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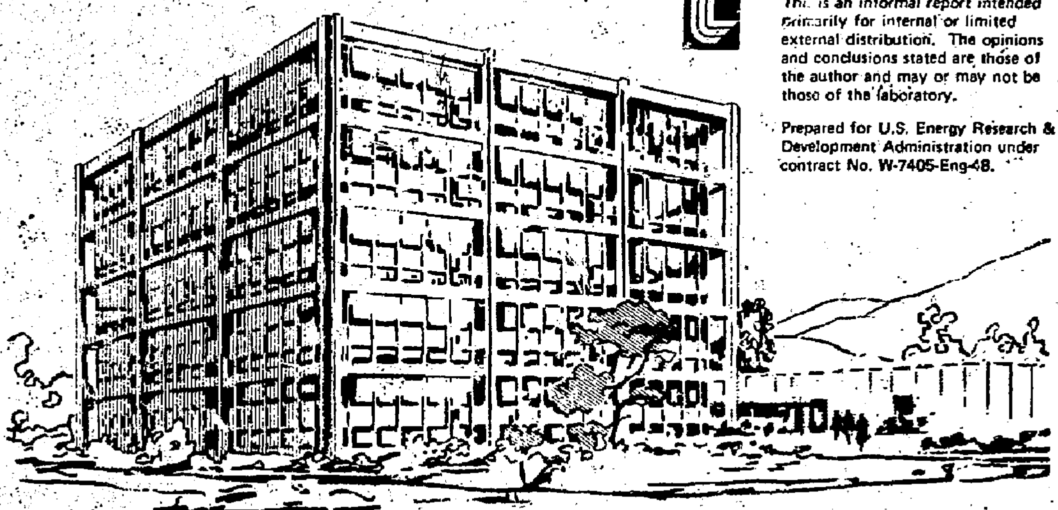

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Differential Equation for Alfvén Ion Cyclotron Waves in Finite-length Plasma*

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ABSTRACT

We find the fourth-order differential equation describing an Alfvén-ion-cyclotron wave propagating along a magnetic field of varying intensity. The equation is self-adjoint and possesses non-trivial turning points. The final form of the equation is checked using MACSYMA, a system for performing algebra on a computer.

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Electrostatic instabilities occurring in mirror-confined ion velocity-distributions have been thoroughly investigated. The study of electromagnetic instabilities occurring in mirror-confined distribution is still in progress. Of greatest concern is the Alfven-ion-cyclotron (AIC) mode.[1,2] This instability can become absolute; further, the shift from convective to absolute[3] occurs at modest β-values and anisotropy-values (e.g. $\beta \sim 0.5$ for $T_{\perp}:T_{\parallel} \sim 2:1$)[4] in homogeneous plasma. In the case of slight anisotropy, stabilization via finite length is achieved by limiting the convective amplification.[5] In the case of strong anisotropy, stabilization via finite length is achieved by spatially confining the mode sufficiently to depress its absolute growthrate.[6] An estimate of such growthrate depression may be arrived at by starting from the homogeneous Vlasov dispersion function and making the WKB approximation. The further approximation that the dependence of wavenumber on position is quadratic simplifies computation.[7] Such an estimate in a sense treats velocity space exactly, coordinate-space roughly. A complementary approach is to start from the inhomogeneous Vlasov equation[8] and carry out the fluid expansion.[9] The result is a differential equation whose coefficients contain moments of the velocity distribution. Such an estimate in a sense treats velocity-space roughly, coordinate-space exactly.

To determine the import of either of the above methods it is useful to first combine both approximations and carry out a WKB treatment of the fluid dispersion relation. When this is done, it is found that a second-order fluid expansion yields only traveling waves.[10] A coupling between these travelling waves may be achieved either by reflecting boundaries, as investigated by Tajima et al,[10] or by a resonance of the second-order modes. In the latter case, in order to determine the spatially-localized growing-mode structure, the fourth-order differential equation is required.

We proceed without further ado to derive the fourth-order differential equation in z, the distance along the axis of an axisymmetric mirror machine, which describes the variation of E_L, the complex electric-field amplitude of a left-hand circularly-polarized wave propagating along the magnetic field.

The scalar differential equation for $E_L(z)$ is obtained by substituting the linearized field $\vec{E}_1$ (x, y, z) into Maxwell's wave equation

$$-\frac{c^2}{\omega^2}\ \nabla \times (\nabla \times \vec{E}_1) + \vec{E}_1 + \frac{i\vec{J}_1}{\varepsilon_0\omega} = 0 \qquad (1)$$

We have taken the time variation $e^{-i\omega t}$. To evaluate the spatial second derivatives on the axis of symmetry, it is necessary to know the off-axis field $\overline{E}_1$ out to second order in x and y. We look for a class of modes which have no perpendicular variation. More strictly, we look for the simplest possible generalization of the uniform field modes, consistent with the curved fluxline geometry.

The electron conductivity is such as to rule out any parallel component of $\vec{E}_1$. We therefore take $\vec{E}_1$ to be everywhere perpendicular to the local $\vec{B}_0$ and in addition assume left-hand circular polarization. We assume no parallel currents in the equilibrium state, so that every point on the z-axis indexes a surface orthogonal to the field lines, namely the surface on which it lies. We fix $\vec{E}_1$ to be "constant" over such a surface, which we regard as a "wavefront." (We repeat that this implies neglect in some sense of any perpendicular mode structure.) More precisely, we assume that the ϕ-component of $\vec{E}_1$ in cylindrical coordinates remains constant as the observation point traverses the wavefront along a line of constant ϕ. This prescription for connecting transverse polarizations at different points on a curved wavefront to the transverse polarization on axis preserves rotational symmetry about the axis. It has the additional virtue that it extends with only minor change to the case where B_0 has a quadrupole component.

