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Theory for Stationary Nonlinear Wave
Propagation in Complex Magnetic Geometry

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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, Japan.

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Abstract

We present our recent efforts to derive a systematic calculation scheme for nonlinear wave propagation in the self-consistent plasma profile in complex magnetic-field geometry.

Basic assumptions and/or approximations are i) use of the collisionless two-fluid model with an equation of state; ii) restriction to a steady state propagation and iii) existence of modified magnetic surface, modification due to Coriolis' force.

We discuss four situations: i) weak-field propagation without static flow, ii) arbitrary field strength with flow in axisymmetric system, iii) weak field limit of case ii) and iv) arbitrary field strength in nonaxisymmetric torus. Except for case iii), we derive a simple variation principle, similar to that of Seligar and Whitham, by introducing appropriate coordinates. In cases i) and iii), we derive explicit results for quasilinear profile modification.

1. Introduction

During the last decade, a considerable progress has been achieved in the understanding of the basic physical effects associated with the propagation of an intense high-frequency wave in a plasma [1]. These include particle trapping, stochastic orbit modification of resonant particles, modification of the average plasma profile due to the ponderomotive force and the associated effects on the wave propagation, sideband excitation by parametric instabilities, formation of a high-energy tail in the particle distribution function, etc. However, most of the investigations concerning these nonlinear effects have been restricted to the case of a plasma in a simple geometrical configuration, such as an isotropic plasma, a plasma in a uniform magnetic field, etc.

The aim of the present research is to extend some of these analyses to the situations where the plasma is confined by a magnetic field of complex geometrical structure. The principal problem here is to derive a self-consistent set of general and relatively simple formulas which are valid for a large class of magnetic field configurations and which are at the same time useful for numerical analyses.

Toward this goal, we have recently been developing a formulation based on the collisionless two-fluid description of the plasma [2,3]. Particle aspects are entirely neglected here and the principal nonlinear effects are those due to the ponderomotive force of the oscillating field. In addition, we make the following assumptions and/or approximations:

1) We assume a steady state propagation of the wave having periodic dependence on time;

2) We assume the existence of a generalized magnetic surface for each species of the particle, the generalization being made in order to include the effect of Coriolis' force due to the plasma flow; and

3) We assume an equation of state which relates the pressure P to the density n, the generalized flux function ψ and the time t for each particle species.

Assumption 1) excludes the sideband instabilities; they are related to the problem of the stability of the steady state obtained. Assumption 3) is introduced to close the fluid equations by equations of continuity and motion. Based on these assumptions and/or approximations, we have investigated several situations which cover a large class of problems relevant to the thermonuclear fusion research. In all cases that we have investigated, we arrive at a surprisingly simple variation principle which is of similar structure to that derived by Seligar and Whitham [4]. The variation principle will be useful for numerical analyses since the differential operations contained in it are of lower order than those of the original set of equations.

Since our work is at a stage of day-by-day progress, we here present our work according to the historical development of the theory. First, in section 2, we consider the situation where the oscillating field is weak and the dominant nonlinear mechanism is the density profile modification due to the ponderomotive force along the magnetic field [2]. The plasma flow is neglected as it is unessential and the pressure is assumed to be a function of the density alone. Three important results obtained from this analysis are i) derivation of a simple general expression for the ponderomotive force for given oscillating field profile, ii) derivation of the stationary density profile obtained by the static force balance along the magnetic field, and iii) derivation of a variation principle to self-consistently determine the oscilating field profile. Then in section 3, by restricting ourselves to an axially symmetric system, we extend the analysis to include other nonlinear effects as well, that is, the modification of the plasma flow, that of the magnetic field and the higher-order nonlinear contributions [3]. Here a measure progress is made by the introduction of the modified scalor and vector potentials which take into account the effects of the centrifugal force and Coriolis' force, respectively, due to the plasma flow. Use of the Clebsch representation for the velocity components then allows us to derive a variation principle similar to ref. [4]. The special case of weak oscillating field in an axisymmetric system is discussed in section 4, where we derive explicit expressions or equations for the quasilinear modification of various physical quantities. An important simplification is achieved here by the introduction of a generalized displacement vector or polarizability. Finally in section 5, we consider a general toroidal system without axial symmetry and derive a similar variation principle by using Hamada-like coordinates in which the magnetic field line becomes straight. This formulation will be applicable not only to nonaxisymmetric confinement systems, such as the stellarator, the heliotron, the bumpy torus, etc., but also to the tokamak subject to a nonaxisymmetric oscillating field. Brief summary is given in the last section.

