the following two assumptions: i) A high density assumption appropriate for thermonuclear fusion plasmas will be made by introducing a small expansion parameter $\omega_{ci}^2/\omega_{poi}^2(0)$ where $\omega_{poi}(x)$ and $\omega_{poi}(0)$ are the field-free local ion plasma frequency and its maximum value given at the mid-plane x=0, respectively. ii) The other practical assumption is such that the field-free density profiles $g_o(x) \equiv N_{o\sigma}(x)/N_{o\sigma}(0) \equiv \omega_{po\sigma}^2(x)/\omega_{po\sigma}^2(0)$ (σ=i,e) are Gaussian with respect to x where $N_{o\sigma}(x)$ are the field-free plasma densities. We will further assume for simplicity that all wave damping effects, including the cyclotron damping of the electrostatic ion cyclotron wave, be neglected. This simultaneously means the neglect of heating effects of plasma particles (especially ions), for example, by means of the resonant power absorption of the wave in a nonuniform magnetic field[7] (note that B_o will be assumed to be uniform in this paper).

By the way, the fluid description for the ion can not be used safely in an inhomogeneous "sheet" plasma because ρ_i is comparable to the density gradient scale length the measure of which is L, requiring the kinetic description for the ion. On the other hand, the fluid description for the ion can approximately be applied to an inhomogeneous "slab" plasma because ρ_i much smaller than L, making small the difference between the density of guiding centers and that of ions themselves, and allowing us to regard ions as a fluid. The fluid description for the electron can as well be applied to the inhomogeneous slab plasma under the condition

$$\frac{|\rho_e|}{\rho_i} = \frac{u_e}{u_i} \cdot \frac{\omega_{ci}}{|\omega_{ce}|} = \sqrt{\frac{Zm_e}{m_i} \cdot \frac{Z\gamma_e T_e}{\gamma_i T_i}} < 1 \quad ,$$

where $\rho_\sigma = u_\sigma/\omega_{c\sigma}$, $u_\sigma^2 = \gamma_\sigma T_\sigma / m_\sigma$, $\omega_{c\sigma} = e_\sigma B_o / m_\sigma c$($\sigma$ denotes either the ion for σ=i or the electron for σ=e), $Z = e_i/|e_e|$ with $e_e = -e$, and the remaining notation are defined as usual. Note here that, even in the inhomogeneous sheet plasma ($L \sim \rho_i$), electrons can be treated in the fluid picture if the condition $|\rho_e|/\rho_i \ll 1$, which is usually true, is satisfied. We should therefore keep in mind the fact that the results which will be obtained in this paper for the inhomogeneous "slab" plasma may not be applied to the inhomogeneous "sheet" plasma. It is just the kinetic theory based on the particle picture that gives correct results for the inhomogeneous sheet plasma, and that tells us how different the results obtained in the fluid picture is from those obtained in the particle picture when the limit $L \to \rho_i$ is taken, though this problem will be left unsolved in this paper.

The next section will be devoted to the foundation of the basic warm fluid equations valid in an inhomogeneous plasma whose scale size L is much greater than $\rho_i (> |\rho_e|)$. Here we note that, in the inhomogeneous "slab" plasma, L yields a measure of the density inhomogeneity as well as half the slab thickness. In §3, we will analyze the dynamic responses of the inhomogenous slab plasma to a pair of potentials $\phi_{ext}(z,t)$ applied externally as stated before. The analysis will be accomplished by introducing such a small expansion parameter $e\Phi_o/\gamma_e T_e$ that $(e\Phi_o/\gamma_e T_e)^2 \ll 1$, in addition to the very small parameter $\omega_{ci}^2/\omega_{poi}^2(0)(\ll 1)$, where Φ_o is the amplitude of $\phi_{ext}(z,t)$. With the help of the method of the Green's function, we will there derive the potential functions of x as well as of z and t for both the fundamental and the second harmonic of the nonlinear electrostatic ion cyclotron wave excited by $\phi_{ext}(z,t)$. Here, note that the linear harmonics appearing due to the kinetic treatment will not be included in our analysis restricted to the fluid treatment. In §4, using both static and dynamic solutions of the basic equations, all nonlinear static forces such as the ponderomotive forces acting mainly on ions along both x- and z-axes will be calculated to show the rf-plugging effect, producing new, field-dependent equilibrium plasma density profiles $N_\sigma(x,z)$. Namely, these rf-plugged density profiles will be expressed in terms of the potential function for the fundamental mode of the externally excited, nonlinear electrostatic ion cyclotron wave. In §5, some interesting results obtained in this paper will be summarized as conclusions.

§2. Foundation of the Basic Equations

We start with the warm fluid equations of motion

$$\frac{\partial \mathbf{v}_\sigma}{\partial t} + \mathbf{v}_\sigma \cdot \frac{\partial \mathbf{v}_\sigma}{\partial \mathbf{r}} = -\frac{e_\sigma}{m_\sigma}\frac{\partial}{\partial \mathbf{r}}(\phi_a + \phi) + \omega_{c\sigma}\mathbf{v}_\sigma \times \mathbf{e}_z - u_\sigma^2 \frac{\partial}{\partial \mathbf{r}} \ln n_\sigma \ , \tag{1}$$

the equations of continuity

$$\frac{\partial n_\sigma}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n_\sigma \mathbf{v}_\sigma) = 0 \ , \tag{2}$$

and the Poisson equation

$$-\frac{\partial^2}{\partial \mathbf{r}^2}(\phi_a + \phi) = 4\pi \sum_{\sigma = i,e} e_\sigma n_\sigma \ , \tag{3}$$

where $\phi_a(x,z)$ and $\phi(x,z,t)$ are respectively the ambipolar static potential and the fluctuating potential, both of which are to be determined self-consistently, and the other notation are usual. And we assume, throughout the present paper, that all quantities be uniform in the y-direction:

$$\frac{\partial}{\partial \mathbf{r}} = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_z \frac{\partial}{\partial z} \ . \tag{4}$$

Let us first separate the fluid velocities $\mathbf{v}_\sigma$ and the total densities n_σ as well into static and fluctuating parts, respectively, as

$$\mathbf{v}_\sigma(x,z,t) = \mathbf{V}_\sigma(x,z) + \delta \mathbf{v}_\sigma(x,z,t) \ , \tag{5}$$

and

$$n_\sigma(x,z,t) = N_\sigma(x,z) + \delta n_\sigma(x,z,t). \tag{6}$$

Approximating

$$\ln n_\sigma = \ln N_\sigma + \ln(1 + \frac{\delta n_\sigma}{N_\sigma}) \approx \ln N_\sigma + \frac{\delta n_\sigma}{N_\sigma} - \frac{1}{2}(\frac{\delta n_\sigma}{N_\sigma})^2 ,$$

under the assumption $(\delta n_\sigma/N_\sigma)^2 \sim (e|\phi|/\gamma_e T_e)^2 \ll 1$, and introducing the ponderomotive forces and potentials, $F_{p\sigma}(x,z)$ and $\phi_{p\sigma}(x,z)$, for both ion and electron fluids through such definitions as

$$\mathbf{F}_{p\sigma} = -\left\langle \delta\mathbf{v}_\sigma \cdot \frac{\partial \delta\mathbf{v}_\sigma}{\partial \mathbf{r}} \right\rangle_t , -\frac{e_\sigma}{m_\sigma} \frac{\partial(R_\sigma \phi_{p\sigma})}{\partial x} = F_{p\sigma x}, -\frac{e_\sigma}{m_\sigma} \frac{\partial \phi_{p\sigma}}{\partial z} = F_{p\sigma z}, \tag{7}$$

we can then separate eqs. (1) and (2) into static and fluctuating parts as

$$\mathbf{V}_\sigma \cdot \frac{\partial \mathbf{V}_\sigma}{\partial \mathbf{r}} = -\frac{e_\sigma \partial \phi_a}{m_\sigma \partial \mathbf{r}} + \mathbf{F}_{p\sigma} + \omega_{c\sigma} \mathbf{V}_\sigma \times \mathbf{e}_z - u_\sigma^2 \frac{\partial}{\partial \mathbf{r}} \left[\ln N_\sigma - \frac{1}{2} \left\langle (\frac{\delta n_\sigma}{N_\sigma})^2 \right\rangle_t \right], \tag{8}$$

$$\frac{\partial}{\partial \mathbf{r}} \cdot \left[N_\sigma \mathbf{V}_\sigma + \langle \delta n_\sigma \delta \mathbf{v}_\sigma \rangle_t \right] = 0 , \tag{9}$$

and

$$\frac{\partial \delta\mathbf{v}_\sigma}{\partial t} + \mathbf{V}_\sigma \cdot \frac{\partial \delta\mathbf{v}_\sigma}{\partial \mathbf{r}} + \delta\mathbf{v}_\sigma \cdot \frac{\partial \mathbf{V}_\sigma}{\partial \mathbf{r}} + \delta\mathbf{v}_\sigma \cdot \frac{\partial \delta\mathbf{v}_\sigma}{\partial \mathbf{r}} - \left\langle \delta\mathbf{v}_\sigma \cdot \frac{\partial \delta\mathbf{v}_\sigma}{\partial \mathbf{r}} \right\rangle_t$$

$$= -\frac{e_\sigma}{m_\sigma} \frac{\partial \phi}{\partial \mathbf{r}} + \omega_{c\sigma} \delta\mathbf{v}_\sigma \times \mathbf{e}_z - u_\sigma^2 \frac{\partial}{\partial \mathbf{r}} (\frac{\delta n_\sigma}{N_\sigma}) + \frac{u_\sigma^2}{2} \frac{\partial}{\partial \mathbf{r}} \left[(\frac{\delta n_\sigma}{N_\sigma})^2 - \left\langle (\frac{\delta N_\sigma}{N_\sigma})^2 \right\rangle_t \right], \tag{10}$$

$$\frac{\partial \delta n_\sigma}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot \left[N_\sigma \delta\mathbf{v}_\sigma + \delta n_\sigma \mathbf{V}_\sigma + \delta n_\sigma \delta\mathbf{v}_\sigma - \langle \delta n_\sigma \delta\mathbf{v}_\sigma \rangle_t \right] = 0 , \tag{11}$$

where $\langle\ \rangle_t$ denotes the time average over one period of fluctuating parts, and R_σ are constants to be determined in §4. The Poisson equation is as well divided into the two parts:

$$-\frac{\partial^2\phi_a}{\partial r^2} = 4\pi \sum_{\sigma=i,e} e_\sigma N_\sigma \ , \qquad (12)$$

and

$$-\frac{\partial^2\phi}{\partial r^2} = 4\pi \sum_{\sigma=i,e} e_\sigma \delta n_\sigma \ . \qquad (13)$$

Next, solving both perpendicular and parallel compoments of eq. (8), and introducing the additional static potentials $\phi_{s\sigma}(x,z)$ as

$$-\frac{e_\sigma}{m_\sigma}\frac{\partial\phi_{s\sigma}}{\partial r} = \frac{u_\sigma^2}{2}\frac{\partial}{\partial r}\left\langle\left(\frac{\delta n_\sigma}{N_\sigma}\right)^2\right\rangle_t \ , \qquad (14)$$

we find

$$V_{\sigma x} = V_{\sigma z} = 0 \ , \qquad (15)$$

$$V_{\sigma y}(x,z) = \frac{u_\sigma^2}{\omega_{c\sigma}}\frac{\partial}{\partial x}\left[\beta_\sigma(\phi_a+R_\sigma\phi_{p\sigma}+\phi_{s\sigma}) + \ln N_\sigma\right] , \qquad (16)$$

and

$$\frac{\partial}{\partial z}\left[\beta_\sigma(\phi_a+\phi_{p\sigma}+\phi_{s\sigma}) + \ln N_\sigma\right] = 0 \ , \qquad (17)$$

where $\beta_\sigma=(e_\sigma/m_\sigma)/u_\sigma^2=e_\sigma/\gamma_\sigma T_\sigma$. An integration of eq. (17) now yields the Boltzmann relations for both ions and electrons:

$$\frac{N_\sigma(x,z)}{N_{o\sigma}(x)} = \exp\left[-\beta_\sigma(\phi_a+\phi_{p\sigma}+\phi_{s\sigma})\right] , \qquad (18)$$

with $N_{o\sigma}(x)$ being independent of z. We can then regard $N_{o\sigma}(x)$ as the original plasma densities which are independent of the fluctuating potential $\phi(x,z,t)$ that nonlinearly produces such static potentials as $\phi_a(x,z)$, $\phi_{p\sigma}(x,z)$ and $\phi_{s\sigma}(x,z)$. Here we require the charge neutrality condition for the field-free densities $N_{o\sigma}(x)$:

$$N_{oe}(x)=ZN_{oi}(x), \text{ i.e., } \sum_{\sigma=i,e} e_\sigma N_{o\sigma}(x)=0 \ . \tag{19}$$

Further, with the help of eq. (18), the diamagnetic drift velocities (16) can be put into the form

$$V_{\sigma y}(x,z) = \frac{u_\sigma^2}{\omega_{c\sigma}} \frac{\partial}{\partial x}\left[(R_\sigma-1)\beta_\sigma\phi_{p\sigma}+\ln N_{o\sigma}\right] . \tag{20}$$

Furthermore, eq. (9) turns out to be trivial if we note that using eqs. (4) and (15) yields $(\partial/\partial\mathbf{r})\cdot N_\sigma\mathbf{V}_\sigma=0$, and also that $(\partial/\partial\mathbf{r})\cdot\langle\delta n_\sigma\delta\mathbf{v}_\sigma\rangle_t=0$ because both $\delta v_{\sigma x}$ and $\delta v_{\sigma z}$ differ from δn_σ by $\pi/2$ in phase (in addition, δv_{ex} and δv_{ez} are negligibly small) as seen in §3. In the same manners as above, we can show that $\mathbf{V}_\sigma\cdot(\partial/\partial\mathbf{r})\delta\mathbf{v}_\sigma=0$ in eq. (10), and that $(\partial/\partial\mathbf{r})\cdot[\delta n_\sigma\mathbf{V}_\sigma-\langle\delta n_\sigma\delta\mathbf{v}_\sigma\rangle_t]=0$ in eq. (11).