For this class of modes the Cartesian components of $\vec{E}_\perp$ are, to second order in x and y,

$$\vec{E}_\perp(x,y,z) = \begin{pmatrix} 1 - \frac{\Omega'^2}{8\Omega^2}\, x(x - iy) - \frac{\Omega'}{4\Omega}(x^2 + y^2)\frac{\partial}{\partial s} \\ -i - \frac{\Omega'^2}{8\Omega^2}\, y(x - iy) + i\,\frac{\Omega'}{4\Omega}(x^2 + y^2)\frac{\partial}{\partial s} \\ \frac{\Omega'}{2\Omega}(x - iy) \end{pmatrix} E_L(s) \qquad (2)$$

Here s indexes the wavefront on which (x,y,z) lies, $E_L(s)$ is a complex scalar, the operator $\partial/\partial s$ affects everything to the right ot it, a prime denotes differentiation with respect to s carried out only on the primed quantity itself, and Ω is the ion gyrofrequency. From this, and the fact that s = z on the axis, it follows that

$$[-\nabla \times (\nabla \times \vec{E}_\perp)]_{x=y=0} = \left[\left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right)\left(\frac{\partial}{\partial z} + \frac{\Omega'}{2\Omega}\right) + \frac{7\Omega'^2}{8\Omega^2}\right] E_L(z)\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad (3)$$

To obtain the perturbed ion current $\vec{J}_1$ we start from the linearized Vlasov equation, written for a point on the axis of symmetry x = y = 0;

$$i\omega f_1 + \Omega \frac{\partial f_1}{\partial \phi} = \vec{v} \cdot \frac{\partial f_1}{\partial \vec{r}} + \frac{q\vec{E}_1}{m} \cdot \frac{\partial f_o}{\partial \vec{v}} + \frac{iq}{m\omega} (\nabla \times \vec{E}_1) \times \vec{v} \cdot \frac{\partial f_o}{\partial \vec{v}} \qquad (4)$$

Here we have normalized to

$$\int d^3v \; f_0 \; (\vec{v}, \vec{r}) = n_0(\vec{r}) \qquad (5)$$

To evaluate the first and last terms on the right-hand-side it is necessary to know the off-axis field $\vec{E}_1$. Within our simple assumptions about mode structure, the first-order effect on f_1 of moving the observation point away from the axis is just a rotation of the local $(v_\perp, v_\parallel)$ reference frame,[8] which is proportional to $\Omega'/2\Omega$, the local rate of flux line convergence. The curl of $\vec{E}_1$ is known from (2). The overall result is that (4) reduces to an equation which is one-dimensional in space:

$$(\Omega - \omega) \; f_L = \frac{iq}{m} \frac{\partial f_o}{\partial v_\perp} \; E_L - \frac{q}{m\omega} \left(v_\parallel \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_\parallel}\right) \left(\frac{\partial}{\partial z} - \frac{\Omega'}{2\Omega}\right) \; E_L$$

$$+ \; i v_\parallel \frac{\partial f_L}{\partial z} + i v_\perp \frac{\Omega'}{2\Omega} \left(v_\parallel \frac{\partial f_L}{\partial v_\perp} - v_\perp \frac{\partial f_L}{\partial v_\parallel}\right) \qquad (6)$$

where

$$f_1 = e^{-i\phi} f_L (v_\perp, v_\parallel) \qquad (7)$$

On solving (6), substitution into (1) and the use of (3) will yield the form

$$\frac{c^2}{\omega^2} \left\{\left(\frac{\partial}{\partial z} - \frac{3}{2}\frac{\Omega'}{\Omega}\right)\left(\frac{\partial}{\partial z} + \frac{\Omega'}{2\Omega}\right) + \frac{7}{8}\left(\frac{\Omega'}{\Omega}\right)^2\right\} E_L(z) + E_L(z) + \frac{iJ_L(z)}{\varepsilon_0 \omega} = 0 \qquad (8)$$

where

$$J_L(z) = \frac{q}{2} \int d\vec{v}\, v_\perp\, f_L(z) \qquad (9)$$

We solve (6), by expanding f_L in powers of the operator $\partial/\partial z$, with the scalar function Ω'/Ω treated as having the same order.[9] The resulting hierarchy of equations is

$$(\Omega - \omega)\, f_L^{(0)} = \frac{iq}{m} \frac{\partial f_o}{\partial v_\perp} E_L \qquad (10)$$

$$(\Omega - \omega)\, f_L^{(1)} = \frac{q}{m\omega} v_\perp \left(\frac{\partial f_o}{\partial v_\parallel} - 2v_\parallel \frac{\partial f_o}{\partial v_\perp^2}\right)\left(\frac{\partial}{\partial z} - \frac{\Omega'}{2\Omega}\right) E_L$$

$$+ \left[iv_\parallel \frac{\partial}{\partial z} - iv_\perp^2 \frac{\Omega'}{2\Omega}\left(\frac{\partial}{\partial v_\parallel} - 2v_\parallel \frac{\partial}{\partial v_\perp^2}\right)\right] f_L^{(0)} \qquad (11)$$

$$(\Omega - \omega)\, f_L^{(j)} = \left[i v_\parallel \frac{\partial}{\partial z} - i v_\perp^2 \frac{\Omega'}{2\Omega} \left(\frac{\partial}{\partial v_\parallel} - 2 v_\parallel \frac{\partial}{\partial v_\perp^2} \right) \right] f_L^{(j-1)} \qquad (12)$$

$j \geq 2$

Assume that $f_o\ (v_\perp, v_\parallel)$ is even in $v_\parallel$. Then by inspection the $f_L^{(n)}$ with n odd are odd in $v_\parallel$ so that the corresponding $J_L^{(n)}$ are zero.