2. Weak Field Propagation in Self-Consistent Density Profile

In this section, we consider the situation where the oscillation field is weak and the dominant nonlinear effect is the density profile modification due to the ponderomotive force along the static magnetic field, as in the case of a sufficiently low-β plasma. As mentioned in section 1, we neglect the static plasma flow and assume that the pressure is a known function of the density alone, P=P(n). Within these approximations and/or assumptions, the formulation presented here is valid for arbitrary magnetic feild configuration, provided that there exist magnetic surfaces determined by the average magnetic flux function $\bar{\psi}_m$ which is defined by the relation

$$\bar{\vec{B}}\cdot\vec{\nabla}\ \bar{\vec{\psi}}_m = 0 \quad , \tag{1}$$

where $\bar{\vec{B}}$ is the static part of the magnetic field; it is equal to the unperturbed magnetic field, $\vec{B}_0$, since we are neglecting the static plasma flow.

Our starting equation is the equation of motion in the collisionless two-fluid model,

$$\frac{\partial}{\partial t} \vec{v} + \vec{v}\cdot\vec{\nabla}\vec{v} = \frac{q}{m} \{ \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} - \frac{1}{q} \vec{\nabla} H(n) \} \quad , \tag{2}$$

where $H(n) = \int dP(n)/n$ and the other notation is self-explanatory. The suffix designating the particle species is suppressed for brevity. Denoting the time average by the bar, we can then write the static force balance as

$$m \overline{\vec{v}\cdot\vec{\nabla}\vec{v}} - (q/c) \overline{\vec{v} \times \vec{B}_1} = - \nabla(q\overline{\Psi} + \overline{H}) \quad , \tag{3}$$

where $\vec{B}_1$ is the oscillating part of the magnetic field and Ψ is the electrostatic potential with $\overline{\vec{E}} = - \vec{\nabla}\overline{\Psi}$. The left-hand side of this equation is the ponderomotive force due to the oscillating field. We calculate it in the lowest order approximation, that is, by using the linear approximation for $\vec{v}$,

$$\dot{\vec{v}} = (q/m) (\vec{E}_1 + \vec{v} \times \vec{B}_0/c - \vec{\nabla}H_1/q) \, , \tag{4}$$

where the dot denotes the time derivative and the suffix 1 depicts the oscillating component. Using the relations,

$$\vec{v}\cdot\vec{\nabla}\vec{v} = \vec{\nabla} |\vec{v}|^2/2 - \vec{v} \times (\vec{\nabla} \times \vec{v}) \quad , \tag{5}$$

$$\overline{\dot{X}Y} = - \overline{X\dot{Y}} \quad , \tag{6}$$

and introducing the oscillating displacement vector $\vec{R}$ by the rations, $\dot{\vec{R}} = \vec{v}$ and $\overline{\vec{R}} = 0$, we get

$$m \overline{\vec{v}\cdot\vec{\nabla}\vec{v}} - (q/c)\overline{\vec{v} \times \vec{B}_1} = m\vec{\nabla} \overline{|\vec{v}|^2}/2 + m\overline{\vec{R}\times\vec{\nabla}\times(\dot{\vec{v}}-q\vec{E}_1/m)}$$

$$= m\vec{\nabla}\overline{|\vec{v}|^2}/2 + (q/c)\overline{\vec{R}\times\vec{\nabla}\times(\vec{v}\times\vec{B}_0)} \; . \tag{7}$$

