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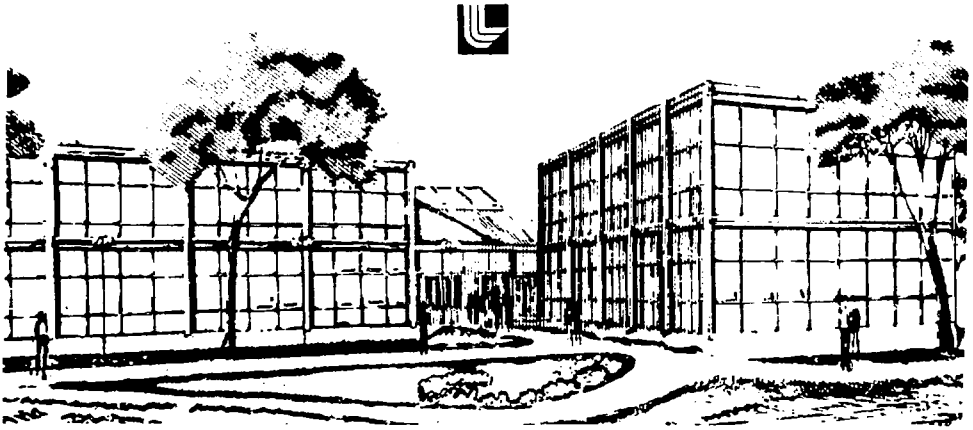

THE ROLE OF STATISTICAL LINEARIZATION IN THE SOLUTION OF NONLINEAR STOCHASTIC EQUATIONS*

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0. Introduction

Throughout the proceedings of this conference the notion of stochasticity as promulgated by Anosov and Sinai and evinced by the KAM theorem has stimulated a great deal of work buttressed by countless examples. Succinctly the results of endless study have led to the notion that stochasticity is commensurate with irregularity in phase space.

This definition is certainly not complete in that it does not directly relate to one's intuitive conception of randomness in nature. We often encounter physical systems which have some built in uncertainty as to their operating conditions. This uncertainty may be inherent, that is the equations of motion ascribed to some macroscopic or microscopic observable are only approximate, or it may be imposed upon the system due to an imperfect understanding of how the ambient environment affects operation. Under most practical circumstances it is usually easier to cope with the former type of ignorance than the latter type since the gross stability of a dynamical system is not necessarily affected by slight structural modifications of its underlying Hamiltonian. On the other hand uncertainty concerning external perturbations may have quite dramatic consequences. Examples extant throughout the physical literature attest to the fact that quite a diversity in behavior occurs in the response of structures to earthquakes, in the propagation of acoustic or electromagnetic waves in a quasi-random material

such as an optical waveguide or fiber, and in atmospheric and hydrodynamic turbulence.[1,2]

Such problems have been modelled by the ordinary or partial differential equations,[3]

$$\frac{\partial}{\partial t} G_j(t) = H_j\left(t,\{G_\ell(s)\}\right) + a_j(\{G_\ell\},t)F_j(t)\ j,\ \ell = 1,..,N\ t_o \leq s \leq t \qquad (0.1)$$

Known as a generalized Langevin equation the resultant solution is referred to as a stochastic process. Here the $\{G_j(t)\}$ are a set of N time-dependent system variables, $H_j\left(t, \{G_\ell(s)\}\right)$ is a linear or nonlinear function which contains the explicit dependence of the rate of change of $G_j(t)$ on the values of the chosen variables $G_\ell(s)$ between some initial time t_o and t, and $F_j(t)$ is a fluctuating external force whose mean value (averaged over some initial distribution of the F_j) obeys the relation

$$< F_j(t) >_o = 0 \qquad (0.2)$$

To date the vast majority of research has particularized F_j to Gaussian white noise, $\zeta(t)$, (the derivate of a Wiener process) or to the secondary response to Gaussian noise

$$\sum \alpha_k \frac{d^k F_j(t)}{dt^k} = \zeta_j(t)\ . \qquad (0.3)$$

This requires the 2-time correlation function of $\zeta(t)$ to be of the form

$$< \zeta_k(t_1)\zeta_j(t_2) > \; = \; d\delta(t_1-t_2)\delta_{kj} \tag{0.4}$$

d expressing the strength of the δ-function. The benefit accrued by this restriction is that associated to the dynamical set of equations (0.1) is a linear partial differential equation, the forward Kolmogorov (FPK) equation,

$$\frac{\partial f}{\partial t}(\underline{G},\, t|\underline{G}_o,s) \; = \; -\sum_{i=1}^{N} \frac{\partial}{\partial G_i}\left(H_i(t,\underline{G})f\right) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial G_i \partial G_j}\left(a_i(t,\underline{G})a_j(t,\underline{G})f\right) \tag{0.5a}$$

with initial condition

$$\lim_{t\downarrow s} f(\underline{G},t|\underline{G}_o,s) \; = \; \delta(\underline{G} - \underline{G}_o) \tag{0.5b}$$

whose solution yields the transition probability density function $f(\underline{G},t|\underline{G}_o,s)$ that the system occupy state $\underline{G}$ at time t given that it had already occupied state $\underline{G}_o$ at time s. Knowledge of f completely characterizes the stochastic process $\underline{G}$. Unfortunately, with the transformation to the FPK formulation come attendant difficulties. Solutions of (0.5) are not all that easy to come by and have only been obtained for 1) all linear systems, 2) some first order nonlinear systems, and 3) for the steady state density corresponding to anharmonic oscillator systems.[2] Nonlinear equations which do not fall into this select class have to be handled via approximation methods. The most important may be classified into three groupings:

a) perturbation theories

b) eigenfunction expansions

and

c) nonlinear optimization procedures.

After a brief interlude on the vicissitudes of using the various techniques germane to a) or to b) I shall devote the rest of my discussion to c) and demonstrate through the method of statistical linearization a relationship to the former two categories.