Finally, we have arrived at the two groups of basic equations: One is composed of the Boltzmann relation (18) with eqs. (7) and (14) and the static part of the Poisson equation, (12), with eq. (19). These will be analyzed consistently in §4. The other is composed of the fluctuating parts of eqs. (1) and (2) which can now be reduced to, by way of eqs. (10) and (11),

$$\frac{\partial \delta \mathbf{v}_\sigma}{\partial t} + \delta \mathbf{v}_\sigma \cdot \frac{\partial \delta \mathbf{v}_\sigma}{\partial \boldsymbol{r}} - \left\langle \delta \mathbf{v}_\sigma \cdot \frac{\partial \delta \mathbf{v}_\sigma}{\partial \boldsymbol{r}} \right\rangle_t = - \frac{e_\sigma}{m_\sigma} \frac{\partial \phi}{\partial \boldsymbol{r}} + \omega_{c\sigma} \delta \mathbf{v}_\sigma \times \mathbf{e}_z$$

$$- \mathbf{e}_y \left(\delta v_{\sigma x} \frac{\partial}{\partial x} + \delta v_{\sigma z} \frac{\partial}{\partial z}\right) V_{\sigma y} - u_\sigma^2 \frac{\partial}{\partial \boldsymbol{r}} \left(\frac{\delta n_\sigma}{N_\sigma}\right) + \frac{u_\sigma^2}{2} \frac{\partial}{\partial \boldsymbol{r}} \left[\left(\frac{\delta n_\sigma}{N_\sigma}\right)^2 - \left\langle \left(\frac{\delta n_\sigma}{N_\sigma}\right)^2 \right\rangle_t \right], \qquad (21)$$

with eqs. (4) and (15) being used, and

$$\frac{\partial}{\partial t}\left(\frac{\delta n_\sigma}{N_\sigma}\right) + \frac{1}{N_\sigma} \frac{\partial}{\partial \boldsymbol{r}} \cdot \left[\delta \mathbf{v}_\sigma N_\sigma \left(1 + \frac{\delta n_\sigma}{N_\sigma}\right) \right] = 0 , \qquad (22)$$

and also of the fluctuating part of the Poisson equation, (13). These basic equations for fluctuating parts will be treated in §3. We now close this section by pointing out that all the basic equations obtained above are valid up to the second order with respect to $\phi(x,z,t)$, and also that even such terms as contain both $(\partial\phi/\partial j)(\partial/\partial k)|\phi|^2$ and $(\partial\phi/\partial j)(\partial/\partial k)|\partial\phi/\partial l|^2 (j,k,l=x,z)$ have implicitly been retained as will be seen in §3.

§3. The Self-Consistent Fluctuating Potential of an Excited Nonlinear Electrostatic Ion Cyclotron Wave

Before analyzing a series of the basic equations established in §2, we prepare the following situation appropriate for the analysis. Namely, the inhomogeneous slab plasma is supposed to be put between the two planes $x=\pm L$ which are parallel to the mid-plane $x=0$, but to be unbounded and homogeneous in the z-direction as well as in the y-direction. The density inhomogeneity is assumed to be only in the x-direction and symmetrical with respect to the mid-plane as shown in Fig. 1. A pair of the external potentials, $\phi_{ext}(z,t)$, will be fed at $x=\pm L$ to the plasma to excite an internal normal mode under consideration.

Let us now make the harmonic expansion for the fluctuating parts as

$$\left.\begin{matrix}\phi(x,z,t)\\ \delta\mathbf{v}_\sigma(x,z,t)\\ \delta n_\sigma(x,z,t)\end{matrix}\right\}=\frac{1}{2}\sum_{\nu=1}\left\{\begin{matrix}\phi^{(\nu)}(x,z)\\ \delta\mathbf{v}_\sigma^{(\nu)}(x,z)\\ \delta n_\sigma^{(\nu)}(x,z)\end{matrix}\right\}\exp(-i\nu\omega_o t)+\text{c.c.}. \qquad (23)$$

Substituting these into eqs. (21), (22) and (13), we obtain a series of coupled differential equations which determine such harmonic components as $\phi^{(\nu)}$, $\delta\mathbf{v}_\sigma^{(\nu)}$ and $\delta n_\sigma^{(\nu)}$. Namely, for the fundamental $\nu=1$, we find

$$-i\omega_o\delta\mathbf{v}_\sigma^{(1)}=-\frac{e_\sigma}{m_\sigma}\frac{\partial\phi^{(1)}}{\partial\mathbf{r}}+\mathbf{e}_x\omega_{c\sigma}\delta v_{\sigma y}^{(1)}-\mathbf{e}_y\omega_{c\sigma}\delta v_{\sigma x}^{(1)}(1+\Delta_\sigma^{(1)})-u_\sigma^2\frac{\partial}{\partial\mathbf{r}}\left(\frac{\delta n_\sigma^{(1)}}{N_\sigma}\right), \qquad (24)$$

$$-i\omega_o\frac{\delta n_\sigma^{(1)}}{N_\sigma}+\frac{1}{N_\sigma}\frac{\partial}{\partial\mathbf{r}}\cdot\delta\mathbf{v}_\sigma^{(1)}N_\sigma=0 , \qquad (25)$$

and

$$-\frac{\partial^2\phi^{(1)}}{\partial \mathbf{r}^2} = 4\pi \sum_{\sigma=i,e} e_\sigma \delta n_\sigma^{(1)} \quad . \tag{26}$$

And for the second harmonic $\nu=2$, we find

$$-2i\omega_o\delta\mathbf{v}_\sigma^{(2)} = -\frac{e_\sigma}{m_\sigma}\frac{\partial\phi^{(2)}}{\partial\mathbf{r}} + \mathbf{e}_x\omega_{c\sigma}\delta v_{\sigma y}^{(2)} - \mathbf{e}_y\omega_{c\sigma}\delta v_{\sigma x}^{(2)}(1+\Delta_\sigma^{(2)})$$

$$-u_\sigma^2\frac{\partial}{\partial\mathbf{r}}\left(\frac{\delta n_\sigma^{(2)}}{N_\sigma}\right) - \frac{1}{2}\,\delta\mathbf{v}_\sigma^{(1)}\cdot\frac{\partial\delta\mathbf{v}_\sigma^{(1)}}{\partial\mathbf{r}} + \frac{u_\sigma^2}{4}\frac{\partial}{\partial\mathbf{r}}\left(\frac{\delta n_\sigma^{(1)}}{N_\sigma}\right)^2 \quad , \tag{27}$$

$$-2i\omega_o\frac{\delta n_\sigma^{(2)}}{N_\sigma} + \frac{1}{N_\sigma}\frac{\partial}{\partial\mathbf{r}}\cdot\delta\mathbf{v}_\sigma^{(2)}N_\sigma + \frac{1}{2N_\sigma}\frac{\partial}{\partial\mathbf{r}}\cdot\delta\mathbf{v}_\sigma^{(1)}\delta n_\sigma^{(1)} = 0, \tag{28}$$

and

$$-\frac{\partial^2\phi^{(2)}}{\partial\mathbf{r}^2} = 4\pi\sum_{\sigma=i,e} e_\sigma\delta n_\sigma^{(2)} \quad . \tag{29}$$

Here we should note that $\Delta_\sigma^{(\nu)} \equiv \omega_{c\sigma}^{-1}[\partial/\partial x+(\delta v_{\sigma z}^{(\nu)}/\delta v_{\sigma x}^{(1)})\partial/\partial z]V_{\sigma y}(\nu=1,2)$ in eqs. (24)and (27) will be hereafter neglected as sufficiently small terms compared with unity, which will be justified after obtaining $\delta v_{\sigma x}^{(\nu)}$ and $\delta v_{\sigma z}^{(\nu)}$ explicitly under the conditions assumed here.

The first task is to solve the set of equations (24) to (26) consistently with the boundary condition that $\phi^{(1)}(\pm L,z)=\pm\Phi_o f_o(z)$, when we assume

$$\phi_{ext}(z,t)=\begin{cases} \frac{1}{2}\,\Phi_o f_o(z)\exp(-i\omega_o t)+c.c. \text{ at } x=L \\ -\frac{1}{2}\,\Phi_o f_o(z)\exp(-i\omega_o t)+c.c. \text{ at } x=-L \end{cases} \quad , \tag{30}$$

where Φ_o is the constant amplitude of $\phi_{ext}(z,t)$, and $f_o(z)$ respresents a slowly varying function of z whose amplitude is just unity and whcse inhomogeneity scale length is h.

And with the use of h, we here introduce a quantity $K_z \equiv h^{-1}$ to measure the parallel wavenumber of $\phi_{ext}(z,t)$. For this purpose, we explicitly write down $\delta \mathbf{v}_\sigma^{(1)}$ as

$$\delta v_{\sigma x}^{(1)} = \frac{i\omega_o}{\omega_{c\sigma}}\delta v_{\sigma y}^{(1)} = -\frac{i\omega_o}{\omega_o^2-\omega_{c\sigma}^2}u_\sigma^2\frac{\partial}{\partial x}(\beta_\sigma\phi^{(1)} + \frac{\delta n_\sigma^{(1)}}{N_\sigma}) , \qquad (31)$$

and

$$\delta v_{\sigma z}^{(1)} = -\frac{i\omega_o}{\omega_o^2}u_\sigma^2\frac{\partial}{\partial z}(\beta_\sigma\phi^{(1)} + \frac{\delta n_\sigma^{(1)}}{N_\sigma}) , \qquad (32)$$

with $\beta_\sigma = e_\sigma/\gamma_\sigma T_\sigma = (e_\sigma/m_\sigma)/u_\sigma^2$. Let us here introduce a convenient function in eqs. (31) and (32) for σ=e:

$$\beta_e\psi(x,z) = \beta_e\phi^{(1)}(x,z) + \delta n_e^{(1)}(x,z)/N_e(x,z). \qquad (33)$$

and a quantity $K_x \equiv d^{-1}$ to measure the perpendicular wavenumber of the excited fluctuations whose inhomogeneity scale length along the x-axis is given by d. And then noting such relations as $|(\partial/\partial j)\ln|\psi|| \sim K_j$ (j=x,z),

$$\left|\frac{\delta v_{ex}^{(1)}}{\delta v_{ez}^{(1)}}\right| = \left|\frac{\omega_o^2}{\omega_o^2-\omega_{ce}^2}\frac{\partial\psi/\partial x}{\partial\psi/\partial z}\right| \sim \frac{\omega_o^2}{\omega_{ce}^2}\frac{K_x}{K_z} = (\frac{\omega_o}{K_z u_e})^2(\rho_e K_x)^2\frac{K_z}{K_x} , \qquad (34)$$

and $|\delta v_{ey}^{(1)}/\delta v_{ez}^{(1)}| \sim (\omega_o/|\omega_{ce}|)(K_x/K_z) = (\omega_o/K_z u_e)|\rho_e K_x|$, we can assume the following approximate form for $\delta \mathbf{v}_e^{(1)}$:

$$\delta v_{ex}^{(1)} = 0, \quad \delta v_{ey}^{(1)} \approx 0 , \qquad (35)$$

and requiring the relation $\psi(x,z) \equiv \psi(z)$ so as to be consistent with $\delta v_{ex}^{(1)} = 0$, i.e., $\partial\psi/\partial x = 0$,

$$\delta v_{ez}^{(1)} = -i(u_e^2/\omega_o)(d/dz)\beta_e\psi(z) \ , \qquad (36)$$

provided that $|\rho_e K_x| < \rho_i K_x \ll 1$ and $(K_z/K_x)^2 \ll 1$ or $d^2/h^2 \ll 1$, and also that K_z, being a measure of the small parallel wave-number of the excited fluctuations as well as that of the exciter $\phi_{ext}(z,t)$, should satisfy the condition

$$u_i \ll \omega_o/K_z \ll u_e . \qquad (37)$$

It is noted here that this condition (37) allows us to neglect the cyclotron damping of a wave component along $\mathbf{B}_o$. Further, we here show, using eqs. (20), (34) and (121), and $R_e \approx -\omega_o^2/\omega_{ce}^2$ from eq. (120), that $|\Delta_e^{(1)}| \sim \rho_e^2 \, |(d^2/dx^2)\ln N_{oe}(x)| \sim \rho_e^2 L_{on}^{-2} \sim 0$ compared with unity since $|\rho_e| < \rho_i \ll L_{on}$ where we have, in the absence of $\phi_{ext}(z,t)$, defined as $L_{on}^{-1}(x) = -(d/dx)\ln N_{o\sigma}(x)$ for both σ=i and e. Now, substituting eqs. (35) and (36) into eq. (25) for σ=e together with eq. (33), we approximately obtain

$$\frac{\delta n_e^{(1)}}{N_e} = - \frac{u_e^2}{\omega_o^2}\frac{1}{N_e}\frac{\partial}{\partial z}\left[N_e \frac{d}{dz} \beta_e \psi(z)\right] \approx -\beta_e \phi^{(1)}, \qquad (38)$$

where $\beta_e\psi(z)$ has been neglected compared with $-\beta_e \phi^{(1)}$ on the r.h.s. of eq. (38) because of the sufficient smallness of $|\psi(z)/\phi^{(1)}| \sim (\omega_o/K_z u_e)^2$ under the condition $\omega_o \ll K_z u_e$. In the above order estimation, we have also used such an inequality as $|L_{nz}|^{-1} \ll K_z$, which can be shown to be satisfied using eq. (125) where $L_{nj}^{-1}(x,z) = -(\partial/\partial j)\ln N_\sigma(x,z)$ or $L_{nj}^{-1}(x,z) = -(\partial/\partial j)\ln \omega_{p\sigma}^2(x,z)(j=x,z)$ with the definition $\omega_{p\sigma}^2(x,z) = 4\pi e_\sigma^2 N_\sigma(x,z)/m_\sigma$

and the requirement (39) being employed. Further, making use of eq. (38) in eq. (26) and requiring the charge neutrality condition in the presence of fluctuations:

$$N_e(x,z) = ZN_i(x,z),\ \text{i.e.,}\ \sum_{\sigma=i,e} e_\sigma N_\sigma(x,z)=0\ , \tag{39}$$ [8]

we can express the ion density fluctuation $\delta n_i^{(1)}$ in terms of $\phi^{(1)}$:

$$\frac{\delta n_i^{(1)}}{N_i} = -(1-\lambda_{De}^2\frac{\partial^2}{\partial r^2})\beta_e\phi^{(1)}\quad , \tag{40}$$

with $\lambda_{D\sigma}^2(x,z)=\gamma_\sigma T_\sigma/[4\pi e_\sigma^2 N_\sigma(x,z)]$. Substituting this into eqs. (31) and (32) for σ=i immediately yields

$$\delta v_{ix}^{(1)} = \frac{i\omega_o}{\omega_{ci}}\delta v_{iy}^{(1)} = -\frac{i\omega_o}{\omega_o^2-\omega_{ci}^2}\frac{e_i}{m_i}\frac{\partial}{\partial x}(1+\frac{\gamma_i T_i}{Z\gamma_e T_e}-\lambda_{Di}^2\frac{\partial^2}{\partial r^2})\phi^{(1)}, \tag{41}$$

and

$$\delta v_{iz}^{(1)} = -\frac{i\omega_o}{\omega_o^2}\frac{e_i}{m_i}\frac{\partial}{\partial z}(1+\frac{\gamma_i T_i}{Z\gamma_e T_e}-\lambda_{Di}^2\frac{\partial^2}{\partial r^2})\,\phi^{(1)}\quad , \tag{42}$$

where we have employed such relations as $\beta_i u_i^2=e_i/m_i$ and $\beta_e\lambda_{De}^2(x,z) = -\beta_i\lambda_{Di}^2(x,z)$. Here we note that the neglect of $\Delta_i^{(1)}$ in eq. (24) turns out to be justified since we can estimate as $\Delta_i^{(1)} \sim \rho_i^2 L_{on}^{-2} \sim 0$, using eq. (122) and $R_i=\omega_o^2/(\omega_o^2-\omega_{ci}^2) \sim O(1)$ from eq. (120). With the help of eqs. (40) to (42), the equation of continuity (25) for the ion leads to a wave equation governing $\phi^{(1)}(x,z)$. That is,

$$\left[1+\frac{C_s^2}{\omega_{pi}^2}\left(\frac{\partial}{\partial x}\frac{\omega_{pi}^2}{\omega_o^2-\omega_{ci}^2}\frac{\partial}{\partial x}+\lambda'\frac{\partial}{\partial z}\frac{\omega_{pi}^2}{\omega_o^2}\frac{\partial}{\partial z}\right)\right]\phi^{(1)}$$

$$=\epsilon_{\ell}\frac{C_{se}^2}{\omega_{pi}^2}\left[1+\left(\frac{\partial}{\partial x}\frac{\omega_{pi}^2}{\omega_o^2-\omega_{ci}^2}\frac{\partial}{\partial x}+\lambda''\frac{\partial}{\partial z}\frac{\omega_{pi}^2}{\omega_o^2}\frac{\partial}{\partial z}\right)\frac{u_i^2}{\omega_{pi}^2}\right]\left(\frac{\partial^2}{\partial x^2}+\lambda'\frac{\partial^2}{\partial z^2}\right)\phi^{(1)}, \tag{43}$$

where we have used $\lambda_{Di}^2(x,z)=u_i^2/\omega_{pi}^2(x,z)$ and $\lambda_{De}^2(x,z)=C_{se}^2/\omega_{pi}^2(x,z)$ together with $C_{se}^2=Z\gamma_e T_e/m_i$ and $C_s^2=C_{se}^2(1+\gamma_i T_i/Z\gamma_e T_e)$. And the terms with a parameter λ' in eq. (43) are assumed to be of the order $(K_z C_s/\omega_o)^2 \ll (K_x C_s/\omega_o)^2 < 1$, i.e., $(K_z/K_x)^2 \ll 1$, and those with another parameter ϵ_{ℓ} to be of the order $(K_x C_{se}/\omega_{pi})^2 = (K_x C_{se}/\omega_{ci})^2\omega_{ci}^2/\omega_{pi}^2 \lesssim \omega_{ci}^2/\omega_{pi}^2 \sim \omega_{ci}^2/\omega_{poi}^2 \ll 1$ with $\omega_{po\sigma}^2(x)=4\pi e_\sigma^2 N_{o\sigma}(x)/m_\sigma$, where both parameters λ' and ϵ_{ℓ} will later be taken equal to unity. The term with $\lambda''(=1)$, being of the order $(K_z u_i/\omega_o)^2$, will be discarded under the assumption (37). By the way, making use of eqs. (125) and (126), we can express the field-dependent, local ion plasma frequency $\omega_{pi}(x,z)$ in terms of $\phi^{(1)}(x,z)$:

$$\frac{\omega_{pi}^2}{\omega_{poi}^2}=\frac{N_i}{N_{oi}}=\exp\left\{-\epsilon_{n\ell}|\beta_e|\left[\phi_{n\ell}(\phi^{(1)},x,z)+C\right]\right\}$$

$$\approx 1-\epsilon_{n\ell}|\beta_e|\left[\phi_{n\ell}(\phi^{(1)},x,z)+C\right], \tag{44}$$

with

$$\phi_{n\ell}(\phi^{(1)},x,z)=\frac{|\beta_e|}{4}\left[\frac{C_s^2}{\omega_o^2-\omega_{ci}^2}\left|\frac{\partial\phi^{(1)}}{\partial x}\right|^2+\lambda'\frac{C_s^2}{\omega_o^2}\left|\frac{\partial\phi^{(1)}}{\partial z}\right|^2-\left|\phi^{(1)}\right|^2\right], \tag{45}$$

where the constant C is at most comparable to $\phi_{n\ell}(\phi^{(1)},x,z)$ or less than it. And the term with $\epsilon_{n\ell}(=1)$ is supposed to be of the order $|\beta_e \phi^{(1)}|^2 \ll 1$. Note that we are not assuming the severer condition $|\beta_e \phi^{(1)}| \ll 1$. Let us now substitute eq. (44) into eq. (43) and then make linearizations with respect to both ϵ_ℓ and $\epsilon_{n\ell}$, not with respect to λ' so as to include terms of such orders as $\epsilon_\ell \lambda'$ and $\epsilon_{n\ell}\lambda'$. Thus, we arrive at a nonlinear partial differential wave equation which can be reduced to the following form solvable by iteration:

$$\mathcal{L}^{(1)}\phi^{(1)}(x,z)=\Lambda_\ell^{(1)}(\phi_o^{(1)},x,z)+\Lambda_{n\ell}^{(1)}(\phi_o^{(1)},x,z) \ , \tag{46}$$

with

$$\Lambda_\ell^{(1)}(\phi_o^{(1)},x,z)=\epsilon_\ell \frac{C_{se}^2}{\omega_{poi}^2(x)}\left\{\frac{u_i^2}{C_s^2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}-\frac{1}{N_{oi}(x)}\frac{dN_{oi}(x)}{dx}\right]+\frac{\omega_o^2-\omega_{ci}^2}{C_s^2}\right\}\left(\frac{\partial^2}{\partial x^2}+\lambda'\frac{\partial^2}{\partial z^2}\right)\phi_o^{(1)}(x,z), \tag{47}$$

and

$$\Lambda_{n\ell}^{(1)}(\phi_o^{(1)},x,z)=\epsilon_{n\ell}\,|\beta_e|\left[\frac{\partial\phi_{n\ell}(\phi_o^{(1)},x,z)}{\partial x}\right]\frac{\partial\phi_o^{(1)}(x,z)}{\partial x} \ , \tag{48}$$

where the lowest order potential $\phi_o^{(1)}(x,z)$ is governed by

$$\mathcal{L}^{(1)}\phi_o^{(1)}(x,z)\equiv\left\{\left[\frac{\partial}{\partial x}+\frac{1}{N_{oi}(x)}\frac{dN_{oi}(x)}{dx}\right]\frac{\partial}{\partial x}+\frac{\omega_o^2-\omega_{ci}^2}{C_s^2}\left(1+\lambda'\frac{C_s^2}{\omega_o^2}\frac{\partial^2}{\partial z^2}\right)\right\}\phi_o^{(1)}(x,z)=0 \ . \tag{49}$$

From now on, let us assume the following, realistic Gaussian forms for the field-free density profiles $g_o(x)$ and the parallel potential profile $f_o(z)$ of the exciter (30).[2-3] Namely, we write

$$g_o(x) \equiv \frac{N_{o\sigma}(x)}{N_{o\sigma}(0)} = \frac{\omega^2_{po\sigma}(x)}{\omega^2_{po\sigma}(0)} = \exp(-\frac{x^2}{2d^2}) \ (\sigma=i,e), \tag{50}$$

and

$$f_o(z) = \exp(-z^2/2h^2) \ (\leq f_o(0)=1) , \tag{51}$$

accompanied with the condition $d^2/h^2 \ll 1$.

And then we assume such forms as

$$\phi^{(1)}(x,z) = \Phi^{(1)}(x,z)f_o(z) , \tag{52}$$

$$\phi_o^{(1)}(x,z) = \Phi_o^{(1)}(x,z)f_o(z) , \tag{53}$$

since the variables of eqs. (46) and (49) are approximately separable* provided that both $\Phi^{(1)}(x,z)$ and $\Phi_o^{(1)}(x,z)$ are much more slowly varying functions of z than $f_o(z)$ under the condition

$$\left|(\partial/\partial z)\ln|\Phi^{(1)}(x,z)|\right| \approx \left|(\partial/\partial z)\ln|\Phi_o^{(1)}(x,z)|\right| \ll K_z , \tag{54}$$

which will later be shown to be true. Here remember that we also have the condition $K_z^2 \ll K_x^2 \sim [(\partial/\partial x)\ln|\Phi^{(1)}(x,z)|]^2 \approx [(\partial/\partial x)\ln|\Phi_o^{(1)}(x,z)|]^2$. Substituting eqs. (50) to (53) into eqs. (46) to (49) with eq. (45), discarding all terms proportional to $|(\partial/\partial z)\ \Phi^{(1)}(x,z)|^2$, $(\partial^2/\partial z^2)\ \Phi^{(1)}(x,z)$ and $(\partial^2/\partial z^2)\Phi_o^{(1)}(x,z)$, and introducing the new variables $\xi = x/d$ and $\zeta = z/h$ so that eqs. (50) and (51) are rewritten as

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*We don't assume the form $f_o(z)=\cos k_z z$, though this form is mathematically ideal since the variables of eq. (49) are completely separable.

$$g_o(\xi) \equiv N_{o\sigma}(\xi)/N_{o\sigma}(0)=\omega^2_{po\sigma}(\xi)/\omega^2_{po\sigma}(0)=\exp(-\xi^2/2) \ , \qquad (50')$$

and

$$f_o(\zeta) = \exp(-\zeta^2/2) \ , \qquad (51')$$

we obtain, setting $\lambda'=1$ and rewriting $\Phi^{(1)}(x,z)$ and $\Phi_o^{(1)}(x,z)$ as $\Phi_\mu^{(1)}(\xi,\zeta)$ and $\Phi_{o\mu}^{(1)}(\xi)$, respectively,

$$\mathcal{L}_\xi^{(1)} \Phi_\mu^{(1)}(\xi,\zeta) = \Lambda_\mu^{(1)}(\Phi_{o\mu}^{(1)},\xi,\zeta) \equiv \Lambda_{l\mu}^{(1)}(\Phi_{o\mu}^{(1)},\xi) + f_o^2(\zeta)\Lambda_{nl\mu}^{(1)}(\Phi_{o\mu}^{(1)},\xi), \qquad (55)$$

with

$$\Lambda_{l\mu}^{(1)}(\Phi_{o\mu}^{(1)},\xi) = \epsilon_l \frac{C_{se}^2}{C_s^2} \frac{\omega_{ci}^2}{\omega_{poi}^2(0)} e^{\frac{\xi^2}{2}} \left[\left(\frac{\rho_i}{d}\right)^2 \left(\frac{\partial^2}{\partial\xi^2}+\xi\frac{\partial}{\partial\xi}+1\right) + \left(\frac{\omega_o}{\omega_{ci}}\right)^2 -1 \right] \left(\frac{\partial^2}{\partial\xi^2} - \frac{d^2}{h^2}\nu\right) \Phi_{o\mu}^{(1)}(\xi), \qquad (56)$$

$$\Lambda_{nl\mu}^{(1)}(\Phi_{o\mu}^{(1)},\xi) = \epsilon_{nl} |\beta_e| \left[\frac{\partial \phi_{nl\mu}(\Phi_{o\mu}^{(1)},\xi)}{\partial\xi} \right] \frac{\partial \Phi_{o\mu}^{(1)}(\xi)}{\partial\xi} \ , \qquad (57)$$

and

$$\phi_{nl\mu}(\Phi_\mu^{(1)},\xi) = \frac{1}{4} \frac{|\beta_e| C_s^2/u_i^2}{(\omega_o/\omega_{ci})^2-1} \left(\frac{\rho_i}{d}\right)^2 \left\{ \left|\frac{\partial \Phi_\mu^{(1)}}{\partial\xi}\right|^2 - \left[\mu - \left(1-\left(\frac{\omega_{ci}}{\omega_o}\right)^2\right)\frac{d^2}{h^2}(1-2\nu)\right] |\Phi_\mu^{(1)}|^2 \right\} , \qquad (58)$$

where $\Phi_{o\mu}^{(1)}(\xi)$ satisfies

$$\mathcal{L}_\xi^{(1)} \Phi_{o\mu}^{(1)}(\xi) \equiv \left(\frac{\partial^2}{\partial\xi^2} - \xi\frac{\partial}{\partial\xi} + \mu\right) \Phi_{o\mu}^{(1)}(\xi) = 0 \ . \qquad (59)$$