Taking the component along $\vec{B}_0$, we find after some algebra [2] the following relation,

$$\vec{B}_0 \cdot m[\overline{\vec{v}\cdot\vec{\nabla}\vec{v}} - (q/c)\overline{\vec{v}\times\vec{B}_1}] = \vec{B}_0\cdot\vec{\nabla}(m/2)\{\overline{|\vec{v}|^2} + (q/mc)\overline{(\dot{\vec{R}}\times\vec{B}_0)\cdot\vec{R}}\} \quad (8)$$

$$= -\vec{B}_0\cdot\vec{\nabla}\,\overline{q\vec{R}\cdot(\vec{E}_1 - \vec{\nabla}H_1/q)}/2 \ , \quad (9)$$

where the second line can be derived by using eq.(4) for $\overline{|\vec{v}|^2} = -\overline{R\cdot\dot{v}}$. We can express the polarization vector $q\vec{R}$ and the pressure perturbation, $H_1 \doteq H'(\bar{n})n_1$, in terms of $\vec{E}_1$ by using the linear approximation. Equation (9) then implies that the ponderomotive force along $\vec{B}_0$ is derivable from a potential which is a known function of oscillating field.

Using eq.(9), we integrate eq.(3) along $\vec{B}_0$ to obtain

$$q\bar{\Psi} + \bar{H} - \overline{\vec{R}\cdot(q\vec{E}_1 - \vec{\nabla}H_1)}/2 = f(\bar{\psi}_m) \ , \quad (10)$$

where f is an arbitrary function of $\bar{\psi}_m$. On the left-hand side, $\bar{\Psi}$ is expressible in terms of $\bar{n}$ by the Poisson equation and $\bar{H}$ can be written, correct to second order, as $\bar{H} = H(\bar{n}) + H''(\bar{n})\overline{n_1{}^2}/2$. Then knowing the equation of state, P=P(n), we can in principle solve eq.(10) for $\bar{n}$ in terms of the oscillating variables.

In the particular case of isothermal variation, i.e. $\bar{P} = \bar{n}\,T$ (T = constant), with the ions singly-ionized and with local charge neutrality, $\bar{n}_e = \bar{n}_i$, we obtain

$$\bar{n} = N(\bar{\psi}_m)\exp\left\{\frac{\sum_s q_s\overline{\vec{R}_s\cdot\vec{E}_1} + \sum_s T_s[\overline{n_{s1}^2}/\bar{n}^2 - \overline{\vec{R}_s\cdot\vec{\nabla}(n_{s1}/\bar{n})}]}{2(T_e + T_i)}\right\}, \quad (11)$$

where $N(\overline{\psi}_m)$ is the density in the absence of the oscillating field and the suffix s denotes the particle species. This is a highly nonlinear equation for $\bar{n}$, but if the oscillating field is sufficiently weak, we can approximate $\bar{n}$ in the exponent by $N(\overline{\psi}_m)$.

We now derive a variation principle for the self-consistent determination of the oscillating field. To this end, we use Ampère's equation with the magnetic field eliminated by Faraday's induction law:

$$\frac{\partial^2}{\partial t^2} \vec{E}_1 + c^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{E}_1) = - 4\pi \sum_s \bar{n}_s q_s \frac{\partial^2}{\partial t^2} \vec{R}_s , \tag{12}$$

where on the right-hand side we retained only those nonlinear effects that arise from the density profile modification. To lowest order, we can replace $\partial^2/\partial t^2$ by $-\omega^2$, where ω is the wave frequency. We then take the scalor product of eq.(12) with the field variation $\delta\vec{E}_1$ and integrate over the entire space. Assuming a boundary condition that the surface integral vanishes, we then obtain,

$$\delta \int d^3\vec{x} \; (\overline{|\vec{E}_1|^2} - \overline{|\vec{B}_1|^2}) = -8\pi \int d^3\vec{x} \sum_s \bar{n}_s \overline{\delta\vec{E}_1 \cdot q_s \vec{R}_s} \; . \tag{13}$$