It proves advantageous to write down the integrals defining the $J_L^{(n)}$, n even, and evaluate them by successive partial integrations over velocity space, working "from the top down", rather than try to find the $f_L^{(n)}$ explicitly.

$$J_L^{(0)} \equiv \frac{q}{2} \int d\vec{v}\, v_\perp\, f_L^{(0)} = \frac{iq^2}{m(\Omega-\omega)} \int d\vec{v}\, v_\perp^2 \frac{\partial f_o}{\partial v_\perp^2}\, E_L$$

$$= - \frac{i\epsilon_o \omega_p^2}{\Omega-\omega}\, E_L \qquad (13)$$

as expected.

$$J_L^{(2)} = \frac{q}{2} \int d\vec{v}\, v_\perp\, f_L^{(2)}$$

$$= \frac{iq}{2(\Omega-\omega)} \frac{\partial}{\partial z} \int d\vec{v}\, v_\perp v_\parallel\, f_L^{(1)}$$

$$- \frac{iq}{2(\Omega-\omega)} \frac{\Omega'}{2\Omega} \int d\vec{v} \left(v_\perp^3 \frac{\partial}{\partial v_\parallel} - 2 v_\parallel v_\perp^3 \frac{\partial}{\partial v_\perp^2} \right) f_L^{(1)} \tag{14}$$

Integrate by parts with respect to $v_\parallel$ and with respect to $v_\perp^2$

$$J_L^{(2)} = \frac{iq}{2(\Omega-\omega)} \left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega} \right) \int d\vec{v}\, v_\parallel v_\perp\, f_L^{(1)} \tag{15}$$

Evaluate the integral in (15) by using (11) and (10), integrating by parts repeatedly with respect to $v_\parallel$ and with respect to $v_\perp^2$.

$$\int d\vec{v}\, v_\parallel v_\perp\, f_L^{(1)}$$

$$= \frac{q}{m\omega(\Omega-\omega)} \left(2n_0 \langle v_\parallel^2 \rangle - n_0 \langle v_\perp^2 \rangle\right) \left(\frac{\partial}{\partial z} - \frac{\Omega'}{2\Omega} \right) E_L$$

$$+ \frac{q}{m(\Omega-\omega)} \frac{\partial}{\partial z}\, 2n_0 \langle v_\parallel^2 \rangle \frac{E_L}{\Omega-\omega}$$

$$- \frac{q}{m(\Omega-\omega)} \frac{\Omega'}{2\Omega} \left(6n_0 \langle v_\parallel^2 \rangle - \Psi n_0 \langle v_\perp^2 \rangle\right) \frac{E_L}{\Omega-\omega} \tag{16}$$

Here $n_0(z)$ is the particle number density and < > denotes the average over the velocity distribution. Now use the relation which derives from adiabatic invariance

$$(n_0 \langle v_\parallel^2\rangle)' = \frac{\Omega'}{\Omega}\left(n_0 \langle v_\parallel^2\rangle - \frac{1}{2} n_0 \langle v_\perp^2\rangle\right) \qquad (17)$$

to rewrite (16) in the form

$$\int d\vec{v}\, v_\parallel v_\perp\, f_L^{(1)} = \frac{q}{m\omega}\left[\frac{2\Omega}{\Omega-\omega}\, n_0 \langle v_\parallel^2\rangle - n_0 \langle v_\perp^2\rangle\right]\left(\frac{\partial}{\partial z} + \frac{\Omega'}{2\Omega}\right)\frac{E_L}{\Omega-\omega} \qquad (18)$$

Substitute in (15)

$$J_L^{(2)} = \frac{i\varepsilon_0}{\omega(\Omega-\omega)}\left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right)\left[\omega_p^2\, \frac{\Omega}{\Omega-\omega}\langle v_\parallel^2\rangle - \frac{\langle v_\perp^2\rangle}{2}\right]\left(\frac{\partial}{\partial z} + \frac{\Omega'}{2\Omega}\right)\frac{E_L}{\Omega-\omega} \qquad (19)$$

Note the similarity between the differential operators appearing in (8) and (19). On to fourth order (compare (14));

$$J_L^{(4)} = \frac{iq}{2(\Omega-\omega)}\, \frac{\partial}{\partial z} \int d\vec{v}\, v_\perp v_\parallel\, f_L^{(3)}$$

$$- \frac{iq}{2(\Omega-\omega)}\, \frac{\Omega'}{2\Omega} \int d\vec{v}\left(v_\perp^3 \frac{\partial}{\partial v_\parallel} - 2\, v_\parallel v_\perp^3 \frac{\partial}{\partial v_\perp^2}\right) f_L^{(3)} \qquad (20)$$

Integrate by parts with respect to $v_\parallel$ and with respect to $v_\perp^2$ (compare (15))

$$J_L^{(4)} = \frac{iq}{2(\Omega-\omega)} \left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right) \int d\vec{v}\, v_\parallel v_\perp\, f_L^{(3)} \tag{21}$$