1. Perturbation Theory

Standard perturbation theory may be applied to nonlinear stochastic equations. As customary this involves isolation of the nonlinearity by marking it with a multiplicative perturbative expansion prefactor, μ say, and then expanding the solution to all orders in μ.[2] As an illustration let us restrict attention to the second order anharmonic oscillator equation

$$L_o G + \mu g(G,\dot{G}) = \xi(t) \tag{1.1a}$$

L_o being the linear differential operator

$$L_o = \frac{d^2}{dt^2} + \gamma_o \frac{d}{dt} + \omega_o^2 \tag{1.1b}$$

and $g(G,\dot{G})$ some nonlinear function of the position and velocity of the oscillator. The coefficients γ_0 and ω_0^2 are respectively the linear damping and internal frequency of the harmonic oscillator.

Substituting the μ expansion of G

$$G(t) = \sum_{j=0}^{\infty} \mu^j G_j(t) \tag{1.2}$$

into (1.1a) yields for each order in μ an infinite set of inhomogeneous linear differential equations with Green's function solutions

$$\begin{aligned} G_0 &= \int_0^{\infty} h(\tau)\zeta(t-\tau)d\tau \\ G_1 &= -\int_0^{\infty} h(\tau)g(G_0(t-\tau),\dot{G}_0(t-\tau))d\tau \\ &\vdots \end{aligned} \tag{1.3}$$

The various statistics of G can then be found by performing the appropriate averages on the perturbation series (1.2), with the expectation and 2-time correlation function $< G(t)G(t') > = R_{GG}(t,t')$, for example, being given by

$$\langle G(t) \rangle = \sum \mu^j \langle G_j(t) \rangle \tag{1.4a}$$

$$R_{GG}(t,t') = R_{G_oG_o}(t,t') + \mu[R_{G_oG_o}(t,t') + R_{G_oG_1}(t,t') + R_{G_1G_o}(t,t')] + O(\mu^2) \tag{1.4b}$$

Since $\zeta(t)$ is Gaussian then by the linearity of the first equation in (1.3) so is G_o. If $\zeta(t)$ is also stationary $R_{GG}(t,t')$ simplifies to a 1-time autocorrelation function $R_{GG}(\tau)$

$$R_{GG}(t,t') = R_{GG}(t-t') = R_{GG}(\tau) \tag{1.5}$$

The spectral density $S_{GG}(\omega)$ is immediately obtained by the Wiener-Khinchine formula as

$$S_{GG}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} R_{GG}(\tau) d\tau \tag{1.6}$$

which becomes by virtue of (1.4b)

$$S_{GG}(\omega) = S_{G_oG_o}(\omega) + 2\mu \mathrm{Re}\left(S_{G_oG_1}(\omega)\right) + O(\mu^2) \tag{1.7}$$

The difficulty with utilizing eqs. (1.4) and (1.7) revolves around a problem endemic to all straightforward perturbation theories - the occurrence of secular terms. To overcome this shortcoming renormalization techniques such as those due to Lindstedt and Poincaré or alternatively in terms of diagrammatic expansions have been developed.[1,4] These latter methods have

been in vogue in quantum field theories but prior to Kraichnan's early work in turbulence had not been applied to nonlinear classical stochastic equations.

In essence, one regroups subsets of all the possible diagrams which arise from the complete perturbation theory. The diagrams in each subset are of similar form and can be summed exactly, with the resulting restructured perturbation equations now referred to as *consolidated equations*. In general they are very difficult to solve since they comprise a set of coupled nonlinear integral equations relating the renormalized spectral density $S(\omega)$ in terms of renormalized propagators (Green's functions) $\hat{G}(\omega)$ and renormalized vertex or interaction functions $\hat{\Gamma}(\omega)$. Due to this complexity very few papers involving this method have appeared in print. One of the most extensive, however, is a computation by Morton and Corrsin on the Duffing anharmonic oscillator equation perturbed by white noise[5]

$$\ddot{G} + \alpha \dot{G} + \omega_0^2 G + \mu G^3 = \zeta(t) \tag{1.8}$$

in which they truncated the consolidated equations to various orders and compared the resulting numerical calculations for $S_{GG}(\omega)$ with the results of an analog computer experiment. Certain formulae from their study will be referred to and made use of in the subsequent sections of this paper. The general renormalized theory for classical dynamical systems has been derived by P. C. Martin et al.[6]

The perturbation story does not have to end on such an abysmal note. A remarkable saving grace occurs if instead of analyzing dynamical equations such as (0.1) one instead performs the perturbation expansion on the FPK eq. (0.5) or for non-white noise excitation on the Liouville equation

$$\frac{\partial \rho}{\partial t}(\underline{u},t) = L\rho(\underline{u},t) \tag{1.9}$$

in which $\rho(\underline{u},t)$ is the N-particle distribution function for the system in phase space $(u_1, \ldots, \underline{u}_N) = (\underline{G}_1,\underline{p}_1; \ldots; \underline{G}_N,\underline{p}_N)$. The diagrammatics involved are the same, but the perturbation theory is now applied to a linear partial differential equation and not to a set of nonlinear ordinary differential equations. For generality our ensuing analysis will be phrased in terms of the Liouville equation. Let us emphasize, however, that it follows through exactly for the FPK equation.