And we have introduced

$$d^2k_x^2(z)=\mu(\zeta)=\left[\left(\frac{\omega_o}{\omega_{ci}}\right)^2-1\right]\frac{u_i^2}{C_s^2}\left(\frac{d}{\rho_i}\right)^2\left[1-\left(\frac{\omega_{ci}}{\omega_o}\right)^2\frac{C_s^2}{u_i^2}\left(\frac{\rho_i}{d}\right)^2\frac{d^2}{h^2}\,\nu(\zeta)\right]\gg\frac{d^2}{h^2}\,, \tag{60}$$

$$h^2k_z^2(z)=\nu(\zeta) = -f_o^{-1}(\zeta)(d^2/d\zeta^2)f_o(\zeta)=1-\zeta^2\ll h^2/d^2, \tag{61}$$

where, in the explicit form, $k_x^2(z)=\left[(\omega_o^2-\omega_{ci}^2)/C_s^2\right]\left[1-C_s^2k_z^2(z)/\omega_o^2\right]$ with $k_z^2(z)=-f_o^{-1}(z)(d^2/dz^2)f_o(z)$. We have also used the relation

$$h^2\left[(d/dz)\ln f_o(z)\right]^2=1-\nu(\zeta)\equiv\zeta^2, \tag{62}$$

and then note that $0\leq\nu(\zeta)\leq 1$ in the region $|\zeta|\leq 1$ ($|z|\leq h$). The first task, being now to solve eqs. (55) to (59), can be accomplished by employing the method of Green's function so as to satisfy the boundary condition $\Phi_\mu^{(1)}(\pm\ell,\zeta)=\pm\Phi_o$ with $\ell=L/d$ for a certain value of $\zeta=z/h$.[9] After some calculations, we arrive at the self-consistent solution for $\Phi_\mu^{(1)}(\xi,\zeta)$:

$$\Phi_\mu^{(1)}(\xi,\zeta)=\Phi_{o\mu}^{(1)}(\xi)+\Phi_oF_e(\mu,\ell)F_o(\mu,\ell)\left\{I_\mu(\xi,\zeta)-\frac{1}{2}\left[\frac{F_e(\mu,\xi)}{F_e(\mu,\ell)}-\frac{F_o(\mu,\xi)}{F_o(\mu,\ell)}\right]I_\mu(-\ell,\zeta)\right\}, \tag{63}$$

accompanied with

$$\Phi_{o\mu}^{(1)}(\xi)=\Phi_oF_o(\mu,\xi)/F_o(\mu,\ell), \tag{64}$$

where

$$I_\mu(\xi,\eta)=\int_\xi^{\ell} d\eta\left[\frac{F_e(\mu,\xi)F_o(\mu,\eta)}{F_e(\mu,\ell)F_o(\mu,\ell)}-\frac{F_e(\mu,\eta)F_o(\mu,\xi)}{F_e(\mu,\ell)F_o(\mu,\ell)}\right]e^{-\frac{\eta^2}{2}}\Lambda_\mu^{(1)}(\Phi_{o\mu}^{(1)}(\eta),\eta,\zeta)/\Phi_o, \tag{65}$$

together with

$$F_e(\mu,\xi)=F(-\mu/2,1/2:\xi^2/2)=F_e(\mu,-\xi), \tag{66}$$

$$F_o(\mu,\xi)=\xi F(1/2-\mu/2,3/2;\xi^2/2)=-F_o(\mu,-\xi). \tag{67}$$

Here, the confluent hypergeometric function $F(\alpha,\gamma;x)$ is as usual defined by

$$F(\alpha,\gamma;x)=\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)}\frac{x^n}{n!}=1+\frac{\alpha}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\cdots. \tag{68}$$

And we see that the condition (54) is satisfied since $\left|(\partial/\partial z)\ln\left|\Phi_{o\mu}^{(1)}\right|\right| = \left|(\partial\zeta/\partial z)(\partial\nu/\partial\zeta)(\partial\mu/\partial\nu)(\partial/\partial\mu)\ln\left|\Phi_{o\mu}^{(1)}\right|\right| = (2|\zeta|/h)[1-(\omega_{ci}/\omega_o)^2](d^2/h^2)\left|(\partial/\partial\mu)\ln\left|\Phi_{o\mu}^{(1)}\right|\right| \ll K_z\left|(\partial/\partial\alpha)\ln\left|F(\alpha,3/2;\xi^2/2)/F(\alpha,3/2;\ell^2/2)\right|\right| \lesssim O(K_z)$ with $\alpha=1/2-\mu/2$, provided that $1/2<\omega_{ci}/\omega_o<1$ and $|\xi|\leq\ell\lesssim O(1)$, but that, in the α-ξ space, all points on and near the locus $\xi=\xi(\alpha)$ satisfying $F(\alpha,3/2,\xi^2/2)=0$ be excluded. If we wish to include such points, we may only write $|(\partial/\partial z)\Phi_o^{(1)}(x,z)/\Phi_o^{(1)}(L,0)| \ll |(d/dz)f_o(z)|$ instead of the condition (54). In any case, because of the sufficient smallness of $\partial\mu/\partial\nu$, our assumption is justified; the z-dependence of $\Phi_o^{(1)}(x,z)$(and also of $\Phi^{(1)}(x,z)$) is much weaker than that of $f_o(z)$. Let us now introduce the following convenient functions; setting $\epsilon_\ell=\epsilon_{n\ell}=1$ in eqs. (56) and (57), neglecting terms of order $(\rho_i/d)^2$ compared to unity and writing $\bar{F}_j(\mu,\xi)=F_j(\mu,\xi)/F_j(\mu,\ell)$ $(j=e,o)$,

$$\sigma_{\ell\mu j}(\xi)=\frac{\omega_{ci}^2}{\omega_{poi}^2(0)}\left[\left(\frac{\omega_o}{\omega_{ci}}\right)^2-1\right]\frac{C_{se}^2}{C_s^2}\int_0^{\xi}d\eta\bar{F}_j(\mu,\eta)\left[\frac{\partial^2}{\partial\eta^2}-\frac{d^2}{h^2}(1-\zeta^2)\right]\bar{F}_o(\mu,\eta), \tag{69}$$

with such properties as $\sigma_{\ell\mu e}(\xi)=\sigma_{\ell\mu e}(-\xi)$ and $\sigma_{\ell\mu o}(\xi)=-\sigma_{\ell\mu o}(-\xi)$, and putting $r=(C_s^2/u_i^2)(\rho_i/d)^2\equiv(1+Z\gamma_eT_e/\gamma_iT_i)(\rho_i/d)^2$,

$$\sigma_{n\ell\mu j}(\xi)=\frac{(\beta_e\Phi_o/2)^2r}{(\omega_o/\omega_{ci})^2-1}\int_0^{\xi}d\eta e^{-\frac{\eta^2}{2}}\bar{F}_j(\mu,\eta)\left[\frac{\partial\bar{F}_o(\mu,\eta)}{\partial\eta}\right]\frac{\partial}{\partial\eta}\left\{\left[\frac{\partial\bar{F}_o(\mu,\eta)}{\partial\eta}\right]^2\right.$$
$$\left.-\left[\mu-\left(1-\left(\frac{\omega_{ci}}{\omega_o}\right)^2\right)\frac{d^2}{h^2}(2\zeta^2-1)\right]\left[\bar{F}_o(\mu,\eta)\right]^2\right\}, \tag{70}$$

with $\sigma_{n\ell\mu e}(\xi)=\sigma_{n\ell\mu e}(-\xi)$ and $\sigma_{n\ell\mu o}(\xi)=-\sigma_{n\ell\mu o}(-\xi)$. Here, eq. (60) is rewritten as, eliminating $\nu(\zeta)$ with the use of eq. (61),

$$\mu(\zeta)=\left[(\omega_o/\omega_{ci})^2-1\right]r^{-1}\left[1-(\omega_{ci}/\omega_o)^2r(d^2/h^2)(1-\zeta^2)\right]. \tag{71}$$

And then the expression (63) can be put into the form

$$\Phi_\mu^{(1)}(\xi,\zeta)/\Phi_o=\bar{F}_o(\mu,\xi)+F_e(\mu,\ell)F_o(\mu,\ell)\left\{\left[\sigma_{\ell\mu e}(\xi)-\sigma_{\ell\mu e}(\ell)+\sigma_{\ell\mu o}(\ell)\right]\bar{F}_o(\mu,\xi)\right.$$
$$\left.-\sigma_{\ell\mu o}(\xi)\bar{F}_e(\mu,\xi)\right\}+F_e(\mu,\ell)F_o(\mu,\ell)f_o^2(\zeta)\left\{\left[\sigma_{n\ell\mu e}(\xi)-\sigma_{n\ell\mu e}(\ell)\right.\right.$$
$$\left.+\sigma_{n\ell\mu o}(\ell)\right]\bar{F}_o(\mu,\xi)$$
$$\left.-\sigma_{n\ell\mu o}(\xi)\bar{F}_e(\mu,\xi)\right\}\equiv\bar{F}_o(\mu,\xi)+F_e(\mu,\ell)F_o(\mu,\ell)\left\{\left[(\sigma_{\ell\mu e}(\xi)-\sigma_{\ell\mu e}(\ell)\right.\right.$$
$$+\sigma_{\ell\mu e}(\ell))$$
$$\left.+f_o^2(\zeta)(\sigma_{n\ell\mu e}(\xi)-\sigma_{n\ell\mu e}(\ell)+\sigma_{n\ell\mu o}(\ell))\right]\bar{F}_o(\mu,\xi)-\left[\sigma_{\ell\mu o}(\xi)\right.$$
$$\left.\left.+f_o^2(\zeta)\sigma_{n\ell\mu o}(\xi)\right]\bar{F}_e(\mu,\xi)\right\}, \tag{72}$$

where we note $f_o^2(\zeta)=\exp(-\zeta^2)$ and that, as easily predicted, $\Phi_\mu^{(1)}(\xi,\zeta)$ is anti-symmetrical with respect to $\xi=x/d$ and symmetrical with respect to $\zeta = z/h$, i.e., $\Phi_\mu^{(1)}(\xi,\zeta) = -\Phi_\mu^{(1)}(-\xi,\zeta) = \Phi_\mu^{(1)}(\xi,-\zeta)$, so that $\Phi_\mu^{(1)}(\xi,\zeta)$ vanishes at $\xi = 0$ for all values of ζ. And we confirm that $\Phi_\mu^{(1)}(\pm l,\zeta) = \Phi_{o\mu}^{(1)}(\pm l) = \Phi_o \bar{F}_o(\mu,\pm l) = \pm \Phi_o$, being independent of ζ and $\mu(\zeta)$.

By the way, we should notice the fact that the expressions for $\Phi_{o\mu}^{(1)}(\xi)$ or $\Phi_o^{(1)}(x,z)$, consequently for $\Phi_\mu^{(1)}(\xi,\zeta)$ or $\Phi^{(1)}(x,z)$, have been derived without any restriction on μ except that μ is a positive real number, including positive integers. In other words, the fundamental mode of the externally excited electrostatic ion cyclotron wave is considered to be an eigenmode in which μ is a continuous eigenvalue of the wavenumber since μ can be varied continuously by controlling the external and plasma parameters as seen in eq. (71). The potential for the fundamental component of the wave, $\phi^{(1)}(x,z)$, in the case $\mu=n$ (at least $\mu \approx n$ where n is a positive integer) may, however, be expected to produce a corresponding electric field $E^{(1)}(x,z) \equiv -(\partial/\partial r)\,\phi^{(1)}(x,z)$ more effectively than the potential in the case $\mu \neq n$ $(\mu > 0)$. Namely, the condition $\mu=n$ can be the optimum one to produce the strongest electric field as may be understood by numerical studies of eq. (72) for various values of μ. Therefore, we here try to reduce eqs. (69), (70) and (72) to suitable expressions for the case $\mu=n$ by employing such formulas as, excluding n=0 since $\mu > 0$,

$$H_n(\xi) = \left\{ \begin{array}{ll} (-1)^{n/2} \cdot (n-1)(n-3)\cdots 3\cdot 1\cdot F_e(n,\xi) & \text{for } n=2,4,6,\cdots \\ (-1)^{(n-1)/2} \cdot n(n-2)\cdots 3\cdot 1\cdot F_o(n,\xi) & \text{for } n=1,3,5,\cdots \end{array} \right\}, \quad (73)$$

and

$$h_n(\xi)=\begin{cases}(-1)^{n/2}\cdot 2^{n/2}\cdot(n/2)!\cdot F_o(n,\xi) & \text{for } n=2,4,6,\cdots \\ (-1)^{(n+1)/2}\cdot 2^{(n-1)/2}\cdot((n-1)/2)!\cdot F_e(n,\xi) & \text{for } n=1,3,5,\cdots\end{cases} \quad (74)$$,

where $H_n(\xi)$ is the Hermite polynomials, $h_n(\xi)$ the Hermite function of the second kind, and

$$h_o(\xi) = \xi F(\tfrac{1}{2}, \tfrac{3}{2}; \tfrac{\xi^2}{2}) = \int_0^{\xi} d\eta \exp(\tfrac{\eta^2}{2}) \; . \quad (75)$$

