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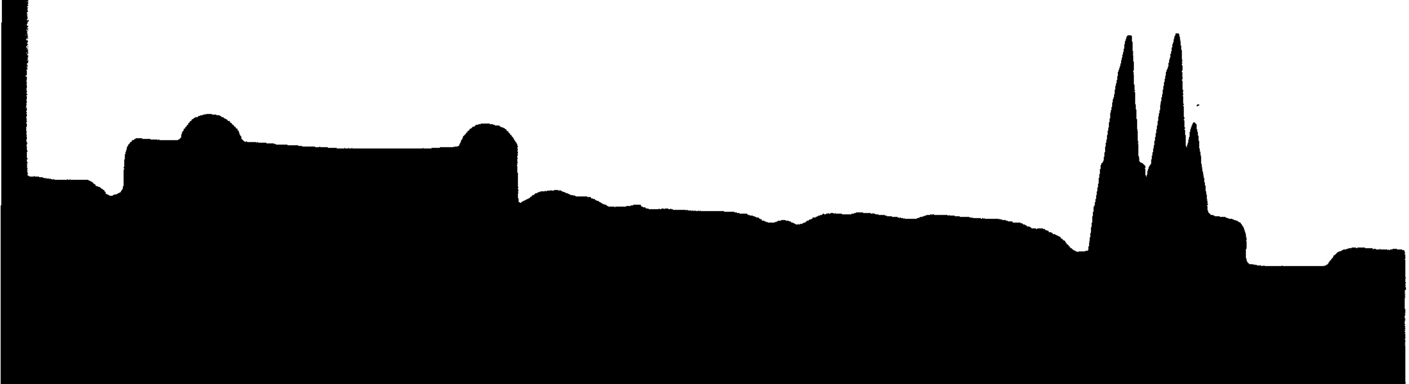

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Department of Tehoretical Electrotechnics
Institute of Technology, Uppsala University
Box 534
S-751 21 Uppsala
Sweden

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Adiabatic and non-adiabatic electron oscillations in a static electric field

Christer Wahlberg
Institute of Technology, Uppsala University, Box 534, S-751 21 Uppsala, Sweden

Abstract. The influence of a static electric field on the oscillations of a one-dimensional stream of electrons is investigated. In the weak field limit the oscillations are adiabatic and mode coupling negligible, but becomes significant if the field is stronger. The latter effect is believed to be of importance for the stability of e.g. potential double layers.

1. Introduction

Recent developments in plasma physics have shown that static electric fields are involved in quite a few plasma phenomena where the classical resistivity is essentially zero. These phenomena include e.g. parallel fields in magnetic mirrors [1], plasma beam-curved magnetic field interaction experiments [2], potential double layers [3] and turbulent resistivity [4]. As regards the further properties of such plasmas very little is known. Specifically, the difficult problem of stability of general BGK-equilibria, of which the one-dimensional double layers form a subclass, is as yet unsolved. The aim of the following analysis is to demonstrate a few basic features of the interaction between a stream of electrons and a static electric field, which might have consequences for the stability problem mentioned.

2. Basic equations

Neglecting the thermal spread, the following equations describe the motion of the electron stream in the electrostatic field $E_o(x)=-d\phi/dx$:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv)=0, \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{eE}{m_e}, \quad \frac{\partial E_1}{\partial x} = -\frac{en_1}{\varepsilon_o} \quad .$$

$n=n_o(x)+n_1(x,t)$ is the electron density, $v=v_o(x)+v_1(x,t)$ the velocity and $E=E_o(x)+E_1(x,t)$ the electric field. In the stationary state it follows that the electron density and velocity are $n_B/G(x)$ and $v_BG(x)$, respectively, where $G(x)=[1+2e\phi(x)/m_ev_B^2]^{1/2}$, and n_B and v_B denote the density and velocity at points where $\phi(x)=0$. It is straightforward to show [5] that the linearized versions of the equations above give the following equation for $E_1(x,t)$:

$$v_B^2\frac{\partial}{\partial x}\left[G^3(x)\frac{\partial E_1}{\partial x}\right]+2v_BG^2(x)\frac{\partial^2 E_1}{\partial x\partial t}+2v_BG(x)G'(x)\frac{\partial E_1}{\partial t}+G(x)\frac{\partial^2 E_1}{\partial t^2}+\omega_B^2E_1=0, \qquad (1)$$

where $\omega_B=(e^2n_B/\varepsilon_om_e)^{1/2}$. Furthermore, the energy conservation equation $\partial W_1/\partial t+\partial F_1/\partial x=0$ associated with eq. (1) involves the following expressions for the energy density and flux, respectively