Given a Hilbert space H we decompose the Liouville equation (1.9) by defining projection operators P and (I - P) with P acting on PH such that $P\rho \equiv \rho_0$ and (I - P) acting on the orthogonal subspace (I - P)H such that $(I - P)\rho \equiv \rho_1$. We further stipulate that for an arbitrary phase function A

$$PLPA = PL_0A = L_0PA \tag{1.10}$$

so that the Liouville operator L separates into two parts

$$L = L_0 + L_1 \tag{1.11}$$

It is thus clear that L_0 is diagonal in the subspace PH, i.e. that L_0 and P commute, and that L_1 connects PH with $(I - P)$. Thus, L_0 is chosen to be the deterministic or unperturbed part of L and L_1 the perturbation, or equivalently the stochastic interaction. It can easily be shown that subject to the initial condition that $(I - P)\rho(o) = 0$ ρ_0 has the Laplace transformed solution

$$\hat{\rho}_0 = (1 - \hat{G}_0\hat{M})^{-1}\hat{G}_0\rho_0(o) = (\hat{G}_0^{-1} - \hat{M})^{-1}\rho_0(o) \tag{1.12}$$

where $\hat{G}_0$ is the free propagator

$$\hat{G}_0 = \frac{1}{s - L_0} \tag{1.13}$$

The formal structure of the operator $\hat{M}$ is given by the series representation

$$\hat{M} = \sum_{1}^{\infty} PL_1[\hat{G}_0(I - P)L_1]^n P \tag{1.14}$$

It begins and ends with a P, but contains only (I - P) in the intermediate state factors. This implies that only the initial and final perturbations connect to the phase space PH with all intermediate state excitations being in the $(I - P)H$ space. In diagrammatic language the operator $\hat{M}$ has the form of a "linked-cluster expansion", i.e., all diagrams are completely linked or connected and are irreducible. This corresponds to expanding in terms of cumulants; all uncorrelated, i.e., unlinked, events are eliminated from the intermediate state. By identifying the propagator $(\hat{G}_0^{-1} - \hat{M})^{-1}$ with the Green's function $\hat{G}$

$$\hat{G} = (s - L_0 - \hat{M})^{-1}$$

$$= \hat{G}_0 + \hat{G}_0\hat{M}\hat{G} . \tag{1.15}$$

$\hat{M}$, therefore, corresponds physically to the mass operator of quantum field theory.[4,7]

By inverse Laplace transforming (1.12) the solution to ρ_0 in configuration space becomes

$$\frac{\partial \rho_0}{\partial t} = L_0\rho_0 + \int_0^t M(\tau)\rho_0(t - \tau)d\tau. \tag{1.16}$$

An important approximation to this equation can be found by truncating $\hat{M}$ at the first term ($\hat{M} = \hat{M}_1$), i.e., neglecting all higher interaction terms

in (1.14). The resulting equation is

$$\frac{\partial \rho_0}{\partial t} = L_0 \rho_0(t) + P L_1 \int_0^t e^{L_0(t - t_1)} (I - P) L_1 P \rho_0(t_1) dt_1 \qquad (1.17)$$

an equation which has appeared in the stochastic theories of fluids and plasmas as the Boltzmann and Landau equations.[1] It has also been referred to as the method of first order smoothing[1] since it is the linear counterpart of the Krylov-Bogoliubov method of averaging for nonlinear differential equations.[8] When generalized to nonlinear systems it is equivalent to a particular truncation of the consolidated equations. We call this the 1st Kraichnan-Wyld approximation since it yields identical results to that obtained by Morton and Corrsin for the Duffing oscillator.

Other consolidated equation approximations can be found by introducing a consolidation operator Q. We add and subtract Q from the (I - P) projection to ρ to obtain

$$\frac{\partial \rho_1}{\partial t} = (I - P)(L_0 + Q)\rho + (I - P)(L_1 - Q)\rho \qquad (1.18)$$

Physically speaking this results in modification of the underlying Hamiltonian so that if

$$H = H_0 + H_1 \tag{1.19}$$

the consolidation operator Q corresponds to adding an average potential $\bar{V}$ to $\mathbf{H}_0$ while subtracting the same potential from $\mathbf{H}_1$

$$\mathbf{H}_0' = \mathbf{H}_0 + \bar{V} \; ; \; \mathbf{H}_1' = \mathbf{H}_1 - \bar{V} \tag{1.20}$$

such that

$$\mathbf{H}_0 + \mathbf{H}_1 = \mathbf{H}_0' + \mathbf{H}_1' \; . \tag{1.21}$$

Thus, in analogy with (1.15) a consolidated mass operator $\hat{M}_Q$ can be derived

$$\hat{M}_Q = P_Q(L_1 - Q)\sum[\hat{G}_Q(I - P_Q)(L_1 - Q)]^n P_Q \tag{1.22}$$

which, when Q is selected to cancel certain prescribed diagrams from the Q = 0 perturbation series, admits consolidated equations other than first order smoothing.

2. Eigenfunction Expansions

The approximation techniques involving the use of eigenfunction expansions are derivable from linear operator theory and have therefore been used in conjunction with the FPK and Liouville partial differential equations either by

investigating the eigenvalue problem associated with their (self)adjoint operators or by particularizing the projection operator methods originally developed by David Hilbert and extended by Grad to the asymptotic theory of the Boltzmann equation.[9] I do not care to say much about the former category other than its most successful approach seems to be a combined eigenfunction-perturbation theory which results in uniformly valid asymptotic expansions for the various statistics of the response.[2] It should be interesting to examine its relationship to the projection-perturbation theory of the previous section.

As for the latter category I shall restrict my discussion to the projection operator formalism introduced into nonequilibrium statistical mechanics by Zwanzig.[10] Deviations of the Liouville phase density ρ or of the FPK transition probability density f from an equilibrium Maxwellian distribution ρ_o or f_o are expanded in terms of a Gaussian basis set (generalized Hermite vector polynomials)

$$\tilde{\rho} = \tilde{\rho}_o\left(b_o H_{eo} + \sum_{i,j} b_j^{(i)} H_{ej}^{(i)}\right) \tag{2.1}$$

where

$$\tilde{\rho}_o = \begin{cases} \rho_o = C\, e^{-\beta \underline{\underline{H}}(\underline{G},\underline{p},t)} \cdot & \text{Liouville} \\ f_o = \left[(2\pi)^{N/2}|\underline{\underline{K}}|^{1/2}\right]^{-1} e^{-1/2(\underline{G},\underline{\underline{K}}^{-1},\underline{G})} & \text{FPK} \end{cases} \tag{2.2}$$