The right-hand side can be written correct to second order as [2],

$$\int d^3\vec{x} \; \overline{n_s} \; \overline{\delta\vec{E}_1 \cdot q_s \vec{R}_s} = \delta \int d^3\vec{x} \; \overline{P}_s , \tag{14}$$

from which we finally obtain the following variation principle,

$$\delta\int d^3\vec{x} \quad \{(\overline{|\vec{E}_1|^2} - \overline{|\vec{B}_1|^2})/8\pi + \sum_s \bar{P}_s\} = 0 . \quad (15)$$

The form of this variation principle has resemblance to that in the fluid theory [4]. However, here the variation is taken with respect to $\vec{E}_1$ only, all the other variables being determined in terms of $\vec{E}_1$ by the linear theory and eq.(10). Therefore, our variation principle will be extremely useful for numerical analyses.

3. General Formulation for Axisymmetric System

In general, an intense oscillating field causes an average plasma flow due to the ponderomotive force across the static magnetic field. Therefore, one cannot neglect the static flow $\bar{\vec{v}}$, particularly for a strong field case. Now, if we note the ralation (5), we find that the effect of the plasma flow can be taken into account by introducing the modified scalor and vector potentials, χ and $\vec{\alpha}$, by the relations

$$\chi_s \equiv \Psi + (m_s/2q_s)|\vec{v}_s|^2 , \quad (16)$$

$$\vec{\alpha}_s \equiv \vec{A} + (m_s c/q_s)\vec{v}_s . \quad (17)$$

The modification of the scalor potential is due to the centrifugal force, while that of the vector potential is due to Coriolis' force. Note that $\vec{\alpha}_s q_s/c$ is nothing but the canonical momentum of the fluid. Using these modified potentials, we can write the equation of motion as

$$\vec{\alpha}_s = - c\vec{\nabla}\chi_s + \vec{v}_s \times \vec{\Omega}_s - (c/q_s n_s)\vec{\nabla} P_s \quad , \tag{18}$$

where $\vec{\Omega}_s$ is the modified magnetic field defined by

$$\vec{\Omega}_s = \vec{\nabla} \times \vec{\alpha}_s \quad , \tag{19}$$

which statisfies the source-free condition,

$$\vec{\nabla}\cdot\vec{\Omega}_s = 0 \quad . \tag{20}$$

To proceed with our argument, we now restrict ourselves to the axisymmetric case, that is, the case where both the plasma and the oscillating field are axially symmetric as in the case of the r.f. end plugging of plasmas in an axially symmetric linear machine, the case of the m=0 mode propagation in a tokamak, etc. In the cylindrical coordinate system, (r,ϕ, z), this implies that all quantities are independent of ϕ. Suppressing the suffix s, we can then write from eq.(20),

$$\vec{\Omega} = [- \frac{1}{2\pi r}\frac{\partial\psi}{\partial z} , \quad \Omega_\phi , \quad \frac{1}{2\pi r}\frac{\partial\psi}{\partial r}] \quad , \tag{21}$$

where ψ is the generalized flux function defined by the relation

$$\vec{\Omega}\cdot\vec{\nabla}\psi = 0 \quad . \tag{22}$$

From eqs.(19) and (21), we have

$$\psi = 2\pi r \, \alpha_\phi + S(t) \quad , \tag{23}$$

where S(t) is an integration constant which is an arbitrary periodic function of time.

We now rewrite the equation of motion (18). First, its ϕ-component yields

$$\dot{\psi} + \vec{v}\cdot\vec{\nabla}\psi = \dot{S}(t)\ . \qquad (24)$$

For the (r,z)-components, we use the Clebsch representation [4], (u, h), defined by the relations

$$\frac{q}{mc}\alpha_r = \partial u/\partial r + h\ \partial\psi/\partial r\ ,\ \frac{q}{mc}\alpha_z = \partial u/\partial z + h\ \partial\psi/\partial z\ . \qquad (25)$$