Note here that, if the boundary condition were such that $\Phi_{on}^{(1)}(\pm\infty)=0$, $h_n(\xi)$ could not be one of a pair of linearly independent solutions of eq. (59) with $\mu=n$. Noting relations such as $\bar{F}_e(n,\xi)=a_{ne}\bar{H}_n(\xi)+a_{no}\bar{h}_n(\xi)$, $\bar{F}_o(n,\xi)=a_{ne}\bar{h}_n(\xi)+a_{no}\bar{H}_n(\xi)$ and $F_e(n,\ell)F_o(n,\ell)=h_n(\ell)H_n(\ell)/n! \equiv A_n$ with definitions $a_{ne}=[1+(-1)^n]/2$, $a_{no}=[1-(-1)^n]/2$, $\bar{H}_n(\xi)=H_n(\xi)/H_n(\ell)$ and $\bar{h}_n(\xi)=h_n(\xi)/h_n(\ell)$, we can then find eqs. (69), (70) and (72) in the form

$$\sigma_{\ell nj}(\xi) = a_{ne}\sigma_{\ell nj}^{(e)}(\xi) + a_{no}\sigma_{\ell nj}^{(o)}(\xi) \; , \quad (76)$$

$$\sigma_{n\ell nj}(\xi) = a_{ne}\,\sigma_{n\ell nj}^{(e)}(\xi) + a_{no}\sigma_{n\ell nj}^{(o)}(\xi) \; , \quad (77)$$

and

$$\Phi_n^{(1)}(\xi,\zeta)=a_{ne}\,\Phi_n^{(1e)}(\xi,\zeta) + a_{no}\,\Phi_n^{(1o)}(\xi,\zeta) \; ,$$

$$\text{i.e., } \Phi_n^{(1)}(\xi,\zeta)=\begin{cases}\Phi_n^{(1e)}(\xi,\zeta) & \text{for } n=2,4,6,\ldots \\ \Phi_n^{(1o)}(\xi,\zeta) & \text{for } n=1,3,5,\ldots\end{cases} \quad (78)$$,

with

$$\Phi_n^{(1e)}(\xi,\zeta)/\Phi_o=\bar{h}_n(\xi)+A_n\Big\{\Big[\sigma_{\ell ne}^{(e)}(\xi)-\sigma_{\ell ne}^{(e)}(\ell)+\sigma_{\ell no}^{(e)}(\ell)\Big]\bar{h}_n(\xi)$$

$$-\sigma_{\ell no}^{(e)}(\xi)\ \bar{H}_n(\xi)\Big\}+A_n f_o^2(\zeta)\Big\{\Big[\sigma_{n\ell ne}^{(e)}(\xi)-\sigma_{n\ell ne}^{(e)}(\ell)+\sigma_{n\ell no}^{(e)}(\ell)\Big]\bar{h}_n(\xi)$$

$$-\sigma_{n\ell no}^{(e)}(\xi)\ \bar{H}_n(\xi)\Big\}, \qquad (79)$$

and

$$\Phi_n^{(1o)}(\xi,\zeta)/\Phi_o=\bar{H}_n(\xi)+A_n\Big\{\Big[\sigma_{\ell ne}^{(o)}(\xi)-\sigma_{\ell ne}^{(o)}(\ell)+\sigma_{\ell no}^{(o)}(\ell)\Big]\bar{H}_n(\xi)$$

$$-\sigma_{\ell no}^{(o)}(\xi)\ \bar{h}_n(\xi)\Big\}+A_n f_o^2(\zeta)\Big\{\Big[\sigma_{n\ell ne}^{(o)}(\xi)-\sigma_{n\ell ne}^{(o)}(\ell)+\sigma_{n\ell no}^{(o)}(\ell)\Big]\bar{H}_n(\xi)$$

$$-\sigma_{n\ell no}^{(o)}(\xi)\ \bar{h}_n(\xi)\Big\}, \qquad (80)$$

where

$$\left.\begin{matrix}\sigma_{\ell ne}^{(e)}(\xi)\\ \sigma_{\ell no}^{(e)}(\xi)\end{matrix}\right\}=B_\ell\int_o^\xi d\eta\left\{\begin{matrix}\bar{H}_n(\eta)\\ \bar{h}_n(\eta)\end{matrix}\right\}\mathcal{L}_n\bar{h}_n(\eta),\quad \left.\begin{matrix}\sigma_{\ell ne}^{(o)}(\xi)\\ \sigma_{\ell no}^{(o)}(\xi)\end{matrix}\right\}=B_\ell\int_o^\xi d\eta\left\{\begin{matrix}\bar{h}_n(\eta)\\ \bar{H}_n(\eta)\end{matrix}\right\}\mathcal{L}_n\bar{H}_n(\eta), \qquad (81)$$

with $B_\ell=\left[\omega_{ci}^2/\omega_{poi}^2(0)\right]\left[(\omega_o/\omega_{ci})^2-1\right](C_{se}^2/C_s^2)$ (note $C_s^2/C_{se}^2=1+\gamma_i T_i/Z\gamma_e T_e$),

$\mathcal{L}_\mu=\partial^2/\partial\eta^2-s_\mu$ and $s_\mu=(d^2/h^2)(1-\zeta^2)\equiv(\omega_o/\omega_{ci})^2\left[r^{-1}-\mu((\omega_o/\omega_{ci})^2-1)^{-1}\right]$,

and

$$\left.\begin{matrix}\sigma_{n\ell ne}^{(e)}(\xi)\\ \sigma_{n\ell no}^{(e)}(\xi)\end{matrix}\right\}=B_{n\ell}\int_o^\xi d\eta\, e^{-\frac{\eta^2}{2}}\left\{\begin{matrix}\bar{H}_n(\eta)\\ \bar{h}_n(\eta)\end{matrix}\right\}\left[\frac{\partial\bar{h}_n(\eta)}{\partial\eta}\right]\frac{\partial}{\partial\eta}\left\{\left[\frac{\partial\bar{h}_n(\eta)}{\partial\eta}\right]^2-n'\left[\bar{h}_n(\eta)\right]^2\right\},$$

$$\left.\begin{matrix}\sigma_{n\ell ne}^{(o)}(\xi)\\ \sigma_{n\ell no}^{(o)}(\xi)\end{matrix}\right\}=B_{n\ell}\int_o^\xi d\eta\, e^{-\frac{\eta^2}{2}}\left\{\begin{matrix}\bar{h}_n(\eta)\\ \bar{H}_n(\eta)\end{matrix}\right\}\left[\frac{\partial\bar{H}_n(\eta)}{\partial\eta}\right]\frac{\partial}{\partial\eta}\left\{\left[\frac{\partial\bar{H}_n(\eta)}{\partial\eta}\right]^2-n'\left[\bar{H}_n(\eta)\right]^2\right\},$$

$$(82)$$

with $B_{n\ell}=(\beta_e \Phi_o/2)^2 r\left[(\omega_o/\omega_{ci})^2-1\right]^{-1}$, $n'=\mu'(n)$ and $\mu'(\mu)=$

$\mu-\left[1-(\omega_{ci}/\omega_o)^2\right](d^2/h^2)(2\zeta^2-1)$

$\equiv -\mu+\left[(\omega_o/\omega_{ci})^2-1\right]r^{-1}\left[2-(\omega_{ci}/\omega_o)^2 r(d^2/h^2)\right]$. In eliminating ζ^2 from the expressions for $\mathcal{D}_\mu$ and $\mu'(\mu)$, we have used

$$\zeta^2=1-(h^2/d^2)(\omega_o/\omega_{ci})^2\left[r^{-1}-\mu((\omega_o/\omega_{ci})^2-1)^{-1}\right]\equiv 1-(h^2/d^2)s_\mu, \quad (83)$$

which is obtained by solving eq. (71) for ζ^2. And note that "$k_z(z)$ is real" $\rightleftarrows \zeta^2 \leq 1 \rightleftarrows \mu r \equiv \mu(1+Z\gamma_e T_e/\gamma_i T_i)(\rho_i/d)^2 \lesseqgtr (\omega_o/\omega_{ci})^2-1$. Further, we should notice here that, since $d^2/h^2 \ll 1$, ζ^2 is very sensitive even to a slight change of μ so that, within the region $|\zeta| \leq 1$ (i.e., $|z| \leq h$), μ may approximately be treated as a constant. Namely, even if μ is fixed, the potential excited on such a pair of planes that ζ^2 is constant may approximately be regarded as the same potential as excited on any other planes (perpendicular to the z-axis) in the region $|\zeta| \leq 1$.

Among such various modes that $\mu=n=1,2,3,\cdots$, the potential $\phi^{(1)}(x,z)$ for $n=1$, being rewritten as $\Phi_1^{(1)}(\xi,\zeta)f_o(\zeta)$ in the ξ-ζ space, can favorably be expected to be excited most effectively under an ideal situation. And so the resultant electric field can then be dominant. Further, in case of sufficiently small values of both B_ℓ and $B_{n\ell}$, we have $\Phi_1^{(1)}(\xi,\zeta)=\Phi_1^{(1o)}(\xi,\zeta) \approx \bar{H}_1(\xi)=\xi/\ell \equiv x/L$, so that the x-component of the corresponding electric field becomes such a dipole field as $-d^{-1}(\partial/\partial\xi)\,\Phi_1^{(1)}(\xi,\zeta)f_o(\zeta)\,\cos\omega_o t \approx -(\Phi_o/L)\left[\exp(-z^2/2h^2)\right]\cos\omega_o t$. Hence, the x-component of the dominant electric field due to $\Phi_\mu^{(1)}(\xi,\zeta)$ with $\mu=1$ or $\mu \approx 1$ can approximately be regarded as a dipole field so long as $B_\ell \ll 1$ and $B_{n\ell} \ll 1$. We should, however, note that such a dipole-like field is not realized, of course, unless $\left[(\omega_o/\omega_{ci})^2-1\right]r^{-1}$ is nearly equal to unity so that $\mu=1$ or $\mu\approx 1$ where $r=(1+Z\gamma_e T_e/\gamma_i T_i)(\rho_i/d)^2$ and μ is given by eq. (71), and also that any other mode can be dominantly excited according to an experimentally controlled value of μ though we have above supposed such an ideal situation that $\mu=1$.

The second task is to solve the set of equations (27) to (29) consistently with the boundary condition that $\phi^{(2)}(\pm L,z)=0$ (see eq. (30)), making use of the solutions of eqs. (24) to (26). To do this, we first solve eq. (27) for $\delta\mathbf{v}_\sigma^{(2)}$, finding the expressions

$$\delta v_{\sigma x}^{(2)} = -\frac{i\Omega_o}{\Omega_o^2-\omega_{c\sigma}^2}\left[u_\sigma^2\frac{\partial}{\partial x}(\beta_\sigma\Psi_\sigma)-\left(1+\frac{\omega_{c\sigma}^2}{\omega_o\Omega_o}\right)f_{\sigma x}\right], \tag{84}$$

$$\delta v_{\sigma y}^{(2)} = -\frac{\omega_{c\sigma}}{\Omega_o^2-\omega_{c\sigma}^2}\left[u_\sigma^2\frac{\partial}{\partial x}(\beta_\sigma\Psi_\sigma)-\left(1+\frac{\Omega_o}{\omega_o}\right)f_{\sigma x}\right], \tag{85}$$

and

$$\delta v_{\sigma z}^{(2)} = -\frac{i\Omega_o}{\Omega_o^2}\left[u_\sigma^2\frac{\partial}{\partial z}(\beta_\sigma\Psi_\sigma)-f_{\sigma z}\right], \tag{86}$$

where $\Omega_o=2\omega_o$,

$$\beta_\sigma\Psi_\sigma(x,z) = \beta_\sigma\phi^{(2)}+\delta n_\sigma^{(2)}/N_\sigma-(\delta n_\sigma^{(1)}/N_\sigma)^2/4, \tag{87}$$

$$\mathbf{f}_\sigma(x,z) = -(1/2)\delta\mathbf{v}_\sigma^{(1)}\cdot(\partial/\partial\mathbf{r})\delta\mathbf{v}_\sigma^{(1)}, \tag{88}$$

and we have used the relation $f_{\sigma y}=(\omega_{c\sigma}/i\omega_o)f_{\sigma x}$. For the electron fluid, in a similar manner as done in obtaining $\delta\mathbf{v}_e^{(1)}$, noting that $f_{ex}\approx 0$, $f_{ey}\approx 0$ and $f_{ez}=-(\partial/\partial z)(\delta v_{ez}^{(1)})^2/4$ with eq. (36), we find

$$\delta v_{ex}^{(2)} \approx 0, \quad \delta v_{ey}^{(2)} \approx 0 , \tag{89}$$

and

$$\delta v_{ez}^{(2)} = -\frac{iu_e^2}{\Omega_o}\frac{\partial}{\partial z}\left[\beta_e\Psi_e(z) - \frac{1}{4}\frac{u_e^2}{\omega_o^2}\left(\frac{d}{dz}\beta_e\phi(z)\right)^2\right], \tag{90}$$

where we have required the relation $\Psi_e(x,z)\equiv\Psi_e(z)$ so as to be consistent with eq. (89), i.e., $\partial\Psi_e/\partial x=0$. Further, employing eqs. (35) to (38) in eqs. (28) and (87) with σ=e, we obtain

$$\frac{\delta n_e^{(2)}}{N_e} = -\frac{u_e^2}{\Omega_o}\frac{1}{N_e}\frac{\partial}{\partial z}\left\{N_e\frac{d}{dz}(\beta_e\Psi_e) - \frac{u_e^2}{\omega_o^2}\frac{N_e}{4}\left[\frac{d}{dz}\left(\frac{d}{dz}\beta_e\phi\right)^2 + \frac{2}{N_e^2}\frac{d}{dz}\left(N_e\frac{d}{dz}\beta_e\phi\right)^2\right]\right\} \approx -\beta_e\phi^{(2)} + (\beta_e\phi^{(1)})^2/4 , \tag{91}$$

where $\beta_e\Psi_e(z)$ has been discarded compared with $-(u_e^2/\Omega_o^2)N_e^{-1}(\partial/\partial z)N_e(d/dz)\beta_e\Psi_e(z) \sim -(K_z u_e/\Omega_o)^2$ since $\Omega_o \equiv 2\omega_o \ll K_z u_e$. Note here that $\phi(z)$ can be expressed in terms of $\phi^{(1)}$ through eq. (38). And so $\Psi_e(z)$ can be expressed in terms of $\phi^{(1)}$ and $\phi^{(2)}$ through eq. (91). Furthermore, using eqs. (39) and (91) in eq. (29), we immediately find

$$\frac{\delta n_i^{(2)}}{N_i} = -\left(1-\lambda_{De}^2\frac{\partial^2}{\partial x^2}\right)\beta_e\phi^{(2)} + \frac{1}{4}(\beta_e\phi^{(1)})^2 \approx \frac{\delta n_e^{(2)}}{N_e}, \tag{92}$$

which is equivalent to $e_i\delta n_i + e_e\delta n_e \approx 0$, since $|\lambda_{De}^2(\partial^2/\partial x^2)\beta_e\phi^{(2)}| \sim (K_x^2C_{se}^2/\omega_{pi}^2)|\beta_e\phi^{(2)}| \lesssim O[(\omega_{ci}^2/\omega_{pi}^2)(\beta_e\Phi_o)^2]$. And then we obtain the second harmonic components of the ion fluid velocity as, approximating as $\beta_i\Psi_i(x,z)\approx(1+\gamma_iT_i/Z\gamma_eT_e)\beta_i\phi^{(2)}$,