Using the same technique repeatedly

$$\int d\vec{v}\, v_\parallel v_\perp\, f_L^{(3)} = \frac{i}{\Omega-\omega}\left[\left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right) \int d\vec{v}\, v_\perp v_\parallel^2\, f_L^{(2)} + \frac{\Omega'}{2\Omega}\int d\vec{v}\, v_\perp^3\, f_L^{(2)}\right] \tag{22}$$

$$\int d\vec{v}\, v_\perp v_\parallel^2\, f_L^{(2)} = \frac{i}{\Omega-\omega}\left[\left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right) \int d\vec{v}\, v_\perp v_\parallel^3\, f_L^{(1)} + \frac{\Omega'}{\Omega}\int d\vec{v}\, v_\perp^3 v_\parallel f_L^{(1)}\right] \tag{23}$$

$$\int d\vec{v}\, v_\perp^3\, f_L^{(2)} = \frac{i}{\Omega-\omega}\left(\frac{\partial}{\partial z} - \frac{5\Omega'}{2\Omega}\right) \int d\vec{v}\, v_\perp^3 v_\parallel\, f_L^{(1)} \tag{24}$$

Use (10), (11) and two further relations derived from adiabatic invariance

$$(n_o\langle v_\parallel^4\rangle)' = \frac{\Omega'}{\Omega}\, n_o(\langle v_\parallel^4\rangle - \frac{3}{2}\langle v_\parallel^2 v_\perp^2\rangle) \tag{25}$$

$$(n_o\langle v_\parallel^2 v_\perp^2\rangle)' = \frac{\Omega'}{\Omega}\, n_o(2\langle v_\parallel^2 v_\perp^2\rangle - \frac{1}{2}\langle v_\perp^4\rangle) \tag{26}$$

to express the integrals on the right-hand-side of (23) and (24) in the form

$$\int d\vec{v}\, v_\perp v_\parallel^3\, f_L^{(1)} = \frac{2q\, n_o}{m\omega}\left[\frac{\Omega\langle v_\parallel^4\rangle}{\Omega-\omega} - \frac{3}{2}\langle v_\perp^2 v_\parallel^2\rangle\right]\left(\frac{\partial}{\partial z} + \frac{\Omega'}{2\Omega}\right)\frac{E_L}{\Omega-\omega} \tag{27}$$

$$\int d\vec{v}\, v_\perp^3 v_\parallel\, f_L^{(1)} = \frac{2q\, n_o}{m\omega}\left[\frac{\Omega\langle v_\perp^2 v_\parallel^2\rangle}{\Omega-\omega} - \frac{1}{2}\langle v_\perp^4\rangle\right]\left(\frac{\partial}{\partial z} + \frac{\Omega'}{2\Omega}\right)\frac{E_L}{\Omega-\omega} \qquad (28)$$

Substituting (22) - (24) and (27), (28) back into (21), obtain

$$J_L^{(4)} = \frac{iq^2}{m\omega}\frac{1}{\Omega-\omega}\left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right)\frac{1}{\Omega-\omega}$$

$$\left\{\left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right)\frac{1}{\Omega-\omega}\left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right) n_o[\Omega\langle v_\parallel^4\rangle - \frac{3}{2}(\Omega-\omega)\langle v_\perp^2 v_\parallel^2\rangle]\right.$$

$$+ \left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right)\frac{1}{\Omega-\omega}\frac{\Omega'}{\Omega}\, n_o[2\Omega\langle v_\perp^2 v_\parallel^2\rangle - \frac{1}{2}(\Omega-\omega)\langle v_\perp^4\rangle]$$

$$\left. + \frac{\Omega'}{2\Omega}\frac{1}{\Omega-\omega}\left(\frac{\partial}{\partial z} - \frac{5\Omega'}{2\Omega}\right) n_o[2\Omega\langle v_\perp^2 v_\parallel^2\rangle - \frac{1}{2}(\Omega-\omega)\langle v_\perp^4\rangle]\right\}$$

$$\frac{1}{\Omega-\omega}\left(\frac{\partial}{\partial z} + \frac{\Omega'}{2\Omega}\right)\frac{1}{\Omega-\omega}\, E_L \qquad (29)$$

Commute the fourth moments back past one layer of derivative operators, to the "center" of the whole expression, using the relations (25) and (26). The composite operator in braces becomes

$$F = \left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right) n_o \left[\frac{\Omega}{\Omega-\omega} \langle v_\parallel^4\rangle - \frac{3}{2}\langle v_\perp^2 v_\parallel^2\rangle\right]\left(\frac{\partial}{\partial z} + \frac{\Omega'}{2\Omega}\right)$$

$$- \left(\frac{\partial}{\partial z} - \frac{3\Omega'}{2\Omega}\right) n_o \left[\frac{2\Omega}{\Omega-\omega} \langle v_\parallel^2 v_\perp^2\rangle - \frac{1}{2}\langle v_\perp^4\rangle\right]\left(\frac{\Omega'}{2\Omega}\right)$$

$$+ \frac{\Omega'}{2\Omega} n_o \left[\frac{2\Omega}{\Omega-\omega} \langle v_\parallel^2 v_\perp^2\rangle - \frac{1}{2}\langle v_\perp^4\rangle\right]\left(\frac{\partial}{\partial z} - \frac{5\Omega'}{2\Omega}\right)$$

$$+ \frac{\Omega'}{2\Omega} \frac{n_o}{\Omega-\omega} \left[2\Omega \langle v_\parallel^2 v_\perp^2\rangle - \frac{1}{2}(\Omega-\omega)\langle v_\perp^4\rangle\right]' \tag{30}$$

Now make use of the operator identity

$$\Omega^{-n} \frac{\partial}{\partial z} \Omega^n \equiv \frac{\partial}{\partial z} + n\frac{\Omega'}{\Omega} \tag{31}$$

to deduce

$$\Omega^{-3/2} F \Omega^{-1/2} = \frac{\partial}{\partial z} n_o \left[\frac{\langle v_\parallel^4\rangle}{\Omega(\Omega-\omega)} - \frac{3}{2}\frac{\langle v_\perp^2 v_\parallel^2\rangle}{\Omega^2}\right]\frac{\partial}{\partial z}$$

$$+ \left[\frac{\Omega'^2}{\Omega(\Omega-\omega)} - \frac{\Omega''}{\Omega}\right] n_o \left[\frac{\langle v_\parallel^2 v_\perp^2\rangle}{\Omega(\Omega-\omega)} - \frac{\langle v_\perp^4\rangle}{4\Omega^2}\right] \tag{32}$$