The coefficients $b_j^{(i)}$ are determined by the inner product relation

$$b_j^{(i)} = \int \tilde{\rho} H_{ej}^{(i)} dV \tag{2.3}$$

with the superscript (i) spanning the N or 2N components of G or of $\underline{u}$, respectively, and the matrix $\underline{\underline{K}}$ is determined from (0.5a) by the integral

$$\underline{\underline{K}} = \lim_{t \to \infty} \int_0^{\infty} e^{\underline{\underline{H}} s} \underline{\underline{a}} \underline{\underline{a}}^{tr} e^{\underline{\underline{H}}^{tr} s} \, ds \tag{2.4}$$

With these definitions in mind we now, as in the previous section, operationally manipulate the Liouville equation (1.9), but note that for all steps ρ may be replaced by f.

Premultiplying (1.9) by P and (I - P), respectively, one obtains a formal solution for $\hat{\rho}_o$ as

$$\hat{\rho}_o = \left[s - L_o - PL_1 \hat{G}(I - P)L_1 \right]^{-1} \rho_o(o) \tag{2.5}$$

where we have introduced the Green's function

$$\hat{G} = \left[s - L_o - (I - P)L_1 \right]^{-1} = \hat{G}_o + \hat{G}_o (I - P)L_1 \hat{G}. \tag{2.6}$$

Upon comparing (2.5) with (1.11) we see that

$$\hat{M} = PL_1\hat{G}(I - P)L_1 \tag{2.7}$$

Thus, the projection operator kernel is identical to the perturbation operator, as it should be, since both are exact when all orders in the perturbation series are kept. In principle, there is really no difference between the two techniques. Just as projection operators restrict the dynamical problem to a subspace of the Hilbert space of probability densities, the consolidated equations reflect this same restriction in the dynamical state space.

Therefore, although the equations defining the 2nd order stationary statistics are linear in those variables from projection operator theory and nonlinear in the consolidated equations from perturbation theory, both can be related via the mass operator $\hat{M}_Q$ and are renormalized to any specific order in μ. In this vein the spectrum $S_{GG}(\omega)$ arising from projection operator theory for the Duffing anharmonic oscillator equation is

$$S_{GG}(\omega) = \frac{\alpha(\omega_o^2 + \hat{M}_Q)}{(\omega_o^2 - \omega^2 + \hat{M}_Q)^2 + \alpha^2\omega^2} \tag{2.8}$$

where

$$\hat{M}_Q = 3 \left\{ \right. + 18 \left\{ \right. + \cdots \tag{2.9}$$

The straight and wavy rectangle and the point are, respectively, a *renormalized propagator*, a *renormalized spectral density*, and a renormalized vertex interaction. For general nonlinearities the only change occurring in (2.8) is in the diagrammatic structure of $\hat{M}_Q$. Which of the two equivalent representations yields the most rapidly converging solutions is probably dictated by the physical problem itself. A partial answer will be offered in the next section.

3. Statistical Linearization

Statistical linearization is a method specifically designed to produce accurate second order statistics for a large variety of nonlinear stochastic differential and integral equations.[3] It is based on the replacement in (0.1) of all nonlinear functions by linear functions so that by requiring that the mean squared error due to this substitution be minimal, one obtains an exact reproduction of the mean and an approximate expression for the dispersion. Procedurally it is akin to the Krylov-Bogoliubov asymptotic analysis of nonlinear deterministic differential equations[8] in that both assume solutions a posteriori and then ascertain the conditions under which they actually exist. Statistical linearization entertains an added dimension since it also lies at the roots of nonlinear optimization theory.

I would like to sketch the proof of this statement since it will be crucial to any future generalizations. To avoid overly complicating the analysis we consider a 1-dimensional version of (0.1) and rewrite it as

$$LG(t) = U(d/dt)F(t) \tag{3.1a}$$

where L is the nonlinear operator

$$L = Q(d/dt)H(G) + R(d/dt)G(t) \tag{3.1b}$$

and Q(d/dt), R(d/dt), and U(d/dt) are linear differential operators of time. The optimization problem involves choosing an optimal transformation L* belonging to a class $\mathbb{C}$, $L \notin \mathbb{C}$, which minimizes the mean squared error ε^2

$$\min \varepsilon^2 = \min \lim_{\Delta t \to \infty} \frac{1}{2\Delta t} \int_{-\Delta t}^{\Delta t} [(L^* - L)G]^2 \, dt$$

$$= \min \lim_{\Delta t \to \infty} \frac{1}{2\Delta t} \int_{-\Delta t}^{\Delta t} J(G,t)dt \tag{3.2}$$

due to replacement of L by L*. The solution is a standard problem in variational calculus. Setting $G(t) = \phi(t) + \kappa\eta(t)$ and constraining ϕ and η to pass through $\pm \Delta t$,

$$I(\kappa) = \lim_{\Delta t \to \infty} \frac{1}{2\Delta t} \int_{-\Delta t}^{\Delta t} J(\phi + \kappa\eta, t)dt \geq J(o) = \varepsilon^2 \tag{3.3}$$