$$W_1(x,t)=\frac{1}{2}\varepsilon_o\omega_B^{-2}G(x)\left(\frac{\partial E_1}{\partial t}\right)^2-\frac{1}{2}\varepsilon_ov_B^2\omega_B^{-2}G^3(x)\left(\frac{\partial E_1}{\partial x}\right)^2+\frac{1}{2}\varepsilon_oE_1^2, \qquad (2a)$$

$$F_1(x,t)=\varepsilon_ov_B^2\omega_B^{-2}G^3(x)\frac{\partial E_1}{\partial x}\frac{\partial E_1}{\partial t}+\varepsilon_ov_B\omega_B^{-2}G^2(x)\left(\frac{\partial E_1}{\partial t}\right)^2. \qquad (2b)$$

In the following we will use two different approaches to the equation.

2.1. Lagrangian description. We take a certain point, fixed in the stream (initially at $x=x_o$) and describe the field at this point as a function of x. This field we denote by $E_1(x;x_o)$. Using the expression for the stationary electron trajectories it is then easily shown [5] that eq. (1) turns into

$$v_B^2\frac{d}{dx}\left[G^3(x)\frac{d}{dx}E_1(x;x_o)\right]+\omega_B^2E_1(x;x_o)=0. \qquad (3)$$

Moreover, the energy density, kinetic plus potential, of the electron oscillations, as observed in the moving coordinate system, is given by

$$W_1(x;x_o)=\frac{1}{2}\varepsilon_ov_B^2\omega_B^{-2}G^3(x)\left[\frac{d}{dx}E_1(x;x_o)\right]^2+\frac{1}{2}\varepsilon_oE_1^2(x;x_o). \qquad (4)$$

2.2. Coupled mode description. We let $E_1(x,t)$ be given by $a(x)\exp(-i\omega t)$ and define the normal modes $a_\pm(x)$ according to

$$a_\pm(x)=(ik_\mp a-da/dx)/(ik_\mp-ik_\pm),$$

where $k_\pm(x)=[\omega\mp\omega_B G^{-1/2}(x)]/v_B G(x)$ are the wave numbers for the fast (pos.energy) and slow (neg.energy) waves. By use of eq. (1) we then find that

$$da_\pm/dx-ik_\pm a_\pm=\frac{3}{4}G'(x)G^{-1}(x)(a_\mp-a_\pm). \tag{5}$$

It is convenient to introduce the normal mode "amplitudes" $A_\pm(x)$ defined by

$$A_\pm(x)=a_\pm(x)G^{3/4}(x)\exp\left[-i\int_0^x k_\pm(u)du\right], \tag{6}$$

in which case the (averaged) energy flux (2b) for the fast and slow modes are proportional to $\pm|A_\pm(x)|^2$, respectively, and the total flux is

$$\langle F_1\rangle=\frac{1}{2}\varepsilon_0\omega v_B\omega_B^{-1}\left(|A_+(x)|^2-|A_-(x)|^2\right).$$

Moreover, we introduce the dimensionless parameter

$$\lambda_B(x)=4\pi eE_0(x)/m_e v_B(x)\omega_B(x), \tag{7}$$

where $v_B(x)$ and $\omega_B(x)$ denote local values of the streaming velocity and plasma frequency, respectively, and the variable

$$y=\omega_B(2\pi v_B)^{-1}\int_0^x G^{-3/2}(u)du.$$

Then the coupled mode equations (5) can be written

$$dA_\pm/dy=-\frac{3}{8}\lambda_B(y)\exp\left[\pm 4\pi iy\right]A_\mp. \tag{8}$$

The parameter λ_B, that essentially determines the strength of the coupling, can be given a simple physical interpretation, namely the relative change in kinetic energy, during one plasma period, for the streaming electrons.