Substitute in (29) and use the operator identity (31) again

$$J_L^{(4)} = -\frac{i\varepsilon_o}{\omega}\frac{\Omega^{3/2}}{\Omega-\omega}\frac{\partial}{\partial z}\frac{1}{\Omega-\omega}$$

$$\left\{\frac{\partial}{\partial z}\ \omega_p^2\left[\frac{\langle v_\parallel^4\rangle}{\Omega(\Omega-\omega)} - \frac{3}{2}\frac{\langle v_\perp^2 v_\parallel^2\rangle}{\Omega^2}\right]\frac{\partial}{\partial z}\right.$$

$$\left. + \left[\frac{\Omega'^2}{\Omega(\Omega-\omega)} - \frac{\Omega''}{\Omega}\right]\omega_p^2\left[\frac{\langle v_\parallel^2 v_\perp^2\rangle}{\Omega(\Omega-\omega)} - \frac{\langle v_\perp^4\rangle}{4\Omega^2}\right]\right\}$$

$$\frac{1}{\Omega-\omega}\frac{\partial}{\partial z}\frac{\Omega^{1/2}}{\Omega-\omega}E_L \tag{33}$$

Use the operator identity (31) on the expression (19) for $J_L^{(2)}$

$$J_L^{(2)} = \frac{i\varepsilon_o}{\omega}\frac{\Omega^{3/2}}{\Omega-\omega}\frac{\partial}{\partial z}\omega_p^2\left[\frac{\langle v_\parallel^2\rangle}{\Omega(\Omega-\omega)} - \frac{\langle v_\perp^2\rangle}{2\Omega^2}\right]\frac{\partial}{\partial z}\frac{\Omega^{1/2}}{\Omega-\omega}E_L \tag{34}$$

Assume cold electrons so that the electron current can be lumped in with $J_L^{(0)}$ using (13);

$$J_L^{(0)} = -\frac{i\varepsilon_o\,\omega_{pi}^2}{\Omega_i-\omega}E_L + \frac{i\varepsilon_o\,\omega_{pe}^2}{\Omega_e+\omega}$$

$$\cong \; - \frac{i\varepsilon_0\, \omega_p^2\, \omega}{\Omega(\Omega-w)} \tag{35}$$

Rewrite the wave equation (8) using the operator identity (31)

$$\frac{c^2}{\omega^2}\; \Omega^{3/2} \left[\frac{\partial}{\partial z}\; \Omega^{-2} \frac{\partial}{\partial z} + \frac{7}{8}\; \frac{\Omega'^2}{\Omega^4}\right] \Omega^{1/2}\; E_L(z) + E_L(z) + \frac{iJ_L(z)}{\varepsilon_0\omega} \;=\; 0 \tag{36}$$

Substitute (33), (34), (35) into (36); this results in the manifestly self-adjoint O.D.E.

$$\left\{\frac{\partial}{\partial z}\; \frac{c^2}{\Omega^2}\; \frac{\partial}{\partial z} + \frac{\omega^2}{\Omega^2} + \frac{7}{8}\; \frac{c^2\Omega'^2}{\Omega^4} + \frac{\omega_p^2\, \omega^2}{\Omega^3(\Omega-\omega)}\right.$$

$$+ \frac{1}{\Omega-\omega}\; \frac{\partial}{\partial z}\; \omega_p^2 \left[\frac{\langle v_\perp{}^2\rangle}{2\Omega^2} - \frac{\langle v_\parallel{}^2\rangle}{\Omega(\Omega-\omega)}\right] \frac{\partial}{\partial z}\; \frac{1}{\Omega-\omega}$$

$$- \frac{1}{\Omega-\omega}\; \frac{\partial}{\partial z}\; \frac{1}{\Omega-\omega}\; \frac{\partial}{\partial z}\; \omega_p^2 \left[\frac{3\langle v_\perp{}^2 v_\parallel{}^2\rangle}{2\Omega^2} - \frac{\langle v_\parallel{}^4\rangle}{\Omega(\Omega-\omega)}\right] \frac{\partial}{\partial z}\; \frac{1}{\Omega-\omega}\; \frac{\partial}{\partial z}\; \frac{1}{\Omega-\omega}$$

$$\left. + \frac{1}{\Omega-\omega}\; \frac{\partial}{\partial z} \left[\frac{\Omega''}{\Omega} - \frac{\Omega'^2}{\Omega(\Omega-\omega)}\right] \frac{\omega_p^2}{(\Omega-\omega)} \left[\frac{\langle v_\perp{}^4\rangle}{4\Omega^2} - \frac{\langle v_\parallel{}^2 v_\perp{}^2\rangle}{\Omega(\Omega-\omega)}\right] \frac{\partial}{\partial z}\; \frac{1}{\Omega-\omega}\right\} \cdot \Omega^{1/2}\; E_L(z) = 0 \tag{37}$$