$I(\kappa)$ therefore takes its minimum value at $\kappa = 0$ and eq. (3.2) is satisfied if

$$I'(0) = \lim_{\kappa \to 0} \frac{I(\kappa) - I(0)}{\kappa}$$

$$= \lim_{\kappa \to 0} \kappa^{-1} \lim_{\Delta t \to \infty} \frac{1}{2\Delta t} \int_{-\Delta t}^{+\Delta t} [\frac{\partial J}{\partial G} \kappa\eta + O(\kappa^2)]dt \qquad (3.4)$$

$$= 2 \langle \eta(L^* - L)G \rangle$$

$$= 0$$

The bracket symbol designates time averaging which for ergodic systems can be replaced by ensemble averaging. Since η is an arbitrary test function we can certainly choose it to be an element of $\mathbb{C}$ and therefore identical to L^*G. Condition (3.4) now becomes

$$\langle (L^*G)^2 \rangle = \langle (L^*G)(LG) \rangle \qquad (3.5)$$

If $\mathbb{C}$ also includes K = constant, $\eta = 1$ belongs to $\mathbb{C}$ and one can simultaneously satisfy (3.5) while preserving the mean

$$\langle L^*G \rangle = \langle LG \rangle \qquad (3.6)$$

For computational purposes one often identifies $\mathbb{C}$ with the set of orthogonal polynomials. The solution $G = (L^*)^{-1}F$ and its autocorrelation function $R_{GG}(\tau)$ are representable in terms of Wiener-Hopf integral equations and are quite laborious to calculate.[10] However, when $F(t)$ obeys Gaussian statistics and/or is harmonic with random phase

$$F(t) = \zeta(t) + a_F^{(1)} \sin(\Omega_F t + \phi) \tag{3.7}$$

these equations simplify considerably since L*G is linear. Under these conditions the optimization theory is exact, since Gaussian and harmonic processes are preserved under linear transformation. The implication of this can be easily seen upon rewriting (3.1) as a linear inhomogeneous differential equation in G(t)

$$[R(d/dt) + \pi Q(d/dt)]\, G(t) = U(d/dt)F(t) + \Phi[G(t)] \tag{3.8}$$

where

$$\Phi[G(t)] = Q(d/dt)[\pi G(t) - H(G)] \tag{3.9}$$

Upon transforming G(t) into a convolution integral equation it is simple to show that

$$R_{GG}(\tau) = \int_0^\infty\!\!\int d\tau_1 d\tau_2 h(\tau_1)h(\tau_2)\,[R_{U(\cdot)FU(\cdot)F}^{(\alpha)} + R_{\Phi U(\cdot)F}^{(\alpha)} + R_{U(\cdot)F\Phi}^{(\alpha)} + R_{\Phi\Phi}(\alpha)] \tag{3.10}$$

where $\alpha = \tau + \tau_2 - \tau_1$. Since Φ is nonlinear the mixed autocorrelation functions $R_{\Phi U(\cdot)F}^{(\alpha)}$, $R_{U(\cdot)F\Phi}^{(\alpha)}$, and $R_{\Phi\Phi}(\alpha)$ exhibit, besides purely Gaussian and harmonic components, non-Gaussian and harmonic distortion terms and noise-harmonic interference terms.[11] It is these latter terms which produce

the higher harmonics in the spectral density $S_{GG}(\omega)$; when small they can be neglected and $R_{GG}(\tau)$ reduces to

$$R_{GG}(\tau) = \iint_0^\infty h(\tau_1)h(\tau_2)R_{U(\cdot);\overset{(\alpha)}{FU(\cdot)F}}\, d\tau_1 d\tau_2 \tag{3.11}$$

This result has been termed statistical-harmonic linearization and could have been derived from (3.8) by setting $\Phi \equiv 0$. Rule of thumb criteria which permit this reduction for (3.1) are[3,11]:

1. (a) $\deg \hat{Q}(i\Omega_F) < \deg \hat{R}'(i\Omega_F)$
   (b) $|\hat{Q}(ik\Omega_F)/R'(ik\Omega_F) << |\hat{Q}(i\Omega_F)/R'(i\Omega_F)|$, $k \geq 2$
   (c) $\lim_{k\to\infty} |\hat{Q}(ik\Omega_F)/\hat{R}'(ik\Omega_F)| \to 0 \quad \forall k$

Conditions 1(b) and 1(c) guarantee the damping out of the higher harmonics of the periodic signal and the higher frequency components of the random noise by the effective linear part of the system $|\hat{Q}(i\Omega_F)/\hat{R}'(i\Omega_F)|$.

2. The polynomials $R'(ik\Omega_F)$ cannot have purely real zeros, $k = 1,2,3,\ldots$. This criterion guarantees stability of the solution, with transients dying out as $t \to \infty$.

3. The function H(G) should have finite partial derivatives with respect to its independent variables, and should not be an explicit function of time. Thus H(G) may belong to both the class of piecewise continuous and discontinuous function of Heaviside type.

The polynomials $\hat{R}'(\cdot)$ and $\hat{Q}(\cdot)$ are the Fourier-transformed

$$R(\cdot) - \frac{a_F(1)}{a}\left(\cos\psi - \frac{\sin\psi}{\Omega_F}\frac{d}{dt}\right) \quad \text{and} \quad Q(\cdot)$$

with $\psi = \omega t + \phi$.

Before considering a few specific examples of the preceding discussion a few comments on the nature of equations (3.10) and (3.11) are in order. Both results are renormalized since the Green's function $h(\tau)$ contains the linearizing coefficient(s) π. Equation (3.11) is still effectively nonlinear since it is linked to the nonlinearity $H(G)$ via the "least squares" fitting coefficient(s) π. This seems to preclude treatment of even nonlinearities but they can be handled effectively by addition of a constant bias term to πG. I shall come to this point again shortly.