After some calculation [3], we then obtain from the (r,z)-components of eq.(18) the following equations:

$$\dot{u} + h(\dot{\psi} - \dot{S}) = -\frac{q}{m}\chi - \frac{1}{m}\int \frac{\partial P}{\partial n}\frac{dn}{n}\ , \qquad (26)$$

$$\dot{h} + \vec{v}\cdot\vec{\nabla}h = \frac{q}{mc}\frac{v_\phi}{2\pi r} + \frac{1}{m}\left[\frac{\partial}{\partial\psi}\int\frac{\partial P}{\partial n}\frac{dn}{n} - \frac{1}{n}\frac{\partial P}{\partial\psi}\right]\ . \qquad (27)$$

Here the pressure P is assumed to be a known function of n, ψ and t, $P = P(n,\ \psi,\ t)$, and v_r, v_ϕ and v_z are determined in terms of u, ψ, h and $\vec{A}$ by using eqs.(17), (23) and (25). Equations (24), (26) and (27), together with the continuity equation,

$$\dot{n} + \vec{\nabla}\cdot(n\vec{v}) = 0\ , \qquad (28)$$

and Maxwell-Poisson system of euqations,

$$\ddot{A}_\phi = c^2 r\vec{\nabla}\left(\frac{1}{r^2}\vec{\nabla}(rA_\phi)\right) + 4\pi c\sum_s q_s\ n_s\ v_{s\phi}\ , \qquad (29)$$

$$\ddot{A}_r = c^2\frac{\partial}{\partial z}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) - c\frac{\partial}{\partial r}\frac{\partial\Psi}{\partial t} + 4\pi c\sum_s q_s n_s v_{sr}\ , \qquad (30)$$

$$\ddot{A}_z = c^2\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right) - c\frac{\partial}{\partial z}\frac{\partial\Psi}{\partial t} + 4\pi c\sum_s q_s n_s v_{sz}\ , \qquad (31)$$

$$\nabla^2\Psi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = 4\pi\sum_s q_s\ n_s\ , \qquad (32)$$

are the basic equations which describe the axisymmetric system. If we use the Coulomb guage, $\vec{\nabla}\cdot\vec{A} = 0$, and denote

$$A_r = -\frac{1}{r}\frac{\partial W}{\partial z}, \qquad A_z = \frac{1}{r}\frac{\partial W}{\partial r}, \tag{33}$$

equations (30) and (31) are combined to make a single equation,

$$\ddot{B}_\phi = c^2 r\vec{\nabla}\left(\frac{1}{r^2}\vec{\nabla}(rB_\phi)\right)+4\pi c\sum_s q_s\left[\frac{\partial}{\partial z}(n_s v_{sr}) - \frac{\partial}{\partial r}(n_s v_{sz})\right], \tag{34}$$

where the ϕ-component of the magnetic field is given by

$$B_\phi = -r\,\vec{\nabla}\cdot\left(\frac{1}{r^2}\vec{\nabla}W\right). \tag{35}$$

We now show that these set of equations can be derived from the variation principle,

$$\delta L[u_s, \psi_s, h_s, A_\phi, W, \Psi] = 0, \tag{36}$$

with

$$L = \int dt \int r dr \int dz \left\{\frac{1}{8\pi}\left(|\vec{E}|^2 - |\vec{B}|^2\right) + \sum_s P_s\right\}. \tag{37}$$

Note the similarity of this variation principle to eq.(15). We first calculate $\delta P = (\partial P/\partial\psi)\delta\psi + (\partial P/\partial n)\delta n$ by using the relation (26) which yields

$$\delta P = mn\left\{-\delta\dot{u} - \delta h(\dot{\psi} - \dot{S}) - (q/m)\delta\Psi - \vec{v}\cdot\delta\vec{v} - h\delta\dot{\psi} + \left(\frac{1}{m}\frac{\partial}{\partial\psi}\int\frac{\partial P}{\partial n}\frac{dn}{n} + \frac{1}{mn}\frac{\partial P}{\partial\psi}\right)\delta\psi\right\}. \tag{38}$$

