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Localized Solutions of Non-Linear Klein-Gordon Equations*

by

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Abstract

Non-dissipative, stationary solutions for a class of non-linear Klein-Gordon equations for a scalar field have been found explicitly. Since the field is different from zero only inside a sphere of definite radius, the solutions are called quantum droplets.

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The particle physicists got recently interested in certain stable, non-dissipative, localized solutions of some non-linear field equations, hoping that they may provide attractive models of extended hadrons or other elementary particles.[1] Such solutions called solitons or lumps have been known since quite a long time,[2] however, only for purely classical fields and in the two-dimensional space time. There are several non-existence[3] and existence[4] theorems but until quite recently there were no explicit solutions for the relativistic equations in the four-dimensional Minkowski space-time. Gaussian shaped solutions of a non-relativistic, non-linear equation have been found by J. Bialynicki-Birula and J. Mycielski[5] who also pointed out recently that essentially the same stationary, Gaussian shaped solutions exist for relativistic, non-linear Schrödinger or Klein-Gordon equations for a complex scalar field. In two previous papers[6] the author has provided droplet-like stationary solutions of positive energy and parity and spin 1/2 for a class of non-linear Dirac spinor equations in the four-dimensional space-time, as well as Gaussian shaped solutions for another class of non-linear Dirac equations.

In this paper we shall see that also certain classes of non-linear Klein-Gordon equations in four-dimensions possess stationary, non-dissipative droplet-like solutions of definite spin and parity. Consider the complex scalar field ϕ described by the Lagrangian density

$$L = \phi^*_{,\alpha}\phi^{,\alpha} - \mu^2 U(\kappa) \quad , \tag{1}$$

where $U(\kappa)$ is a dimensionless function of the dimensionless invariant

$$\kappa = A^{-2}\phi^*\phi \quad . \tag{2}$$

The normalization constant A has the dimension of ϕ, and the constant μ has dimension of cm^{-1}.

The corresponding equation of motion has the form[6]

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + V(\kappa)\right)\phi(x) = 0 \tag{3}$$

with

$$V(\kappa) = \mu^2 \frac{dU(\kappa)}{d\kappa} \quad . \tag{4}$$

We are looking for solutions which are stationary, with positive energy E, and are eigenfunctions of the angular momentum. Such solutions must have the following general form

$$\phi(x) = A \exp(-iEt)\mathcal{R}_\ell(r)V^\ell_m(\theta,\phi) \quad . \tag{5}$$

Expressions of this form with definite ℓ and m can be solutions of (3) only if $V(\kappa)$ is spherically symmetric. This is so if κ itself is spherically symmetric which happens only for $\ell = 0$. Only then

$$\kappa = A^{-2}R^2(r) = \kappa(r) \tag{6}$$

and our equation (3) can be reduced to an ordinary but non-linear differential equation for the radial function

$$r^{-2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = (V(\kappa(r)) - E^2)R \tag{7}$$

Guided by the droplet solutions found for the Dirac spinor case[6] we have tried the following class of non-linear potentials

$$V = \mu^2(1 - b\kappa^{-\lambda})^2 \tag{8}$$

where b and λ are some non-negative constants. It turns out that our equation (7) has indeed droplet-like solutions of the form

$$R = \begin{cases} A(1 - \alpha^2 r^2)^{n/2} & \text{for } 0 \leq \alpha r \leq 1 \\ 0 & \text{for } \alpha r > 1 \end{cases} \tag{9}$$

provided that

$$E = \mu \;, \quad n = \lambda^{-1} \;, \quad b = 2(n-2)(n+1)^{-1}$$
$$\alpha^2 = \mu^2 4(n-2)n^{-1}(n+1)^{-2} \;, \quad n > 2 \quad . \tag{10}$$

The last condition (or $0 < \lambda < 1/2$) guarantees that not only $R(r)$ but also the first derivative $R'(r)$ are continuous at $r = \alpha^{-1} = a$. We see that the constant b is not free but is a definite function of the exponent λ (or n). Thus the final form of the potential $V(\kappa)$ is:

$$V = \mu^2\{1 - 2(1-2\lambda)(1+\lambda)^{-1}\kappa^{-\lambda}\}^2 \qquad (11)$$

The corresponding expression for $U(\kappa)$ is

$$U(\kappa) = (\kappa + c\kappa^{1-\lambda} + d\kappa^{1-2\lambda})A^2 , \qquad (12)$$

with

$$c = -4(1-2\lambda)(1-\lambda^2)^{-1} , \quad d = 4(1-2\lambda)(1+\lambda)^{-2} . \qquad (13)$$

Although $V(\kappa)$ has a weak singularity at $\kappa = 0$, there is no singularity in the Lagrangian density or energy density, as can be seen immediately from (12), due to the restriction: $0 < \lambda < 1/2$. The potential $V(\kappa)$ is thus infinite at $\kappa = 0$, then decreases rapidly with increasing κ, reaches the minimum value equal to zero at the point $\kappa = b^n$ and then increases monotonically to the asymptotic value μ^2.

It follows from (6) and (9) that

$$\kappa(r) = (1 - \alpha^2r^2)^n . \qquad (14)$$

Inserting this expression into (8) we find the following form of the self-consistent potential as a function of r:

$$V = \mu^2(1 - b(1 - \alpha^2r^2)^{-1})^2 \qquad (15)$$

with b depending of course on the choice of $n = \lambda^{-1}$. It can easily be checked that for $2 < n < 5$ the minimum value of (15) is equal to zero but the function is not monotonic, as it has a maximum at $r = 0$. For $n \geq 5$ the function is monotonic with the minimum lying at $r = 0$. However for $n > 5$ the minimum value is greater than zero. With r approaching the value a from below, the potential (15) increases rapidly to infinity producing what may be called an infinite well or a bag which keeps the field inside the sphere like in the case of a classical droplet. However, in contrast to the classical droplet, our quantum droplet has smooth, continuous distribution of matter with no sharp edges.

Our solution corresponds to a scalar droplet at rest. Applying suitable Lorentz transformations one can obtain arbitrarily moving droplets. A linear superposition of two non-overlapping droplet solutions is again a rigorous solution of our non-linear equation of motion.

Similar droplet-like solutions of non-linear equations can be also found for non-linear Klein-Gordon equations for fields with higher spin values. More detailed investigation of various physical aspects of these and other localized solutions will be presented in another paper.

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