4. Examples

A. Nonlinear stochastic systems exhibiting sustained oscillations (Van der Pol oscillator) or forced oscillations (Duffings oscillator with $F(t)$ given by (3.7))can be assumed to have an a posteriori solution

$$G(t) = G_h(t) + G_\zeta(t)$$

$$= a \sin \omega t + G_\zeta(t) \qquad (4.1)$$

with G_h, according to rule of thumb criterion 2, asymptotically stable. By linearizing H(G) according to the prescription in (3.5) and (3.6)

$$H(G) = L_1 G_h(t) + L_2 G_\zeta(t) \tag{4.2}$$

we find for L_1 and L_2

$$L_1(a,\Omega_F,\sigma_G^2) = (1/\pi a) \int_0^{2\pi} d\psi \sin\psi \int_{-\infty}^{\infty} H(G_\zeta + a \sin\psi) p(G_\zeta) dG_\zeta \tag{4.3a}$$

$$L_2(a,\Omega_F,\sigma_G^2) = (1/2\pi\sigma_G^2) \int_0^{2\pi} d\psi \int_{-\infty}^{\infty} G_\zeta H(G_\zeta + a \sin\psi) p(G_\zeta) dG \tag{4.3b}$$

For the Duffing oscillator (1.8) these become

$$L_1 = 3\beta(\tfrac{1}{4} a^2 + \sigma_G^2)$$

$$L_2 = 3\beta(\tfrac{1}{2} a^2 + \sigma_G^2) \tag{4.4}$$

Thus, the effect of a harmonic signal on noise is to shift the dispersion. L_1 and L_2 represent the gain of an effective linear equation to (1.8), since

$$\gamma^2 = \omega_0^2 + L_1 + L_2 \tag{4.5}$$

The autocorrelation function $R_{GG}(t)$ in (3.10) can be shown to be[11]

$$R_{GG}(t) = R_{GG}^{(0)}(t) + \frac{R_{G_hG_h}(t)}{(\gamma^2 - \Omega_F^2)^2 + \alpha^2\Omega_F^2} + \mu^2\left(\frac{a_F^6 \cos(3\Omega_F t)}{32[(\gamma^2 - 9\Omega_F^2)^2 + 9\alpha^2\Omega_F^2]}\right.$$

$$+ 6\int_{-\infty}^{+\infty} dt' R_{GG}^{(0)}(t')\left\{R^3_{G_\zeta G_\zeta}(t - t') + 3[R^2_{G_\zeta G_\zeta}(t - t')R_{G_hG_h}(t - t')\right.$$

$$\left.\left. + R_{G_\zeta G_\zeta}(t - t')\frac{1}{16} a_F^4 \cos 2\Omega_F(t - t')]\right\}\right) \tag{4.6}$$

with the statistical-harmonic linearized solution (3.11) deleting all terms multiplied by μ^2 and where $R_{GG}^{(0)}(t)$ is given by

$$R_{GG}^{(0)}(t) = \frac{d}{2\alpha\gamma^2} e^{-(\alpha/2)|t|}\left(\cos \Psi|t| + \frac{\alpha}{2\Psi} \sin \Psi|t|\right) \tag{4.7a}$$

with

$$\Psi = (\gamma^2 - \alpha^2/4)^{1/2} \tag{4.7b}$$

and spectral density $S_{GG}^{(0)}(\omega)$ as

$$S_{GG}^{(0)}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} R_{GG}^{(0)}(t)dt = \frac{d}{(\gamma^2 - \omega^2)^2 + \omega^2\alpha^2} \tag{4.7c}$$

In the absence of the harmonic excitation, $L_1 \equiv a_F^{(1)} \equiv 0$, $R_{GG}^{(0)}(t)$ and $R_{GG}(t)$ have been shown to be identical with the 1st Kraichnan-Wyld and quasi-linear Green's function approximations to the consolidated equations of Morton and Corrsin.[5,12] Fig. 1 depicts representative spectral density results for both an overdamped and an underdamped oscillator. Although not shown in this figure, the 1st K-W approximation (statistical linearization) always lies between the quasi-linear Green's function and cumulant discard curves. Note that $S_{GG}(\omega)$ diverges when expanded to second order in a perturbation series.

A marked improvement in statistical linearization is obtained by fixing the renormalized frequency γ^2 with the FPK variance σ_{GFPK}^2 (c.f. 4.7a)

$$\sigma_{GFPK}^2 = \frac{1}{2\alpha\gamma^2} \tag{4.8}$$

This method automatically gives the exact variance and corrects $R_{GG}(t)$ in the region where the approximate results differ most from the exact ones, i.e., near $t = 0$. Table I presents comparisons of $S_{GG}(\omega)$ as computed from its various approximate forms with the "exact" 2nd-K-W approximation of the consolidated equations (identical to an analog computer experiment) of Morton and Corrsin. The mean squared derivation D is given by

$$D \equiv \frac{1}{11} \sum_{i=1}^{11} [S_{GG}(\omega_i) - S_{GG}^{exact}(\omega_i)]^2$$

We find for the case $\alpha = 2$, $\mu = 1$,

$$D(\text{SL0}) = 2.02 \times 10^{-4}$$

$$D(\text{SL1}) = 1.37 \times 10^{-4}$$

$$D(\text{modified SL0}) = 3.36 \times 10^{-5}$$

The modified SL0 is by this criterion 4.1 times better than SL1 and 6.0 times better than SL0. For $\alpha = 0.5$, $\mu = 0.1$ we find

$$D(\text{SL0}) = 1.76 \times 10^{-3}$$

$$D(\text{SL1}) = 1.48 \times 10^{-3}$$

$$D(\text{modified SL0}) = 8.33 \times 10^{-4}$$

Here the modified SL0 is an improvement by a factor of 1.8 over SL1 and by a factor of 2.1 over SL0. In both cases, the modified statistical linearization gave better agreement with the exact results, as judged by the mean square deviation, than the quasilinear Green's function method (SL1).