The velocity variation $\delta\vec{v}$ can be calculated from eqs. (17) and (25) as

$$-\vec{v}\cdot\delta\vec{v} = -\vec{v}\cdot\vec{\nabla}\delta u - \vec{v}\cdot\vec{\nabla}\psi\delta h - (h\vec{v}\cdot\vec{\nabla} + \frac{qv_\phi}{2\pi mcr})\delta\psi$$

$$+ \frac{q}{mc}(\frac{v_z}{r}\frac{\partial}{\partial r} - \frac{v_r}{r}\frac{\partial}{\partial z})\delta W + \frac{q}{mc}v_\phi \delta A_\phi . \qquad (39)$$

Using the relation, $\vec{E} = -\vec{\nabla}\Psi - c^{-1}\dot{\vec{A}}$, we can calculate $\delta|\vec{E}|^2$ as

$$\frac{1}{2}\delta|\vec{E}|^2 = \frac{1}{c^2}\dot{A}_\phi \delta\dot{A}_\phi + \vec{\nabla}\Psi\cdot\vec{\nabla}\delta\Psi + \frac{1}{r^2c^2}\vec{\nabla}\dot{W}\cdot\vec{\nabla}\delta\dot{W}$$

$$- \frac{1}{c}\delta[\frac{\partial\Psi}{\partial r}\frac{\partial\dot{W}}{\partial z} - \frac{\partial\Psi}{\partial z}\frac{\partial\dot{W}}{\partial r}] . \qquad (40)$$

A similar calculation for $\delta|\vec{B}|^2$ yields

$$\frac{1}{2}\delta|\vec{B}|^2 = \frac{1}{r^2}\vec{\nabla}(rA_\phi)\cdot\vec{\nabla}(r\delta A_\phi) - B_\phi[\frac{\partial}{\partial z}\frac{1}{r}\frac{\partial\delta W}{\partial z} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\delta W}{\partial r}] . \qquad (41)$$

Using these results, we can show that $\delta L/\delta u$ yields the continuity equation (28), $\delta L/\delta\psi$ yields eq.(24), $\delta L/\delta h$ yields eq.(27), $\delta L/\delta A_\phi$ yields eq.(29), $\delta L/\delta W$ yields eq.(34) and $\delta L/\delta\Psi$ yields eq.(32). Note that eq.(26) is used to calculate δP. If the equation of state is given by the relation $P=nT(\psi)$, then we have from eq.(26)

$$P = N(\psi)T(\psi)\exp[-\frac{q\chi + m\dot{u} + mh(\dot{\psi} - \dot{S})}{T(\psi)}] \qquad (42)$$

which is a generalization of eq.(11).

We conclude this section by a remark that the variation principle derived here is valid for arbitrary strength of the oscillating field, provided that the field is periodic in time and that the generalized magnetic surface defined by eq.(22) can exist.

4. Quasilinear Approximation for Axisymmetric System

In order to draw a more explicit result, we now restrict ourselves to the weak-field limit and derive the expressions or equations for the quasilinear modification of various physical quantities in an axisymmetric system. As in section 2, we divide the variable into the static part, denoted by bar, and the oscillating part, denoted by subscript 1. The unperturbed part of the static components will be designated by subscript 0, and we now have $\vec{\bar{B}}\neq\vec{B}_0$ and $\vec{\bar{v}}\neq\vec{v}_0$, in contrast to the case of section 2. Instead of using the Clebsch representation, we here use the more familiar cylindrical coordinate representation.

General two-fluid formulation of an axisymmetric system without oscillating field, that is, the unperturbed system, is presented in ref. [5]. The unperturbed fluid velocity can be written in the form

$$\vec{v}_0 = [-\frac{F(\psi_0)}{n_0 r}\frac{\partial\psi_0}{\partial z}, \quad v_{\phi_0}, \quad \frac{F(\psi_0)}{n_0 r}\frac{\partial\psi_0}{\partial r}] , \qquad (43)$$

where $F(\psi_0)$ is an arbitrary function of the unperturbed generalized flux function ψ_0 which from eq. (23) can be written as

$$\psi_0 = 2\pi r\ [A_{\phi_0} + (mc/q)v_{\phi_0}] = \psi_{m_0} + 2\pi r(mc/q)v_{\phi_0}, \qquad (44)$$

where ψ_m is the magnetic flux function. The unperturbed force balance along $\vec{\Omega}_0$, together with the condition $P_0 = P_0(n_0)$, yeilds,

$$\chi_0 + H_0/q = f(\psi_0) \quad , \tag{45}$$

where f is another arbitrary function of ψ_c (cf. eq.(10)).