3. Weak field. When $|\lambda_B(x)| \ll 1$ the mode coupling is clearly negligible, and the situation is rather similar to the field-free case. However, in order to conserve the energy flux $\pm|A_\pm(x)|^2$, the amplitude of the oscillations must vary slowly according to $|a_\pm(x)| \sim G^{-3/4}(x)$, by virtue of (6). This result can be given another physical explanation by use of the Lagrangian description in sec. 2.1. One observes that the expression (4) for the energy density can be given the form of a Hamiltonian for a harmonic oscillator, i.e.

$$W_1(x;x_0)=p^2/2M(x)+ \frac{1}{2}M(x)\Omega^2(x)q^2, \tag{9}$$

where

$$q=E_1(x;x_0), \quad p=\varepsilon_0 v_B{}^2\omega_B{}^{-2}G^3(x)\frac{d}{dx}E_1(x;x_0),$$

$$M(x)=\varepsilon_0 v_B{}^2\omega_B{}^{-2}G^3(x), \quad \Omega(x)=\omega_B v_B{}^{-1}G^{-3/2}(x).$$

Moreover, Hamilton's equations resulting from (9) are equivalent to eq. (3). Thus, for sufficiently slow variation of $M(x)$ and $\Omega(x)$, we can directly exploit the result from classical mechanics which states that the quantity W_1/Ω, i.e.

$$Q=W_1(x;x_0)G^{3/2}(x) \tag{10}$$

is an adiabatic invariant. It is easy to show that the conditions for slow variation, $2\pi|d\Omega/dx| \ll \Omega^2$ and $2\pi|dM/dx| \ll M\Omega$, equivalently can be formulated $|\lambda_B(x)| \ll 1$, λ_B given by (7). If adiabatic conditions prevail, the energy density is locally proportional to the square of the amplitude $a(x;x_0)$, so that the invariance of Q requires $a(x;x_0) \sim G^{-3/4}(x)$, as we found before. This result explains some earlier calculations on electron oscillations on accelerated streams, e.g. [6].

4. Strong field. If the adiabatic condition $|\lambda_B(x)| \ll 1$ is not fulfilled, the invariant (10) is in general violated, and at the same time mode coupling becomes significant. We now examine the details of the coupling in two special cases. (A similar treatment, with application to microwave tubes, has been given in [7].)

4.1. <u>Constant λ_B potential.</u> Such a potential can be realized by choosing

$$\phi(x)=m_e v_B^2\left[(1-3\omega_B\lambda_B x/8\pi v_B)^{4/3}-1\right]/2e.$$

In this case eqs. (8) are easily solved, and the solution, with $A_-(0)=0$, is:

$$\lambda_B<\lambda_B^C: |A_+(y)/A_+(0)|^2=1+(\beta^{-2}-1)\sin^2 2\pi\beta y,$$

$$\lambda_B=\lambda_B^C: |A_+(y)/A_+(0)|^2=1+4\pi^2 y^2 ,$$

$$\lambda_B>\lambda_B^C: |A_+(y)/A_+(0)|^2=1+(\beta^{-2}+1)\sinh^2 2\pi\beta y$$

where $\lambda_B^C=16\pi/3\approx17$, $\beta=|1-(\lambda_B/\lambda_B^C)^2|^{1/2}$.

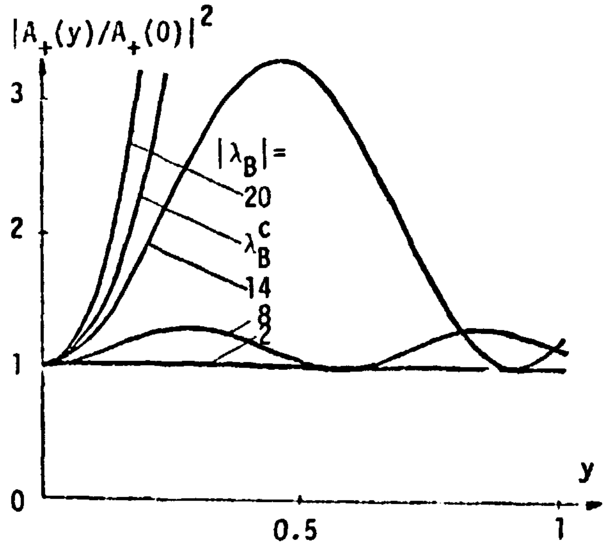

Fig.1. Energy flux for the fast beam mode in a potential with λ_B = constant.

In fig. 1 $|A_+(y)/A_+(0)|^2$ is shown for a few values of λ_B. If $\lambda_B<\lambda_B^C$ the energy flux is oscillating between the fast and slow modes ($|A_-(y)|^2=|A_+(y)|^2-|A_+(0)|^2$), and the amplitude of the oscillations increases with λ_B. If $\lambda_B>\lambda_B^C$ The energy flux increases exponentially, and the situation is similar to an ordinary convective instability. The critical accelerating field, above which this instability occurs is