B. For our second example consider the centered nonlinear Schrödinger equation in the presence of a Gaussian white noise excitation

$$\frac{i\partial G}{\partial \tau} - \frac{\partial^2 G}{\partial x^2} - \frac{2}{A_0^2} |G|^2 G = \zeta(t) \tag{4.9}$$

Its homogeneous solution is an envelope soliton

$$G_s = A_o \operatorname{sech}(x) \tag{4.10}$$

whose structure would not be preserved by statistically linearizing (4.9). We therefore choose as our a posteriori solution

$$G = G_s + \delta G \tag{4.11}$$

and rewrite (4.9) as a soliton equation and a fluctuating equation

$$\frac{i\partial G_s}{\partial \tau} - \frac{\partial^2 G_s}{\partial x^2} - \frac{2}{A_o^2} |G_s|^2 G_s = 0 \tag{4.12a}$$

$$\frac{i\partial \delta G}{\partial \tau} - \frac{\partial^2 \delta G}{\partial x^2} - \frac{2}{A_o^2} \left[|G|^2 G - |G_s|^2 G_s \right] = \varepsilon(t) \tag{4.12b}$$

We cannot directly statistically linearize (4.12b) since its nonlinear term contains even and odd functions of δG. Conditions (3.5) and (3.6) can still be satisfied, however, if we add a constant bias to δG, i.e., $\delta G = G_o + \delta\tilde{G}$ such that $< \delta G > = G_o$. Eq. (4.12b) can then be separated into

$$\frac{i\partial G_o}{\partial \tau} - \frac{\partial^2 G_o}{\partial x^2} - \frac{2A_o^2}{B_o^2} |G_o|^2 G_o = 0 \qquad (4.13a)$$

$$\frac{i\partial \delta\tilde{G}}{\partial \tau} - \frac{\partial^2 \delta\tilde{G}}{\partial x^2} - \frac{2}{A_o^2}\left[|G|^2 G - |G_s|^2 G_s - \frac{A_o^4}{B_o^2} |G_o|^2 G_o \right] = \zeta(t) \qquad (4.13b)$$

with G_o determined by solving

$$\frac{i\partial G_o}{\partial \tau} - \frac{\partial^2 G_o}{\partial x^2} - \frac{2}{A_o^2} \langle |G|^2 G - |G_s|^2 G_s \rangle = 0 \qquad (4.13c)$$

Computations on these equations are in the process of being completed.[13] They will be compared with the numerical solution of (4.9), of which some characteristic results are presented in Figs. (2-4). In these calculations the initial soliton at t = 0 has carrier wave number K = .2516 cm^{-1} and slope m = 0.064. The ambient field $\zeta(t)$ is given by a group of five modes with amplitude two orders of magnitude below that of the soliton and with random phases. An ensemble of perturbation calculations is generated with the phases of the ambient field changed randomly from realization to realization. As can easily be seen, although the soliton maintains its form in time, instabilities seem to be growing and becoming more important. They will eventually lead to dissolution of the soliton.

5. Discussion

In the past few sections we have discussed three types of approximation techniques for obtaining the statistical properties of nonlinear Langevin equations. In Fig. 5 I have constructed a flow chart indicating their interrelationships. The method of statistical linearization is obtained upon truncating the Dyson mass operator $\hat{M}$ at its first term. By choosing the consolidation operator Q to produce the exact mean-squared response at $t = 0$, truncation of its corresponding mass operator $\hat{M}_Q$ yields the FPK-statistical linearization result. Corrections to these first order schemes are a by-product of including higher order terms in $\hat{M}$ and $\hat{M}_Q$. These of course are necessary when the low passband criterion for the effective linear transfer function $|\hat{Q}(i\omega)/\hat{R}'(i\omega)|$ is not satisfied. The great advantage of statistical linearization and of its modification with the FPK variance lies in its great computational simplicity. Nonetheless, utilization of all the various techniques in conjunction with one another should produce powerful diagnostics when attempting to decipher the type of approximation methods needed for an accurate description of the dynamics of a nonlinear stochastic system.

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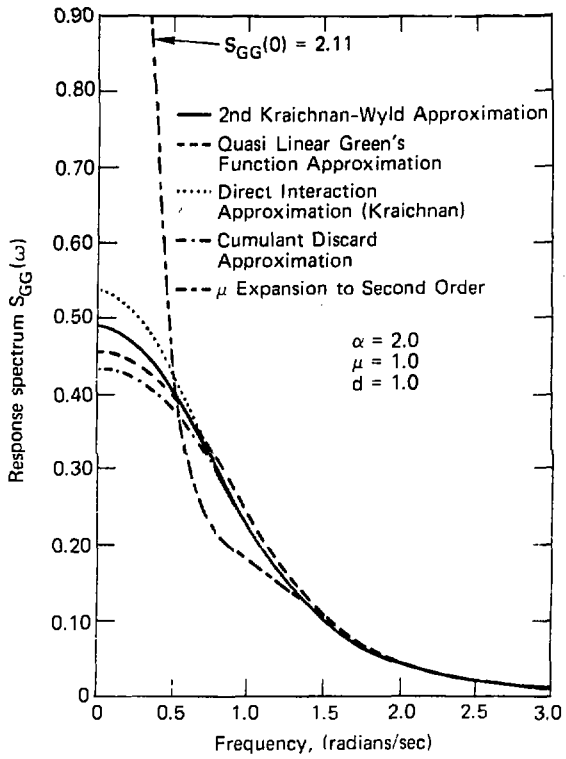
Calculated values of $S_{GG}(\omega)$

$g(G)=G^3$, $\alpha=2$, $\mu=1$

ω	MC KWII	STATISTICAL LINEARIZATION	STATISTICAL LINEARIZATION CORRECTED	STATISTICAL LINEARIZATION WITH FOKKER-PLANCK
0.0	0.48863	0.44444	0.45501	0.47646
0.5	0.40282	0.39024	0.40080	0.41035
1.0	0.22556	0.23529	0.24323	0.23802
1.5	0.10062	0.10458	0.10812	0.10371
2.0	0.04438	0.04494	0.04622	0.04443
2.5	0.02108	0.02102	0.02149	0.02081
3.0	0.01092	0.01084	0.01102	0.01075
3.5	0.00613	0.00608	0.00616	0.00604
4.0	0.00367	0.00365	0.00368	0.00363
4.5	0.00232	0.00231	0.00233	0.00230
5.0	0.00154	0.00153	0.00154	0.00153