Note the strong family resemblance between three of the objects in square brackets. Indeed, when the ion distribution is bimaxwellian

$$\frac{3\langle v_\perp^2 v_\parallel^2\rangle}{2\Omega^2} - \frac{\langle v_\parallel^4\rangle}{\Omega(\Omega-\omega)} = 3\langle v_\parallel^2\rangle \left[\frac{\langle v_\perp^2\rangle}{2\Omega^2} - \frac{\langle v_\parallel^2\rangle}{\Omega(\Omega-\omega)}\right] \qquad (38)$$

$$\frac{\langle v_\perp^4\rangle}{4\Omega^2} - \frac{\langle v_\parallel^2 v_\perp^2\rangle}{\Omega(\Omega-\omega)} = \langle v_\perp^2\rangle \left[\frac{\langle v_\perp^2\rangle}{2\Omega^2} - \frac{\langle v_\parallel^2\rangle}{\Omega(\Omega-\omega)}\right] . \qquad (39)$$

The equation (37) is not yet in a form suitable for numerical computation. Define the following normalized quantities in terms of the midplane ($z = 0$) plasma parameters:

$$\begin{aligned} b(z) &= \Omega(z)/\Omega(0) \\ w &= \omega/\Omega(0) \\ c_{A0} &\equiv c\,\Omega(0)/\omega_{pi}(0) \\ s &\equiv z\,\omega_{pi}(0)/c \end{aligned} \qquad (40)$$

$$\beta_\perp^0(z) \equiv \omega_{pi}^2(z) <v_\perp^2> (z)/\Omega^2(0)\, c^2$$

$$\beta_\parallel^0(z) \equiv 2\, \omega_{pi}^2\, (z) <v_\parallel^2> (z)/\Omega^2(0)\, c^2 \tag{41}$$

$$N(z) = n(z)/n(0)$$

$$\beta_{\perp\perp}^0(z) \equiv \omega_{pi}^2\, (z)\, \omega_{pi}^2\, (0) <v_\perp^4>(z)/2\Omega^4(0)\, c^4$$

$$\beta_{\perp\parallel}^0(z) \equiv 2\, \omega_{pi}^2\, (z)\, \omega_{pi}^2\, (0) <v_\perp^2 v_\parallel^2> (z)/\Omega^4(0)\, c^4 \tag{42}$$

$$\beta_{\parallel\parallel}^0(z) \equiv 4\, \omega_{pi}^2\, (z)\, \omega_{pi}^2\, (0) <v_\parallel^4> (z)/3\Omega^4(0)\, c^4$$

Note that in case the ion distribution is bimaxwellian

$$\beta_{\perp\perp}^0 = (\beta_\perp^0)^2/N$$

$$\beta_{\perp\parallel}^0 = \beta_\perp^0\, \beta_\parallel^0/N \tag{43}$$

$$\beta_{\parallel\parallel}^0 = (\beta_\parallel^0)^2/N$$

$$\beta_\perp^{\,0} = N^2\, \beta_\perp(0)$$

$$\beta_\parallel^{\,0} = N\, \beta_\parallel(0)$$

$$N = (1-r)\, b/(b-r) \qquad (44)$$

$$r = [\beta_\perp(0) - \beta_\parallel(0)]/\beta_\perp(0)$$

Multiply (37) by $\Omega^{3/2}(0)/\omega_{pi}^2(0)$;

$$\left\{\frac{\partial}{\partial s}\ \frac{1}{b^2}\ \frac{\partial}{\partial s} + \frac{w^2}{b^2}\ \frac{c_{A0}^2}{c^2} + \frac{7}{8}\,\frac{b'^2}{b^4} + \frac{N\,w^2}{b^3(b-w)}\right.$$

$$+ \frac{1}{2}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\left[\frac{\beta_\perp^{\,0}}{b^2} - \frac{\beta_\parallel^{\,0}}{b(b-w)}\right]\frac{\partial}{\partial s}\ \frac{1}{b-w}$$

$$- \frac{3}{4}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\left[\frac{\beta_{\perp\parallel}^{\,0}}{b^2} - \frac{\beta_{\parallel\parallel}^{\,0}}{b(b-w)}\right]\frac{\partial}{\partial s}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\ \frac{1}{b-w}$$

$$\left. + \frac{1}{2}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\left[\frac{b''}{b} - \frac{b'^2}{b(b-w)}\right]\frac{1}{(b-w)^2}\left[\frac{\beta_{\perp\parallel}^{\,0}}{b^2} - \frac{\beta_{\parallel\parallel}^{\,0}}{b(b-w)}\right]\frac{\partial}{\partial s}\ \frac{1}{b-w}\right\}\ b^{1/2}\, E_L = 0 \qquad (45)$$

where now primes denote derivatives with respect to s.

In case the ion distribution is bimaxwellian

$$\left\{\frac{\partial}{\partial s}\ \frac{1}{b^2}\ \frac{\partial}{\partial s} + \frac{w^2}{b^2}\ \frac{c_{AO}^2}{c^2} + \frac{7}{8}\ \frac{b'^2}{b^4} + \frac{w^2}{b^2}\ \frac{1-r}{(b-r)(b-w)}\right.$$

$$+ \frac{1}{2}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\ \beta_\parallel(0)\ \frac{1-r}{(b-r)^2}\ \frac{r-w}{b-w}\ \frac{\partial}{\partial s}\ \frac{1}{b-w}$$

$$- \frac{3}{4}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\ \beta_\parallel^2(0)\ \frac{1-r}{(b-r)^2}\ \frac{r-w}{b-w}\ \frac{\partial}{\partial s}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\ \frac{1}{b-w}$$

$$\left. + \frac{1}{2}\ \frac{1}{b-w}\ \frac{\partial}{\partial s}\ \beta_\parallel^2(0)\ \frac{1-r}{(b-r)^3}\ \frac{r-w}{(b-w)^4}\ \left[b''(b-w) - b'^2\right] \frac{\partial}{\partial s}\ \frac{1}{b-w}\right\} \cdot b^{1/2}\ E_L = 0 \qquad (46)$$