$$E_o^C=4m_e v_B\omega_B/3e$$

4.2. Constant field. Choosing $\phi(x)=-E_0x$ we can write $\lambda_B(y)=\lambda_{B0}/(1-\frac{1}{4}\lambda_{B0}y)$. In this case the solution to the coupled mode eqs. (8) can be given in terms of Bessel functions. With $z=1-\frac{1}{4}\lambda_{B0}y$, one finds that (again choosing $A_-(y=0)=0$)

$$|A_+(z)/A_+(1)|^2=\pi^2\sigma^2z\Big\{\Big[Y_1(\sigma)J_1(\sigma z)+$$

$$+Y_2(\sigma)J_2(\sigma z)-J_1(\sigma)Y_1(\sigma z)-J_2(\sigma)Y_2(\sigma z)\Big]^2+$$

$$+\Big[Y_1(\sigma)J_2(\sigma z)-Y_2(\sigma)J_1(\sigma z)-J_1(\sigma)Y_2(\sigma z)+$$

$$+J_2(\sigma)Y_1(\sigma z)\Big]^2\Big\}/16,$$

where $\sigma=8\pi/|\lambda_{B0}|$.

In fig. 2 this solution is shown together with $\lambda_B(z)=\lambda_{B0}/z$. σ is 3/2 in the figure, which means that $|\lambda_{B0}|=\lambda_B^C$. Thus $z\leq1$ corresponds to the unstable range $\lambda_B\geq\lambda_B^C$ found in sec. 4.1., whereas $z>1$ corresponds to $\lambda_B<\lambda_B^C$, where the oscillating character of the energy flux is observed, and the amplitude of the oscillations decreases together with λ_B.

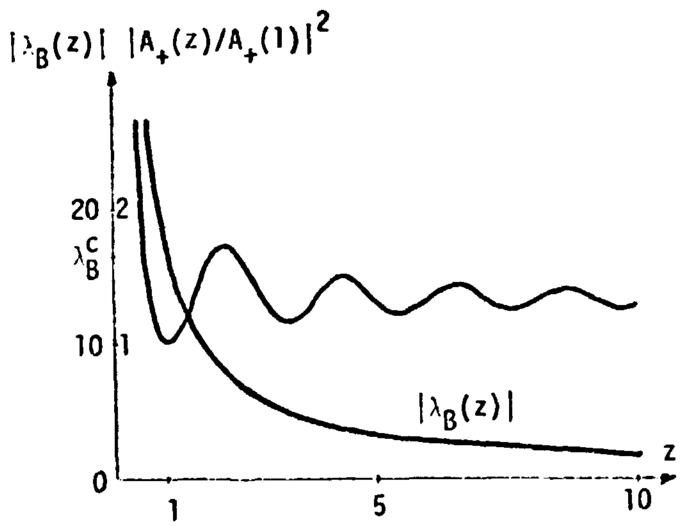

Fig.2. Energy flux for the fast beam mode, and local λ_B in a constant electric field.

5. Stability discussion. It is well known that certain microinstabilities can be analysed within the framework of coupled mode theory. For instance is it possible to interpret the ordinary two-stream instability as being caused by the coupling between the negative energy wave in the stream and the positive energy wave in the plasma [8]. However, if the plasma is penetrated by an electric field, maintained e.g. by one of the mechanisms mentioned in sec. 1, it would of course be necessary to include the additional mode coupling, that we have dealt with here, in a stability

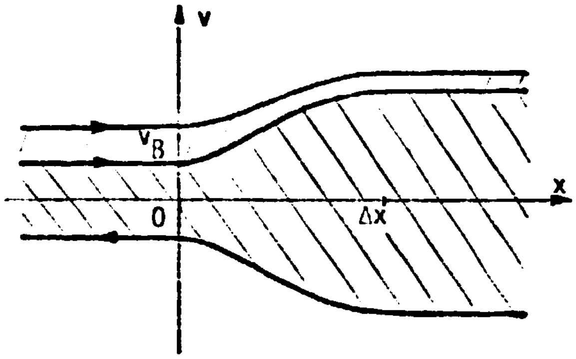

Fig.3. Schematic picture of the phase space distribution of electrons in a typical double layer.