$g(G)=G^3$, $\alpha=0.5$, $\mu=0.1$

0.0	0.71639	0.64867	0.66660	0.66841
0.5	1.06344	0.95620	0.98518	0.99057
1.0	3.29662	3.24280	3.36620	3.33560
1.5	0.61345	0.63318	0.66654	0.61846
2.0	0.11607	0.11616	0.12007	0.11480
2.5	0.03765	0.03753	0.03855	0.03727
3.0	0.01614	0.01602	0.01642	0.01594
3.5	0.00813	0.00805	0.00820	0.00802
4.0	0.00453	0.00451	0.00456	0.00450
4.5	0.00273	0.00273	0.00275	0.00272
5.0	0.00175	0.00175	0.00177	0.00175



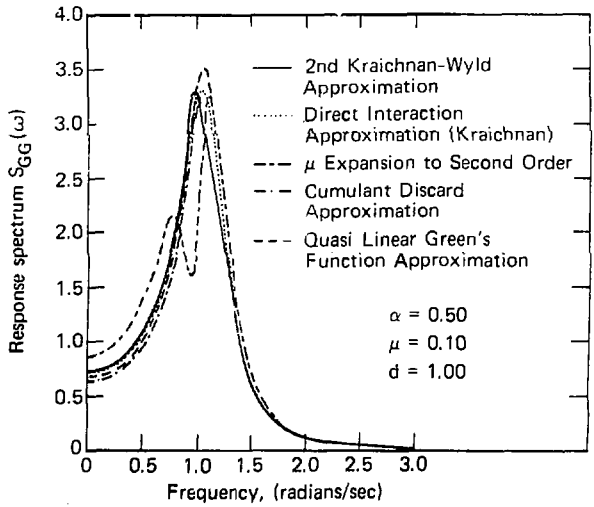


Figure 1:

Spectral density $S_{GG}(\omega)$ for overdamped (upper) and underdamped (lower) Duffing oscillator.

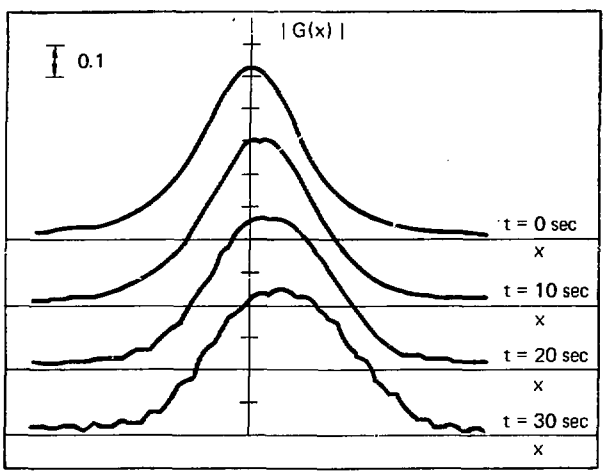


Fig. 2: Time snapshots of numerical solution to $|G(x)|$ from (4.9).

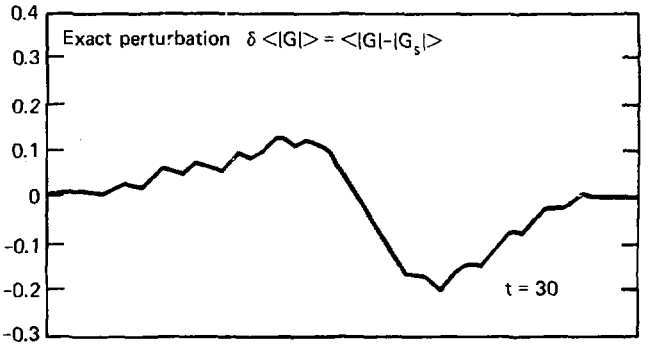


Fig. 3: Numerical solution at t = 30 for the exact perturbation $\delta \langle |G(x)| \rangle$.

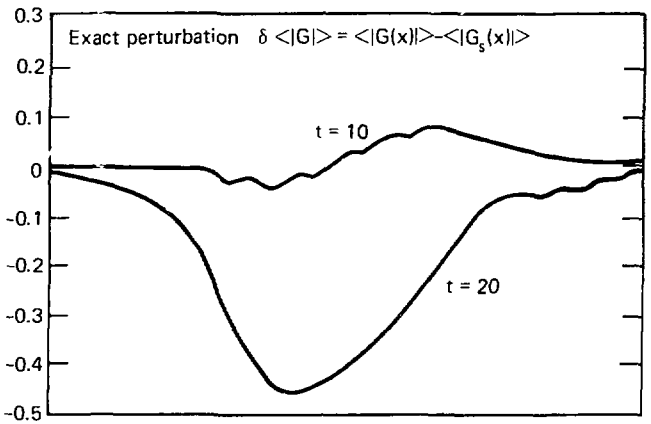


Fig. 4: Numerical solution at t = 10 and t = 20 for the exact perturbation δ <|G(x)|>. Phase information averaged out.

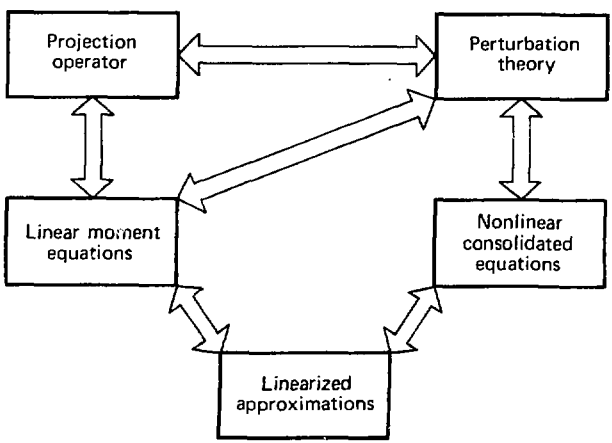


Fig. 5: Flowchart depicting perturbation, projection operator, and consolidated equation hierarchies.