$$\delta v_{ix}^{(2)} = -\frac{i\Omega_o}{\Omega_o^2-\omega_{ci}^2}\left[(1+\frac{\gamma_i T_i}{Z\gamma_e T_e})\frac{e_i}{m_i}\frac{\partial\phi^{(2)}}{\partial x} -(1+\frac{\omega_{ci}^2}{\omega_o\Omega_o})f_{ix}\right] , \quad (93)$$

$$\delta v_{iy}^{(2)} = -\frac{\omega_{ci}}{\Omega_o^2-\omega_{ci}^2}\left[(1+\frac{\gamma_i T_i}{Z\gamma_e T_e})\frac{e_i}{m_i}\frac{\partial\phi^{(2)}}{\partial x} -(1+\frac{\Omega_o}{\omega_o})f_{ix}\right] , \quad (94)$$

and

$$\delta v_{iz}^{(2)} = -\frac{i\Omega_o}{\Omega_o^2}\left[(1+\frac{\gamma_i T_i}{Z\gamma_e T_e})\frac{e_i}{m_i}\frac{\partial\phi^{(2)}}{\partial z} - f_{iz}\right] , \quad (95)$$

where $f_{ix}(x,z)$ and $f_{iz}(x,z)$ take the approximate form

$$\left.\begin{matrix} f_{ix} \\ f_{iz} \end{matrix}\right\} = \frac{1}{4\omega_o^2}(1+\frac{\gamma_i T_i}{Z\gamma_e T_e})^2(\frac{e_i}{m_i})^2\left\{\begin{matrix} R_i\partial/\partial x \\ \partial/\partial z \end{matrix}\right\}\left[R_i(\frac{\partial\phi^{(1)}}{\partial x})^2+(\frac{\partial\phi^{(1)}}{\partial z})^2\right] , \quad (96)$$

with $R_i=\omega_o^2/(\omega_o^2-\omega_{ci}^2)$. And note such relations as $u_i^2\beta_i=e_i/m_i=|\beta_e|C_{se}^2$ and $(1+\gamma_i T_i/Z\gamma_e T_e)e_i/m_i=|\beta_e|C_s^2$. Here we point out that the neglect of $\Delta_\sigma^{(2)}$ in eq. (27) can also be justified since $\Delta_\sigma^{(2)} \sim \rho_\sigma^2 L_{on}^{-2} \sim 0$ in similar manners as done for $\Delta_\sigma^{(1)}$ below eqs. (37) and (42). With the help of eqs. (40) to (42) and (91) to (95), the equation of continuity (28) for the ion thus leads to a wave equation governing $\phi^{(2)}(x,z)$. That is,

$$\mathcal{L}^{(2)}\phi^{(2)}(x,z) \equiv \left\{\left[\frac{\partial}{\partial x}+\frac{1}{N_{oi}(x)}\frac{dN_{oi}(x)}{dx}\right]\frac{\partial}{\partial x}\right.$$

$$\left. + \frac{\Omega_o^2-\omega_{ci}^2}{C_s^2}(1+\frac{C_s^2}{\Omega_o^2}\frac{\partial^2}{\partial z^2})\right\}\phi^{(2)}(x,z) = \Lambda^{(2)}(\phi_o^{(1)},x,z), \quad (97)$$

with

$$\Lambda^{(2)}(\phi_o^{(1)},x,z)=\frac{|\beta_e|c_s^2}{4\omega_o^2}\Big\{(1+\frac{\omega_{ci}^2}{\omega_o\Omega_o})\frac{\omega_o^2}{\omega_o^2-\omega_{ci}^2}\Big[\frac{\partial}{\partial x}+$$

$$+\frac{1}{N_{oi}(x)}\frac{dN_{oi}(x)}{dx}\Big]\frac{\partial}{\partial x}$$

$$-\frac{\Omega_o^2-\omega_{ci}^2}{c_s^2}(1-\frac{c_s^2}{\Omega_o^2}\frac{\partial^2}{\partial z^2})\Big\}\Big[\frac{\omega_o^2}{\omega_o^2-\omega_{ci}^2}\Big(\frac{\partial\phi_o^{(1)}}{\partial x}\Big)^2+\Big(\frac{\partial\phi_o^{(1)}}{\partial z}\Big)^2\Big], \tag{98}$$

where we have approximated as $N_i(x,z)\approx N_{oi}(x)$, and retained only the lowest order terms with respect to $\omega_{ci}^2/\omega_{poi}^2(0)\ll 1$ so that $\phi^{(1)}(x,z)\approx\phi_o^{(1)}(x,z)$ in the inhomogeneous term $\Lambda^{(2)}$ of eq. (97). Now we assume such a form as

$$\phi^{(2)}(x,z)=\Phi^{(2)}(x,z)f_o(z)\ , \tag{99}$$

substitute eqs. (99) and (53) into the wave equation (97) with eq. (98), and rewrite this equation in the $\xi-\zeta$ space as, using eqs. (50′) and (51′) and rewriting $\Phi^{(2)}(x,z)$ as $\Phi_{\lambda\mu}^{(2)}(\xi,\zeta)$,

$$\mathcal{L}_\xi^{(2)}\ \Phi_{\lambda\mu}^{(2)}(\xi,\zeta)\equiv(\frac{\partial^2}{\partial\xi^2}-\xi\frac{\partial}{\partial\xi}+\lambda)\ \Phi_{\lambda\mu}^{(2)}(\xi,\zeta)=\Lambda_{\lambda\mu}^{(2)}(\Phi_{o\mu}^{(1)},\xi,\zeta), \tag{100}$$

with

$$\Lambda_{\lambda\mu}^{(2)}(\Phi_{o\mu}^{(1)},\xi,\zeta)=|\beta_e|\,f_o(\zeta)P(\frac{\partial^2}{\partial\xi^2}-\xi\frac{\partial}{\partial\xi}-Q_\lambda)\Big[\Big(\frac{\partial\Phi_{o\mu}^{(1)}}{\partial\xi}\Big)^2+S_\lambda\Big(\Phi_{o\mu}^{(1)}\Big)^2\Big], \tag{101}$$

where

$$\lambda(\zeta)=\big[(2\omega_o/\omega_{ci})^2-1\big]r^{-1}\big[1-(\omega_{ci}/2\omega_o)^2r(d^2/h^2)(1-\zeta^2)\big]\quad>0, \tag{102}$$

or

$$\zeta^2=1-(h^2/d^2)(2\omega_o/\omega_{ci})^2\left[r^{-1}-\lambda((2\omega_o/\omega_{ci})^2-1)^{-1}\right], \qquad (102')$$

and

$$P=(\omega_{ci}/2\omega_o)^2\left[1+(\omega_{ci}/\omega_o)^2/2\right]\left[1-(\omega_{ci}/\omega_o)^2\right]^{-2}\cdot r>0, \qquad (103)$$

$$Q_\lambda=\left[1+\frac{1}{2}\left(\frac{\omega_{ci}}{\omega_o}\right)^2\right]^{-1}\left[1-\left(\frac{\omega_{ci}}{\omega_o}\right)^2\right]\left[\left(\frac{2\omega_{ci}}{\omega_o}\right)^2-1\right]\frac{1}{r}\left[1+\frac{1}{2}\left(\frac{\omega_{ci}}{\omega_o}\right)^2 r\frac{d^2}{h^2}(1-2\zeta^2)\right]$$

$$\approx 6\left[1+\frac{1}{2}\left(\frac{\omega_{ci}}{\omega_o}\right)^2\right]^{-1}\left[1-\left(\frac{\omega_{ci}}{\omega_o}\right)^2\right]\lambda\left[1-\frac{5}{6}\lambda r\left(\left(\frac{2\omega_o}{\omega_{ci}}\right)^2-1\right)^{-1}-\frac{1}{3}\left(\frac{\omega_{ci}}{2\omega_o}\right)^2 r\frac{d^2}{h^2}\right]>0, \qquad (104)$$

and also

$$S_\lambda=\left[1-\left(\frac{\omega_{ci}}{\omega_o}\right)^2\right]\frac{d^2}{h^2}\zeta^2=\left[1-\left(\frac{\omega_{ci}}{\omega_o}\right)^2\right]\left\{\left(\frac{2\omega_o}{\omega_{ci}}\right)^2\left[\lambda\left(\left(\frac{2\omega_o}{\omega_{ci}}\right)^2-1\right)^{-1}-r^{-1}\right]+\frac{d^2}{h^2}\right\}>0. \qquad (105)$$

Further, r is defined by $r=(C_s^2/u_i^2)(\rho_i/d)^2$ as before, and $\Phi_{o\mu}^{(1)}(\xi)$ is given by eq. (64). Note that eq. (102) takes such an explicit form as $\lambda(z)/d^2=\left[(\Omega_o^2-\omega_{ci}^2)/C_s^2\right]\left[1-C_s^2k_z^2(z)/\Omega_o^2\right]$ with $\Omega_o=2\omega_o$. The second task can be accomplished by solving eq. (100) with the boundary condition $\Phi_{\lambda\mu}^{(2)}(\pm\ell,\zeta)=0$ on the basis of the method of Green's function.[9] After similar calculations as done in finding $\Phi_\mu^{(1)}(\xi,\zeta)$, we obtain

$$\Phi_{\lambda\mu}^{(2)}(\xi,\zeta)=|\beta_e|\Phi_o^2F_e(\lambda,\ell)F_o(\lambda,\ell)\Big\{I_{\lambda\mu}(\xi,\zeta)$$
$$-(1/2)\left[\bar{F}_e(\lambda,\xi)-\bar{F}_o(\lambda,\xi)\right]I_{\lambda\mu}(-\ell,\zeta)\Big\}, \qquad (106)$$

together with

$$I_{\lambda\mu}(\xi,\zeta)=\int_\xi^\ell d\eta\left[\bar{F}_e(\lambda,\xi)\bar{F}_o(\lambda,\eta)-\bar{F}_e(\lambda,\eta)\bar{F}_o(\lambda,\xi)\right]e^{-\frac{\eta^2}{2}}\Lambda_{\lambda\mu}^{(2)}(\Phi_{o\mu}^{(1)}(\eta),\eta,\zeta)/(|\beta_e|\Phi_o^2), \qquad (107)$$

where $\bar{F}_j(\lambda,\xi)=F_j(\lambda,\xi)/F_j(\lambda,\ell)(j=e,o)$, and $F_j(\lambda,\xi)$ are defined by eqs. (66) and (67). Let us now introduce two even and odd functions of ξ as

$$\left.\begin{matrix}\sigma_{\lambda\mu e}(\xi)\\ \sigma_{\lambda\mu o}(\xi)\end{matrix}\right\}=\int_0^{\xi} d\eta\, e^{-\frac{\eta^2}{2}}\left\{\begin{matrix}\bar{F}_o(\lambda,\eta)\\ \bar{F}_e(\lambda,\eta)\end{matrix}\right\} U_{\lambda\mu}(\eta)=\left\{\begin{matrix}\sigma_{\lambda\mu e}(-\xi)\\ -\sigma_{\lambda\mu o}(-\xi)\end{matrix}\right\} , \tag{108}$$

respectively, and define the even function $U_{\lambda\mu}(\eta)$ by

$$U_{\lambda\mu}(\eta)=P\left(\frac{\partial^2}{\partial\eta^2}-\eta\frac{\partial}{\partial\eta}-Q_\lambda\right)\left[\left(\frac{\partial\bar{F}_o(\mu,\eta)}{\partial\eta}\right)^2+S_\lambda(\bar{F}_o(\mu,\eta))^2\right]. \tag{109}$$

And then the solution (106) can be put into the form

$$\Phi^{(2)}_{\lambda\mu}(\xi,\zeta)/\Phi_o=|\beta_e|\Phi_o F_e(\lambda,\ell)F_o(\lambda,\ell)f_o(\zeta)\Big\{\sigma_{\lambda\mu o}(\xi)\bar{F}_o(\lambda,\xi)$$
$$-\left[\sigma_{\lambda\mu e}(\xi)-\sigma_{\lambda\mu e}(\ell)+\sigma_{\lambda\mu o}(\ell)\right]\bar{F}_e(\lambda,\xi)\Big\} . \tag{110}$$

Here we see that $\Phi^{(2)}_{\lambda\mu}(\xi,\zeta)$ is symmetrical with respect to both ξ and ζ, i.e., $\Phi^{(2)}_{\lambda\mu}(\xi,\zeta)=\Phi^{(2)}_{\lambda\mu}(-\xi,\zeta)=\Phi^{(2)}_{\lambda\mu}(\xi,-\zeta)$ (note $\lambda(\zeta)==\lambda(-\zeta)$ as well as $\mu(\zeta)=\mu(-\zeta)$), and confirm that $\Phi^{(2)}_{\lambda\mu}(\pm\ell,\zeta)=0$.