analysis. As a particular example let us take the double layer [3]. In fig. 3 a schematic picture of the phase space distribution of electrons for a typical layer is shown. The potential (not shown) is assumed to increase monotonically from a constant value for $x<0$ to another constant value for $x>\Delta x$. The upper dashed region shows a population of beam-like accelerated electrons, and the lower region a population of thermal electrons, of which some are reflected by the potential. It is experimentally observed [3] that the critical drift velocity between the electrons and ions, above which the layer is formed, is of the order of the electron thermal velocity. Thus, in order to make an order of magnitude estimate of the parameter λ_B (7), let us use $v_B \sim (kT_e/m_e)^{1/2}$, which gives a minimum value for the drift velocity of the accelerated electron component. Moreover, the thickness of the layer is observed [3] to be of the order of a few Debye lengths, and we can, for convenience, use the estimate $E_0 \sim \Delta\phi/4\pi\lambda_D$, where $\Delta\phi$ is the total potential drop across the layer, and λ_D the Debye length. Then, using $\omega_B = (kT_e/m_e)^{1/2}/\lambda_D$, we find that

$$\lambda_B \sim e\Delta\phi/kT_e.$$

However, a characteristic feature of the double layer, inside which quasi-neutrality is not valid, is that $e\Delta\phi/kTe \sim O(1)$ or even $\gg 1$ for a strong shock [9] and we see that this in fact is an example of a situation where the electron oscillations are non-adiabatic, and mode coupling significant. In view of this result it seems doubtful whether local Penrose-stability [o] is sufficient for stability of this electron distribution. Indeed, it is quite possible that even if the Penrose-criterion predicts stability for all x, the following feedback mechanism can drive an absolute instability anyway: Let us consider a very simple case with a cold accelerated componenet and a flat ("water-bag") distribution of trapped electrons with $v_{th} \sim v_B$ (i.e. $f_e(x,v)$ = constant in the lower dashed region in fig 3). In fig 4 the local

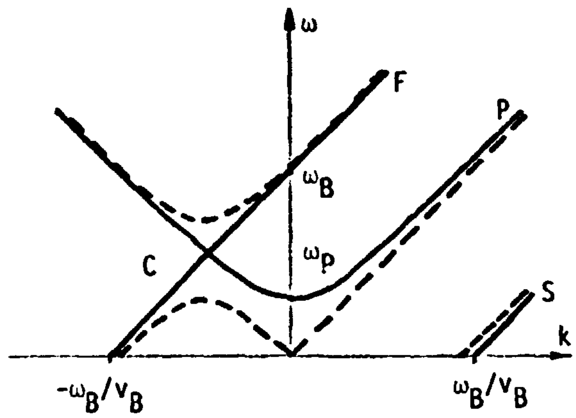

Fig.4. Local dispersion relation for the fast (F) and slow (S) beam modes and the plasma (P) modes. The dashed curves show dispersion relation for the coupled beam-plasma system.

dispersion relation is shown for i) the accelerated electron component (assumed cold) and the thermal component ($v_{th} \sim v_B$) separately (solid lines), and ii) the entire system (dashed lines). The point C shows where the fast beam mode (F) couples to the plasma mode (P). This coupling is of evanescent type and the system is locally stable. However, it is interesting to evaluate the frequency at the coupling point C as a function of the potential ϕ in the plasma. It is very easy to show (5) that situations can exist such that the coupling frequency is the same in regions $x < 0$ and $x > \Delta x$ of the plasma, but lower within the layer ($0 < x < \Delta x$). This result also applies for the cut-off frequency, i.e. the frequency at the minimum of the dashed curve above the point C in fig 4. Then assume that such an equilibrium is perturbed so that e.g. the fast mode, at (or close to) the coupling frequency ω_0, carries the energy flux F_B into the layer. Here the coupling to the thermal component is weak (if $\omega_0 >$ local cut-off), but the coupling to the slow mode is strong, if λ_B is large. Since the slow mode has negative energy, the flux carried by the fast mode when it passes $x = \Delta x$ is always larger than F_B e.g. gF_B, $g > 1$. Due to the evanescent type of coupling for $x > \Delta x$ this flux will be entirely transferred to the plasma wave, which has negative group velocity and thus sends the flux gF_B back into the layer. In general one must expect that some of this flux will be reflected (i.e. coupled to the plasma wave with positive group velocity) and we can assume that the amount tgF_B, $t < 1$, penetrates the layer entirely. However, this flux will also be entirely transferred to the fast beam mode and carried into the layer again. Clearly, if $tg > 1$, this feedback loop might increase the energy flux exponentially, and the configuration is absolutely unstable. On the other hand, if $tg < 1$, the flux is likely to decay exponentially. Preliminary calculations (5) indicate that one always has $tg < 1$ in the limit $\Delta x \to 0$, whereas $tg > 1$ indeed can be fulfilled for finite Δx and a properly chosen form of ϕ (x). However, a rigorous mathematical treatment of this phenomenon becomes very complicated, and our main effort at present is directed to finding a suitable approach that reduces the difficulties to a manageable level.

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