We now consider the perturbation. As in section 2, we calculate the oscillating parts in the linear approximation. A considerable simplification can be achieved if one introduces a generalized displacement vector $\vec{Q}$ defined by the relation

$$\dot{\vec{Q}} \equiv \dot{\vec{R}} - \frac{2\pi F(\psi_0)}{n_0}\vec{\Omega}_1 + \frac{2\pi}{n_0}[\frac{n_1}{n_0}F(\psi_0) - \psi_1 F'(\psi_0)]\vec{\Omega}_0$$
$$- 2\pi r^2 c\, f''(\psi_0)\psi_1\vec{\nabla}\phi, \tag{46}$$

with $\overline{\vec{Q}}=0$. Indeed, a straightforward calculation yields [3]

$$\vec{v}_1 \times \vec{\Omega}_0 + \vec{v}_0 \times \vec{\Omega}_1 = \dot{\vec{Q}}\times\vec{\Omega}_0 + c\,\vec{\nabla}[f'(\psi_0)\psi_1] \quad , \tag{47}$$

$$n_0\vec{v}_1 + n_1\vec{v}_0 = n_0\dot{\vec{Q}} + \vec{\nabla} \times (n_0\vec{Q} \times \vec{v}_0) \quad , \tag{48}$$

from which we obtain

$$\vec{\Omega}_1 = \vec{\nabla} \times \vec{\alpha}_1 = \vec{\nabla} \times (\vec{Q} \times \vec{\Omega}_0) \quad , \tag{49}$$

$$n_1 = - \int^t dt'\, \vec{\nabla}\cdot(n_0\vec{v}_1 + n_1\vec{v}_0) = -\vec{\nabla}\cdot(n_0\vec{Q}) \quad . \tag{50}$$

These equations are similar to those obtained in the absence of the plasma flow.

Using these relations, we can now calculate the second-order quasilinear modification of various variables, $\bar{n}$, $\bar{\Psi}$, $\overline{\vec{v}}$ and $\overline{\vec{B}}$. We first calculate these quantities in terms of $\bar{\psi}$ and the first-order variables, and then derive an equation to determine $\bar{\psi}$.

First we consider $\overline{n}$ and $\overline{\Psi}$. They can be obtained from the force balance along $\vec{\overline{\Omega}}$ and the Poisson equation. Correct to the second order, the static force balance along $\vec{\overline{\Omega}}$ can be calculated as follows:

$$c\vec{\overline{\Omega}} \cdot \vec{\nabla}(\overline{\chi} + \overline{H}/q) = (\vec{\overline{v}} \times \vec{\overline{\Omega}} + \overline{\vec{v}_1 \times \vec{\Omega}_1}) \cdot \vec{\overline{\Omega}} = (\overline{\vec{v}_1 \times \vec{\Omega}_1}) \cdot \vec{\overline{\Omega}}$$

$$= -\overline{[(\vec{v}_1 \times \vec{\overline{\Omega}}) + (\vec{v}_0 \times \vec{\Omega}_1)] \cdot \vec{\Omega}_1}$$

$$= -\overline{(\dot{\vec{Q}} \times \vec{\overline{\Omega}}) \cdot [\vec{\nabla} \times (\vec{Q} \times \vec{\overline{\Omega}})]} - c\vec{\nabla} \cdot \overline{[f'(\overline{\psi})\psi_1]\vec{\Omega}_1}$$

$$= -(1/2)\ \vec{\nabla} \cdot \overline{[(\vec{Q} \times \vec{\overline{\Omega}}) \times (\dot{\vec{Q}} \times \vec{\overline{\Omega}})]} - c\ f''(\overline{\psi})\overline{\psi_1 \vec{\nabla}\overline{\psi} \cdot \vec{\Omega}_1}$$