As a way of connecting back with the physical idea that the anisotropy provides the driving term for the instability, recall that the dispersion function for the parallel-propagating AIC mode in a bimaxwellian plasma is[1]

$$D = -\frac{k^2c^2}{\omega^2} + 1 + \frac{\omega_p^2}{\Omega(\Omega-\omega)} - \frac{\omega_p^2}{2\omega^2}\left(\frac{\langle v_\perp^2\rangle}{2\langle v_\parallel^2\rangle} - \frac{\Omega}{\Omega-\omega}\right) Z'\left(\frac{-\Omega+\omega}{k\sqrt{2\langle v_\parallel^2\rangle}}\right) \qquad (47)$$

Stability in this case is determined solely by the sign of the factor preceding the Z'-function. This sign does not change along a field line, provided that the mode frequency is below the midplane ion gyrofrequency. Multiply (47) by $\Omega^2(0)/\omega_{pi}^2(0)$ and make use of the normalized variables (40) - (44); define $ip \equiv \partial/\partial s$;

$$D = -\frac{p^2}{w^2} + \frac{c_{A0}^2}{c^2} + \frac{1-r}{(b-w)(b-r)} - \frac{(1-r)(r-w)b^2}{2(b-r)^2(b-w)w^2} Z'\left(\frac{-b+w}{p\sqrt{\beta_\parallel(0)}}\right) \tag{48}$$

Stability for any value of $b \geq 1$ is determined by the sign of r-w which is the anisotropy driving term evaluated at the midplane.