By the way, λ of the second harmonic mode $\Phi^{(2)}_{\lambda\mu}(\xi,\zeta)$, as well as μ of the fundamental mode $\Phi^{(1)}_{\mu}(\xi,\zeta)$, is regarded as a continuous eigenvalue of the wavenumber since λ, being a positive real number, can also be varied continuously controlling the external and plasma parameters as seen in eq. (102). Roughly speaking, since $d^2/h^2\ll 1$, $\lambda/\mu\approx[(2\omega_o/\omega_{ci})^2-1]/[(\omega_o/\omega_{ci})^2-1]\gg 1$ for the case $1<\omega_o/\omega_{ci}<2$. If we consider a case $\omega_o/\omega_{ci}\approx\sqrt{2}$ for example, $\lambda/\mu\approx 7$ and then λ takes a positive integer when μ is a positive integer. Or, in general, even in such cases as $\omega_o/\omega_{ci}\neq\sqrt{2}$, λ can be positive integers for appropriate values of μ. Hence, for the purpose of practical use, we rewrite eqs. (108)

and (110) for the case $\lambda=m$ (m is a positive integer). The results are written as, employing eqs. (73) and (74),

$$\Phi^{(2)}_{m\mu}(\xi,\zeta)/\Phi_o=|\beta_e|\,\Phi_o\,A_m f_o(\zeta)\Big\{\sigma_{m\mu f}(\xi)\bar{h}_m(\xi)-\sigma_{m\mu s}(\xi)\bar{H}_m(\xi)$$

$$-\big[\sigma_{m\mu f}(\ell)-\sigma_{m\mu s}(\ell)\big]\big[a_{me}\bar{H}_m(\xi)+a_{mo}\bar{h}_m(\xi)\big]\Big\}\ , \qquad (111)$$

with

$$\left.\begin{matrix}\sigma_{m\mu f}(\xi)\\ \sigma_{m\mu s}(\xi)\end{matrix}\right\}=\int_0^{\xi}d\eta\, e^{-\frac{\eta^2}{2}}\left\{\begin{matrix}\bar{H}_m(\eta)\\ \bar{h}_m(\eta)\end{matrix}\right\}U_{m\mu}(\eta)=\left\{\begin{matrix}-(-1)^m\sigma_{m\mu f}(-\xi)\\ (-1)^m\sigma_{m\mu s}(-\xi)\end{matrix}\right.\ , \qquad (112)$$

where $a_{me}=\left[1+(-1)^m\right]/2$, $a_{mo}=\left[1-(-1)^m\right]/2$, $\bar{H}_m(\xi)=H_m(\xi)/H_m(\ell)$ and $\bar{h}_m(\xi)=h_m(\xi)/h_m(\ell)$. In addition, let us suppose μ to be a positive integer n, too. And the even function (109) is then rewritten as

$$U_{mn}(\eta)=a_{no}U_{mnf}(\eta)+a_{ne}U_{mns}(\eta)\ , \qquad (113)$$

together with

$$\left.\begin{matrix}U_{mnf}(\eta)\\ U_{mns}(\eta)\end{matrix}\right\}=P\Big(\frac{\partial^2}{\partial\eta^2}-\eta\frac{\partial}{\partial\eta}-Q_m\Big)\left\{\begin{matrix}\left[(\partial\bar{H}_n(\eta)/\partial\eta)^2+S_m(\bar{H}_n(\eta))^2\right]\\ \left[(\partial\bar{h}_n(\eta)/\partial\eta)^2+S_m(\bar{h}_n(\eta))^2\right]\end{matrix}\right\}. \qquad (114)$$

We have thus calculated the expressions for both fundamental and second harmonic components of the externally excited, non-linear electrostatic ion cyclotron wave potential in the stationary state. We then find that an appropriate electrostatic ion cyclotron wave is dominantly formed according to experimentally controlled values of μ and λ depending strongly on ω_o/ω_{ci} and $r=(1+Z\gamma_e T_e/\gamma_i T_i)(\rho_i/d)^2$ but weakly on d^2/h^2 and $\zeta=z/h$. Further,

μ and λ are considered to be continuous eigenvalues for fundamental and second harmonic components, respectively. However, when μ and λ are divided into distinct sections $\mu_j < \mu < \mu_{j+1}$ and $\lambda_k < \lambda < \lambda_{k+1}$ (j, k=0,1,2,$\cdots$, and $\mu_o > 0$ and $\lambda_o > 0$) so that, in each section, the number of roots of $\Phi_{\mu}^{(1)}(\xi,\zeta)=0$ or $\Phi_{\lambda\mu}^{(2)}(\xi,\zeta)=0$ for ξ in the region $|\xi| \leq \ell$ ($|x| \leq L$) is unchanged, the wavenumber also remains unchanged in each section and so μ and λ play effectively similar roles as the discrete eigenvalue of wavenumber, provided that $\ell(=L/d)$ be fixed at a certain value fairly greater than unity. Here note that the number of such roots depends on the half width ℓ because ℓ changes the signs of normalization factors such as $F_e(\mu,\ell)$ and $F_o(\mu,\ell)$, especially the sign of $F_o(\mu,\ell)$ as will be understood since $\Phi_{\mu}^{(1)}(\xi,\zeta)/\Phi_o \approx \bar{F}_o(\mu,\xi) \equiv F_o(\mu,\xi)/F_o(\mu,\ell)$. Furthermore, it is shown in this section that the x-component of the fundamental electric field of the wave can be a dipole-like field so long as the parameter $[(\omega_o/\omega_{ci})^2-1]r^{-1}$ is exactly or nearly equal to unity (note that the present theory is limited to the case $(\rho_i/d)^2 \ll 1$).

In the last place of this section, we should point out an interesting result that the condition $\omega_o \ll K_z u_e$ leads to the usual Boltzmann relation for the electron fluid even in inhomogeneous plasmas. Namely, making use of eqs. (23), (38) and (91), and noting such a relation as $\beta_e^2 \phi^2(x,z,t)/2$ $\approx [(\beta_e \phi^{(1)}/2)^2 \exp(i2\omega_o t)+(\beta_e \phi^{(1)*}/2)^2 \exp(-i2\omega_o t)]/2+\beta_e^2 |\phi^{(1)}|^2/4$, we can derive

$$\frac{\delta n_e(x,z,t)}{N_e(x,z)} \approx -\beta_e \phi(x,z,t) + \frac{1}{2}\beta_e^2 \phi^2(x,z,t) - \frac{1}{4}\beta_e^2 |\phi^{(1)}(x,z)|^2$$

$$\approx \exp[-\beta_e \phi(x,z,t)] - \langle \exp[-\beta_e \phi(x,z,t)] \rangle_t , \qquad (115)$$

which is accurate at least up to the second order with respect to $\beta_e\phi^{(1)}(x,z)$. Moreover, we wish to pay attention to other interesting features which, as well as $-\langle\exp(-\beta_e\phi)\rangle_t$ in eq. (115), have their origins in the consideration of $(u_\sigma^2/2)(\partial/\partial\mathbf{r})\langle(\delta n_\sigma/N_\sigma)^2\rangle_t$ in the pressure gradient term of eq. (8). Namely, we first obtain from eqs. (125) and (126)

$$\frac{\delta N_i}{N_{oi}} \equiv \frac{N_i - N_{oi}}{N_{oi}} \approx -\frac{\beta_e^2}{4}\left[\frac{C_s^2}{\omega_o^2-\omega_{ci}^2}\left|\frac{\partial\phi^{(1)}}{\partial x}\right|^2 + \frac{C_s^2}{\omega_o^2}\left|\frac{\partial\phi^{(1)}}{\partial z}\right|^2 - \left|\phi^{(1)}\right|^2 + C'\right], \quad (116)$$

with C' being constant in space and time. This is different from the result $\delta N/N_o \propto -|\partial\phi^{(1)}/\partial x|^2$+Const. which has previously been obtained, by computing only the ponderomotive forces and dropping $|\partial\phi^{(1)}/\partial z|^2$ for simplicity, for other waves such as lower-hybrid,[10] upper-hybrid[11] and Langmuir waves[12] in a point that $\delta N_i/N_{oi}$ contains an additional, unnegligible term proportional to $|\phi^{(1)}|^2$. This point is a new feature found in this paper at least for the non-linear electrostatic ion cyclotron wave. Next, we can accordingly expect this new term to bring about a corresponding term in the differential equation governing the nonlinear wave. Namely, taking the high-density limit $\omega_{ci}^2/\omega_{poi}^2(0)\rightarrow 0$ and neglecting $(\partial/\partial z)\omega_{pi}^2(x,z)$ in eq. (43), we obtain

$$\left\{\left[\frac{\partial}{\partial x} + \frac{1}{N_{oi}}\frac{dN_{oi}}{dx}\right]\frac{\partial}{\partial x} + \frac{\omega_o^2-\omega_{ci}^2}{C_s^2}\left(1 + \frac{C_s^2}{\omega_o^2}\frac{\partial^2}{\partial z^2}\right)\right\}\phi^{(1)}$$
$$-\frac{\beta_e^2}{4}\frac{\partial\phi^{(1)}}{\partial x}\frac{\partial}{\partial x}\left[\frac{C_s^2}{\omega_o^2-\omega_{ci}^2}\left|\frac{\partial\phi^{(1)}}{\partial x}\right|^2 + \frac{C_s^2}{\omega_o^2}\left|\frac{\partial\phi^{(1)}}{\partial z}\right|^2 - \left|\phi^{(1)}\right|^2\right] = 0. \quad (117)$$

This has in this section been solved analytically under a practical situation by treating the nonlinear terms as perturbations, but will be considered differently in a separate paper.

§4. The Density Profiles RF-Plugged by the Nonlinear Static Forces

In this section, we first calculate the nonlinear static forces $\mathbf{F}_{p\sigma}$ and $-(e_\sigma/m_\sigma)(\partial/\partial\mathbf{r})\phi_{s\sigma}$ in order to express $\phi_{p\sigma}(x,z)$ and $\phi_{s\sigma}(x,z)$ in terms of the fundamental mode potential $\phi^{(1)}(x,z)$. Starting from the definition (7), we find that $F_{p\sigma y}=0$,

$$F_{p\sigma x} \equiv -\frac{e_\sigma}{m_\sigma}\frac{\partial(R_\sigma\phi_{p\sigma})}{\partial x} = -\frac{1}{4}\frac{\partial}{\partial x}\left[|\delta v^{(1)}_{\sigma x}|^2+\frac{\omega_o^2}{\omega_o^2-\omega_{c\sigma}^2}|\delta v^{(1)}_{\sigma z}|^2\right], \qquad (118)$$

and

$$F_{p\sigma z} \equiv -\frac{e_\sigma}{m_\sigma}\frac{\partial\phi_{p\sigma}}{\partial z} = -\frac{1}{4}\frac{\omega_o^2-\omega_{c\sigma}^2}{\omega_o^2}\frac{\partial}{\partial z}\left[|\delta v^{(1)}_{\sigma x}|^2+\frac{\omega_o^2}{\omega_o^2-\omega_{c\sigma}^2}|\delta v^{(1)}_{\sigma z}|^2\right], \qquad (119)$$

where we have used such relations that $\delta v^{(1)}_{\sigma y}=(\omega_{c\sigma}/i\omega_o)\delta v^{(1)}_{\sigma x}$ and $(\partial/\partial z)\delta v^{(1)}_{\sigma x}=\left[\omega_o^2/(\omega_o^2-\omega_{c\sigma}^2)\right](\partial/\partial x)\delta v^{(1)}_{\sigma z}$, being obtained from eqs. (31) and (32). And we immediately find from eqs. (118) and (119) that

$$R_\sigma=\omega_o^2/(\omega_o^2-\omega_{c\sigma}^2)\ , \qquad (120)$$

and that $\phi_{p\sigma}=(m_\sigma/e_\sigma)(1/4R_\sigma)\left[|\delta v^{(1)}_{\sigma x}|^2+R_\sigma|\delta v^{(1)}_{\sigma z}|^2\right]+\text{Const.}$ With the help of eqs. (35), (36), (41) and (42), the latter equation now takes the following expressions:

$$\phi_{pe}=(m_e/e_e)(u_e^4/4\omega_o^2)\,|(d/dz)\beta_e\psi(z)|^2+\text{Const.}, \qquad (121)$$

and

$$\phi_{pi}=\frac{e_i}{m_i}\left(1+\frac{\gamma_i T_i}{Z\gamma_e T_e}\right)^2\frac{1}{4\omega_o^2}\left[R_i\left|\frac{\partial\phi^{(1)}}{\partial x}\right|^2+\left|\frac{\partial\phi^{(1)}}{\partial z}\right|^2\right]+\text{Const.}, \qquad (122)$$

where we have neglected terms of the order $\lambda^2_{Di}|(\partial^2\phi^{(1)}/\partial\mathbf{r}^2)/\phi^{(1)}| \sim K_x^2\lambda_{Di}^2=K_x^2u_i^2/\omega_{pi}^2<K_x^2C_s^2/\omega_{pi}^2 \lesssim \omega_{ci}^2/\omega_{pi}^2 \ll 1$. Note here that $\psi(z)$ is connected with $\phi^{(1)}(x,z)$ through eq. (38), and that ϕ_{pe} is

negligibly small as compared with ϕ_{pi} since $|\phi_{pe}/\phi_{pi}| \sim O(\omega_o^2/K_z^2 u_e^2)$ under the assumption $\omega_o \ll K_z u_e$. Further, with the use of eqs. (38) and (40), the definition (14) can approximately be reduced to

$$-(e_\sigma/m_\sigma)\partial\phi_{s\sigma}/\partial\mathbf{r} = (\beta_e^2/4)u_\sigma^2(\partial/\partial\mathbf{r})|\phi^{(1)}|^2 , \tag{123}$$

which yields

$$\phi_{s\sigma} = -(\beta_e^2/4\beta_\sigma)|\phi^{(1)}|^2 + \text{Const.} \tag{124}$$

Next, we wish to express in terms of $\phi^{(1)}(x,z)$ the density profiles rf-plugged by these nonlinear static forces. For this purpose, let us substitute eq. (18) into the requirement $N_i/N_{oi}=N_e/N_{oe}$ derived from eqs. (19) and (39) and employ eqs. (122) and (124). And then, we obtain the field-dependent density profiles

$$\frac{N_i(x,z)}{N_{oi}(x)} = \frac{N_e(x,z)}{N_{oe}(x)} = \exp\left[\frac{\beta_i\beta_e}{\beta_i-\beta_e}(\phi_{pi}-\phi_{pe}+\phi_{si}-\phi_{se})\right]$$
$$=\exp\left\{-|\beta_e|\left[\phi_{nl}(\phi^{(1)},x,z)+C\right]\right\} \approx 1-|\beta_e|\left[\phi_{nl}(\phi^{(1)},x,z)+C\right], \tag{125}$$

accompanied with

$$\phi_{nl}(\phi^{(1)},x,z) = \frac{|\beta_e|}{4}\left\{\frac{C_s^2}{\omega_o^2}\left[R_i\left|\frac{\partial\phi^{(1)}}{\partial x}\right|^2+\left|\frac{\partial\phi^{(1)}}{\partial z}\right|^2\right]-|\phi^{(1)}|^2\right\}, \tag{126}$$

where ϕ_{pe} has been discarded as a result, and the integration constant C is to be determined by a kind of boundary condition, for example, by such a condition that the particle number is conserved within the entire volume V under consideration, i.e.,

$$\int_V d\tau\,\delta N_i(x,z) \equiv \int_V d\tau\left[N_i(x,z)-N_{oi}(x)\right]=0 \quad (d\tau=dxdydz). \tag{127}$$