The sequence of computations leading from the hierarchy of equations (10), (11), (12) to the currents (33), (34), (35) has been duplicated on a computer using the (highly interactive) symbolic manipulation system MACSYMA.[12] The end-products of this manipulation process are expressions in which the differentiations with respect to z are carried out as far as possible. These expressions in the case of $J_L^{(2)}$, $J_L^{(4)}$ have very many terms and are not displayed, merely stored. To carry out the check, we type in the hand-obtained compact forms (33), (34) and let the computer carry out all the differentiations with respect to z; then we command the computer to subtract these forms from the previously-stored expressions. When the difference is zero, we have confidence that all bugs have been removed both from the symbolic manipulation process and from the hand calculation. The system commands, together with explanatory paragraphs, are displayed in the printout which occupies the succeeding few pages.

```
/* 8/22/77 Problem from Duncan C. Watson, LLL.
    Linearized Vlasov equation describing the AIC mode expanding
    f[L] in terms of the small parameter d/dz  .
```

```
/* In order to see the time taken for each command, set the MACSYMA "flag"
SHOWTIME to "all". The times will be given in milliseconds of processor time,
with the additional information of how much of that time was spent in storage
reclamation (gc or "garbage collection"). */
showtime:all$

/* Because this problem requires storage of several expressions whose size
cannot be predicted beforehand, we wish to specify to MACSYMA that it should
allocate storage dynamically, as needed, up to the full machine capacity,
as the calculation proceeds. Without this setting, the system will
periodically ask the user if he wishes more storage. */
dynamalloc:true$

/* A number of names will be used to designate functions from the original
notes. The fact that certain of these names are functions of some of the others
must be made explicit, so that, for example, MACSYMA can compute the derivative
of omega (=om, here) with respect to z, and represent it internally as
"diff(om,z)", whereas small omega (= w, here) when differentiated with respect
to z, results in 0. A shorthand for omega-w which we introduce for the first
time in the next command, is dom=omega-w. This abbreviates the input, output,
and most importantly, the intermediate expressions. */
dependencies(dom(z),om(z),f0(z,vr,vl),n(z),el(z))$
/*vr = Velocity perpendiculaR, vl = Velocity paralleL */

/* The zeroth order is given */
fl[0]:-q/m*diff(f0,vr)*el/%i/dom$

/* Before we forget, we had better tell the simplifier that the derivative
of dom wrt z is the same as the derivative of om wrt z, and should be converted
to the latter form whenever generated. */
tellsimp('diff(dom,z),'diff(om,z))$

/* the "n" array is not really an array, but a set of subscripted
names, where we denote by n[i,j] the quantity <vl^i*vr^j>. Using a
functional notation for these moments would result in rather nasty
circumlocutions in MACSYMA. For example, moment(vl^2*vr^2) is a function of
neither vl nor vr, but of z! The simplifications based on physical
considerations
are embodied in these transformation rules given to the simplifer. */
tellsimp('diff(n[4,0],z),'diff(om,z)/om*(n[4,0]-3/2*n[2,2]))$
tellsimp('diff(n[2,2],z),'diff(om,z)/om*(2*n[2,2]-1/2*n[0,4]))$
tellsimp('diff(n[2,0],z),'diff(om,z)/om*(n[2,0]-n[0,2]/2))$

/* The next iteration for fl can be picked out by sight, and since
it must be typed in one way or another, we simply set it up here */

fl[1]:(-%i*q/m*(diff(f0,vr)*vl/w*diff(el,z)-diff(f0,vl)*vr/w*diff(el,z))
+%i*q/m/w*diff(om,z)/2/om*(vl*diff(f0,vr)-vr*diff(f0,vl))*el
-vl*diff(fl[0],z)-diff(om,z)/2/om*vr*(vl*diff(fl[0],vr)-vr*diff(fl[0],vl)))
    /%i/dom$

/* For all j >=2 we can iterate this equation. However, the only contributions
are made from the last part, which is reproduced here: */

fl[j]:=(-vl*diff(fl[j-1],z)-diff(om,z)/2/om*vr*
   (vl*diff(fl[j-1],vr)-vr*diff(fl[j-1],vl)))/%i/dom$

/* In order to conveniently apply the integration rules, we first
must counteract MACSYMA's built-in facility which transforms multiple
derivatives into a briefer form than we wish. For example, diff(diff(om,z),z)
is changed to 'diff(om,z,2). This is changed, by the application of rule rd
defined below, to d[z](d[z](om)). The program "diffsimp" does
most of the work, and uses various facilities in MACSYMA to change the internal
form of the expression, when "rd" is used. */
```

```
rd(x):=if atom(x)or inpart(x,0)#nounify(diff) then false else diffsimp
(substinpart("[",x,0))$
translate(rd)$

diffsimp(z):=block([e,x1,n1],
if rest(z)=[] then return(z[1]),
e:z[1],x1:z[2],n1:z[3],
if n1=0 then return(diffsimp(cons(e,rest(z,3)))),
return(d[x1](diffsimp(cons(e,cons(x1,cons(n1-1,rest(z,3))))))))$



/* Pattern matching will be used to "pass" the J operation, which integrates,
from left to right over each expression.  The use of any1, any2, or any3
in the rule definitions signals a pattern variable matching "anything".
The pattern variable nonv will match a subexpression free of vr, vl and f0. */

matchdeclare([any1,any2,any3],true)$
matchdeclare(nonv,freeof(vr,vl,f0))$
defrule(r2a,j(vr^any1*vl^any2*nonv*d[z](f0)),
   nonv*d[z](j(vr^any1*vl^any2*f0)))$
defrule(r2b,j(vr^any1*vl^any2*nonv*d[z](d[z](f0))),
   nonv*d[z].d[z](j(vr^any1*vl^any2*f0))))$
defrule(r2c,j(vr^any1*vl^any2*nonv*d[z](d[z](d[z](f0)))),
   nonv*d[z](d[z](d[z](j(vr^any1*vl^any2*f0)))))$
defrule(r2d,j(vr^any1*vl^any2*nonv*d[z](d[z](d[z](d[z](f0))))),
  nonv*d[z](d[z](d[z](d[z](j(vr^any1*vl^any2*f0))))))$
defrule(r3,j(vr^any1*vl^any2*f0*nonv),n[any2,any1+1]*nonv/2)$

/* Declaring J linear means j(a+b) ==> j(a)+j(b) */
declare(j,linear)$
defrule(r4,j(d[vr](any1)*any2), -j(diff(vr^2*any2,vr)*any1/vr^2) )$
defrule(r5,j(d[vl](any1)*any2), -j(diff(any2,vl)*any1) )$

/* The next two lines simplify moments for certain expressions. */
matchdeclare(oddnum,?oddp)$
defrule(r6,n[oddnum,any1],0)$

/* The next rule changes d[z]'s back into derivatives. */
defrule(r8,d[z](any1),diff(any1,z))$

/* The setting of derivabbrev to true simply changes the display
of derivatives to make partial derivatives look like subscripts. */
derivabbrev:true$


/* items of interest are  (- %i*m/q) * j(fl[n]) for n=0,1,2,...
  these correspond to (%i*q^2/(2*m)) * handcomputed values */

/* APRULES  is a program which applies the rules
defined above, in a systematic order. */

aprules(mm):=block([expop,temp],
expop:6,
temp:apply1( ev((-%i*m/q)*mm,expand,diff),rd),
     /* change diff(a,b) to d[b](a) */
temp: apply2(
 j(temp)
     ,r2a,r2b,r2c,r2d,r3,r4,r5,r6),  /* "integrate" */
return(applyb1(temp,r8)  /* change back to "diff" */))$
```

```
ttyoff:false$
j0:aprules(fl[0]);
j1:aprules(fl[1]);  /* This is identically zero. */
j2:aprules(fl[2]);  /* This has 26 distinct terms. */
j3:aprules(fl[3]);  /* This is identically zero. */

/* for fourth order, we have to work harder since fl[4] is huge.
In fact, fl[3] has 173 distinct terms.  Our strategy is to
push each part of f[3] through the following formula
separately. */

fourthterm(h):=1/(%i*dom)*
(-vl*diff(h,z)-diff(om,z)/2/om*vr*vl*diff(h,vr))$

pap(h):=block([term],
term:aprules(fourthterm(h)),
print(term),
loadfile(dun4,ans), /* has a value for ans, fl3 */
ans:ans+term,
fl3:substinpart(0,fl3,1),
save([dun4,ans],fl3,ans))$
/* The programmed application of rules (pap) above, saves a file
at each iteration, which can be restarted in case of a computer
crash. (the time to complete this computation is several cpu minutes) */
/*These two commands initialize the "dump" file. */



ans:0$
save([dun4,ans],ans)$


fl3:ev(fl[3],expand,diff)$

/* We remove from storage those items we do not need to proceed. */
kill(labels,j0,j1,j2,j3,fl)$

/* DOIT is the main loop iterating through fl3. */
doit():=unless fl3=0 do pap(inpart(fl3,1))$

doitall():=block(doit(),logout())$

/* At the conclusion of the "Batching" of this file, one may
invoke DOITALL()$
and then type <control w><control z><control p>:DISOWN

to enter the job into the background operation of the MACSYMA system.
At some time in the future, the job will finish, log itself out, and leave
behind the file dun4 ans.  Two items will be stored on the file: ans
(which is the computed j4), and fl3 (which should be 0).  If fl3 is not
zero, the job did not finish (probably due to a computer crash).
The program can be restarted using "DOIT".

    The hand-computed formulas were typed in to the fi   duntes >,
and the operations carried out by MACSYMA (e.g. differentiation, expansion)
so as to make the results comparable.  The difference between the
two sets of expressions, simplified appropriately, is 0 for
j2 and j4 (j0, j1, and j3 are trivial).  */
*
```

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