Let us hereafter try to consider this example in some detail. Substituting eq. (125) into the condition (127) yields

$$N_i(x,z)/N_{oi}(x) \simeq 1-|\beta_e|\left[\phi_{nl}(\phi^{(1)},x,z)-\langle\phi_{nl}(\phi^{(1)},x,z)\rangle_s\right], \tag{128}$$

where the space average notation is defined by $\langle F(x,z)\rangle_s = \int_V d\tau F(x,z)N_{oi}(x)/\int_V d\tau N_{oi}(x)$. Now, approximating as $\phi^{(1)}(x,z) \approx \phi_o^{(1)}(x,z) \equiv \Phi_o^{(1)}(x,z)f_o(z)$ rewritten as $\Phi_{o\mu}^{(1)}(\xi)f_o(\zeta)$ in the ξ-ζ space, and assuming a realistic situation in which $N_{oi}(x)$ and $f_o(z)$ take the Gaussian forms (50) and (51), we can, in the ξ-ζ space, put eqs. (125) and (126) into the form

$$N_i/N_{oi} \approx 1-|\beta_e| f_o^2(\zeta)\left[\phi_{nl\mu}(\Phi_{o\mu}^{(1)},\xi)-\langle_{nl\mu}(\Phi_{o\mu}^{(1)},\xi)\rangle_s\right], \tag{129}$$

where $f_o(\zeta)$, $\phi_{nl\mu}(\Phi_\mu^{(1)},\xi)$ and $\Phi_{o\mu}^{(1)}(\xi)$ are defined by eqs. (51'), (58) and (64), respectively. If we consider a special case in which $\mu=1$ and so $\Phi_{o1}^{(1)}(\xi)=\Phi_o \bar{H}_1(\xi)=\Phi_o\xi/l$, we can further reduce eq. (129) to

$$N_i/N_{oi} \approx 1-B_{nl}\left[\alpha_c^2-(\xi/l)^2\right]\exp(-\zeta^2)=1-B_{nl}\left[\alpha_c^2-(x/L)^2\right]\exp(-z^2/h^2), \tag{130}$$

with

$$\alpha_c^2 \equiv \left\langle\left(\frac{\xi}{l}\right)^2\right\rangle_s = \frac{1}{l^2}\int_{-l}^{l}d\xi\,\xi^2 e^{-\frac{\xi^2}{2}}\Big/\int_{-l}^{l}d\xi\, e^{-\frac{\xi^2}{2}}=\left(\frac{d}{L}\right)^2\left\{1-\left[\sum_{n=0}^{\infty}\frac{(L/d)^{2n}}{(2n+1)!!}\right]^{-1}\right\}, \tag{131}$$

where we have taken the limit $d^2/h^2 \to 0$ in $\phi_{nl1}(\Phi_{o1}^{(1)},\xi)$. And $B_{nl}=(\beta_e\Phi_o/2)^2 r\left[(\omega_o/\omega_{ci})^2-1\right]^{-1}$ with $r \approx (\omega_o/\omega_{ci})^2-1$, and $(2n+1)!! = (2n+1)(2n-1)\cdots 5\cdot 3\cdot 1$. We then see from eq. (130) that $\delta N_i \equiv N_i-N_{oi} < 0$ in the core region where $|x/L| < \alpha_c$. In practical situations, however, if we apply our theory to such a local volume that $|x| \leq L$ and $|z| \leq H$, and note that, for an arbitrary value of $\mu(\approx[(\omega_o/\omega_{ci})^2-1]r^{-1} > 0)$, $|\delta N_i/N_{oi}|$ is proportional to $f_o^2(z) \equiv \exp(-z^2/h^2)$, taking the maximum at $z=0$, it is not necessary to require the total particle number conservation (127). Because some of the plasma particles which are contained in the local volume before applying $\phi_{ext}(z,t) = \pm \Phi_o \cos \omega_o t \cdot \exp(-z^2/2h^2)$ at $x = \pm L$ are expected to be removed along $\mathbf{B}_o$ into such ambient regions that $|x| \leq L$ and $|z| > H$ after the excitation of the electrostatic ion cyclotron wave by $\phi_{ext}(z,t)$. For example, if $C \geq 0$, we have $N_i/N_{oi} \approx 1 - B_{n\ell}[(d'/L)^2-(x/L)^2]\exp(-z^2/h^2)$ with $d' \equiv d'(d,C) \geq d$ for the special case $\mu=1(r \approx (\omega_o/\omega_{ci})^2-1)$.

§5. Summary and Conclusions

We will here review some interesting results obtained in this paper. First, we have derived the functional form of the nonlinear electrostatic ion cyclotron wave potential $\phi(x,z,t)$ excited by the externally controllable exciter $\phi_{ext}(z,t) = \pm \Phi_0 \cos\omega_0 t \cdot \exp(-z^2/2h^2)$ at $x = \pm L$. The result is written as

$$\phi(x,z,t)=\exp(-z^2/2h^2)\Big[\Phi_{\mu}^{(1)}(x/d,z/h)\cos\omega_0 t + \Phi_{\lambda\mu}^{(2)}(x/d,z/h)\cos 2\omega_0 t + \cdots\Big] , \qquad (132)$$

where $\Phi_{\mu}^{(1)}(\xi,\zeta)$ and $\Phi_{\lambda\mu}^{(2)}(\xi,\zeta)$ are respectively given by eqs. (72) and (110) for the case in which μ and λ are continuous eigenvalues, and reduced to eqs. (78) and (111) for the special case in which μ and λ are posive integers.

In the core region where $|\xi| \ll \ell$ (or $|x| \ll L$), eqs. (72) and (110) are respectively approximated as, for the former case,

$$\Phi_{\mu}^{(1)}(\xi,\zeta) \approx \Phi_0 \bar{F}_0(\mu,\xi)\Big\{1+F_e(\mu,\ell)F_0(\mu,\ell)\Big[\sigma_{\ell\mu o}(\ell)-\sigma_{\ell\mu e}(\ell) + f_0^2(\zeta)(\sigma_{n\ell\mu o}(\ell)-\sigma_{n\ell\mu e}(\ell))\Big]\Big\} , \qquad (133)$$

and

$$\Phi_{\lambda\mu}^{(2)}(\xi,\zeta) \approx -|\beta_e|\, \Phi_0^2 F_e(\lambda,\ell)F_0(\lambda,\ell)\Big[\sigma_{\lambda\mu o}(\ell)-\sigma_{\lambda\mu e}(\ell)\Big]f_0(\zeta)\bar{F}_e(\lambda,\xi), \qquad (134)$$

and eqs. (78) and (111)(with eqs. (79) and (80)) respectively as, for the latter case (μ=n, λ=m),

$$\Phi_n^{(1)}(\xi,\zeta)\approx\begin{cases}\Phi_o\bar{h}_n(\xi)\Big\{1+A_n\Big[\sigma_{\ell no}^{(e)}(\ell)-\sigma_{\ell ne}^{(e)}(\ell)\\ \quad+f_o^2(\zeta)(\sigma_{n\ell no}^{(e)}(\ell)-\sigma_{n\ell ne}^{(e)}(\ell))\Big]\Big\}\text{ for } n=2,4,\dots,\\ \Phi_o\bar{H}_n(\xi)\Big\{1+A_n\Big[\sigma_{\ell no}^{(o)}(\ell)-\sigma_{\ell ne}^{(o)}(\ell)\\ \quad+f_o^2(\zeta)(\sigma_{n\ell no}^{(o)}(\ell)-\sigma_{n\ell ne}^{(o)}(\ell))\Big]\Big\}\text{ for } n=1,3,\dots,\end{cases}\tag{135}$$

and

$$\Phi_{mn}^{(2)}(\xi,\zeta)\approx-|\beta_e|\,\Phi_o^2\,A_m\big[\sigma_{mnf}(\ell)-\sigma_{mns}(\ell)\big]f_o(\zeta)\big[a_{me}\bar{H}_m(\xi)+a_{mo}\bar{h}_m(\xi)\big],\tag{136}$$

with eqs. (112) to (114). Further, in a special case in which $|1/2-\mu/2|\ll 3/2$ and $|\xi|=|x|/d<1$ ($|\xi|\ll\ell$), so that $F_o(\mu,\xi)\approx\xi$ (note eqs. (67) and (68)), the fundamental potential of the externally excited nonlinear electrostatic ion cyclotron wave approximately produces a dipole field perpendicular to the contant magnetic field $\mathbf{B}_o$.

Next, starting from the basic equations founded in §2, we have shown that, under the assumption $\omega_o/K_z\equiv\omega_o h\ll u_e$, the Boltzmann relation for the electron fluid, eq. (115), can be employed even in inhomogeneous plasmas as well as in homogeneous ones, with an accuracy at least up to the second order with respect to $\beta_e\,\phi^{(1)}(x,z)$. Therefore, we can make use of the Boltzmann relation (115) instead of a set of equations of motion and continuity for the electron fluid when analyzing an inhomogeneous plasma.

In the last place, we wish to point out that when expanding eq. (116), the expression for the density depletion $\delta N_i\equiv N_i(x,z)-N_{oi}(x)$ contains a new term proportional to $|\phi^{(1)}(x,z)|^2$

in addition to the usual terms proportional to $-|\partial\phi^{(1)}(x,z)/\partial x|^2$ and $-|\partial\phi^{(1)}(x,z)/\partial z|^2$. The contribution from $|\phi^{(1)}(x,z)|^2$ has appeared because of the inclusion of $-(\delta n_\sigma/N_\sigma)^2/2$ in expanding $\ln n_\sigma$ in a power series of $|\delta n_\sigma/N_\sigma| \sim |\beta_e\phi|$ as done below eq. (6), with the simultaneous use of the relation $\delta n_e(x,z)/N_e(x,z) \approx -\beta_e\,\phi^{(1)}(x,z)$ based on the assumption $\omega_o/K_z \ll u_e$. And this contribution becomes relatively important in the edge regions of the plasma slab. On the other hand, the contribution from $-|\partial\phi^{(1)}(x,z)/\partial x|^2$, as well as that from $-|\partial\phi^{(1)}(x,z)/\partial z|^2$, has its origin obviously in the ponderomotive force, and dominates over that from $|\phi^{(1)}(x,z)|^2$ in the core region of the slab ($|x| \ll L$).

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12) There are so many works on the nonlinear Langmuir waves. For example, we find some of them in the papers cited in Ref. 11.

Figure Caption

Fig. 1. The density profile $g_0(x)$ is shown for the case in which a pair of rf-potentials are absent. Here $g_0(x)=N_{o\sigma}(x)/N_{o\sigma}(0)=g_0(-x)$.

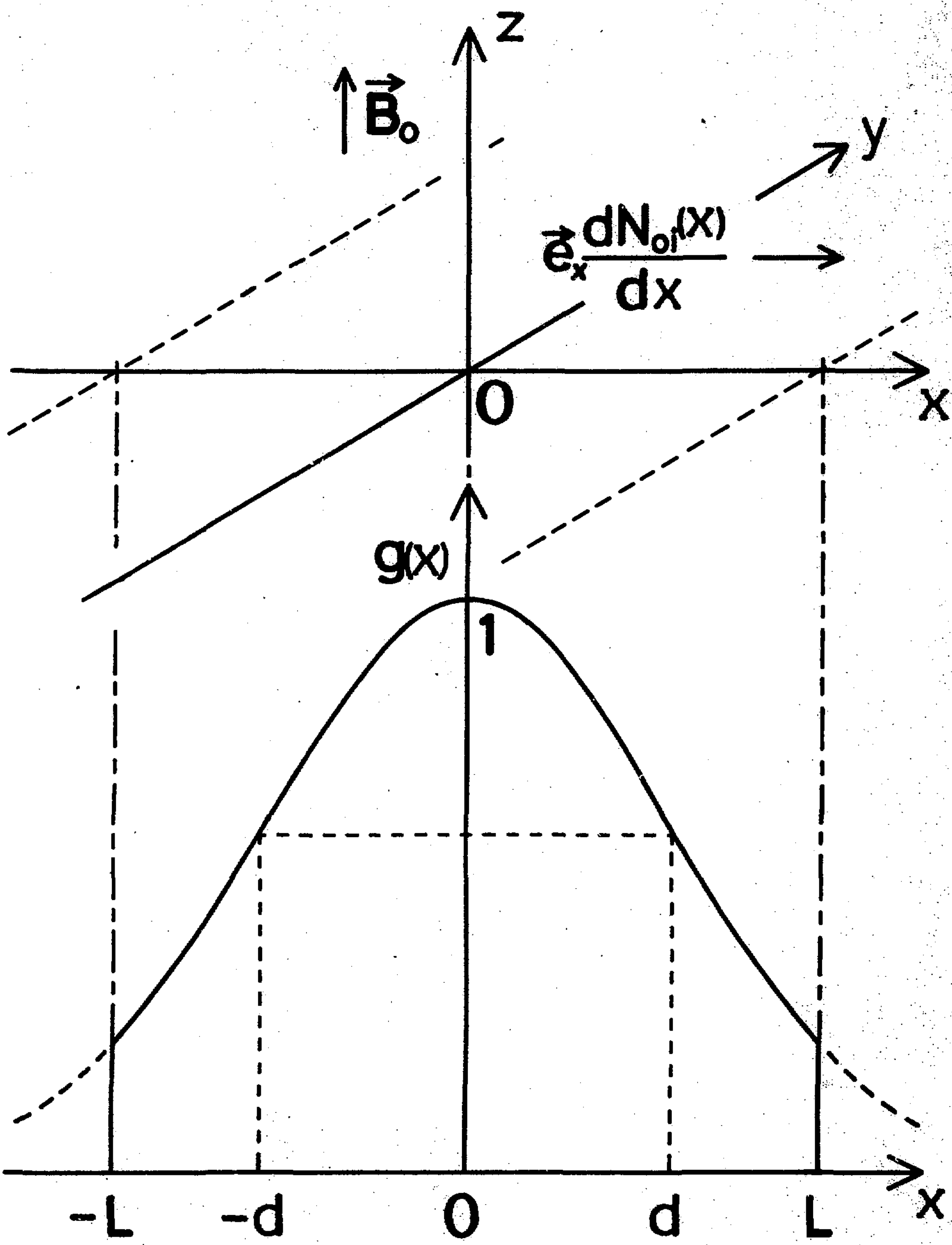

Fig. 1.