$$= \vec{\overline{\Omega}} \cdot \vec{\nabla}\ [\overline{(\vec{Q} \times \vec{\overline{\Omega}}) \cdot \dot{\vec{Q}}}/2 + cf''(\overline{\psi})\overline{\psi_1^{\,2}}]\ , \qquad (51)$$

where we have ignored the difference between $\vec{\Omega}_0$, ψ_0 and $\vec{\overline{\Omega}}$, $\overline{\psi}$, as it gives only higher order corrections, and in the last two equations we used the relations

$$\overline{\vec{\Omega} \cdot \vec{\nabla}\psi} = 0\ ; \qquad \vec{\Omega}_1 \cdot \vec{\nabla}\overline{\psi} + \vec{\overline{\Omega}} \cdot \vec{\nabla}\psi_1 = 0\ , \qquad (52)$$

which follow from the definition of the generalized flux function (22). Equation (51) can be compared with eq.(8); apart from the additional term which arised from the quasilinear modification of the flux function, the right-hand side of eq.(51) is of the same form as that of eq.(8). Integration along $\vec{\overline{\Omega}}$ then yields

$$\overline{\chi} + \overline{H}/q = f(\overline{\psi}) + f''(\overline{\psi})\overline{\psi_1^{\,2}}/2 + \overline{(\dot{\vec{Q}} \times \vec{Q})} \cdot \vec{\overline{\Omega}}/2c\ . \qquad (53)$$

Combining this equation with the average Poisson equation,

$$\nabla^2\overline{\Psi} = -4\pi \sum_s q_s \overline{n}_s\ , \qquad (54)$$

and the assumed equation of state, $P = P(n, \psi, t)$, we can determine $\overline{n}$ and $\overline{\Psi}$ in terms of $\overline{\psi}$, $\overline{\vec{v}}$, $\overline{\vec{B}}$ and the first order quantities.

To calculate $\overline{\vec{v}}$ and $\overline{\vec{B}}$, we first note the relation, $\vec{\nabla}\cdot\overline{(n\vec{v})} = 0$ and introduce the flow flux function Γ by the relation

$$\overline{n\vec{v}} = [\,-\frac{1}{r}\frac{\partial\Gamma}{\partial z}\,,\quad (\overline{nv})_\phi\,,\quad \frac{1}{r}\frac{\partial\Gamma}{\partial r}\,]\ . \tag{55}$$

Using the flux conservation, $\overline{\vec{v}}\cdot\vec{\nabla}\overline{\psi} + \overline{\vec{v}_1\cdot\vec{\nabla}\psi_1} = 0$ (see eq.(24)), we can write the component of $\overline{n\vec{v}}$ across the average generalized magnetic surface in the form,

$$\overline{n\vec{v}}\cdot\vec{\nabla}\overline{\psi} = (\overline{n}\overline{\vec{v}} + \overline{n_1\vec{v}_1})\cdot\vec{\nabla}\overline{\psi} = -\,\overline{n}\ \overline{\vec{v}_1\cdot\vec{\nabla}\psi_1} + \overline{n_1\vec{v}_1}\cdot\vec{\nabla}\overline{\psi}\ , \tag{56}$$

From eqs.(21) and (55), the left-hand side of this equation can be written as

$$\overline{n\vec{v}}\cdot\vec{\nabla}\overline{\psi} = -\ 2\pi\overline{\vec{\Omega}}\cdot\vec{\nabla}\Gamma\ , \tag{57}$$

while the right-hand side of eq.(56) can be calculated, using eqs. (46) and (52), as

$$-\ \overline{n}\cdot\overline{\vec{v}_1\cdot\vec{\nabla}\psi_1} + \overline{n_1\vec{v}_1}\cdot\vec{\nabla}\overline{\psi}$$

$$= \vec{\Omega}\cdot\vec{\nabla}\ [-\pi n\overline{(\vec{Q}\times\dot{\vec{Q}})}r^2\vec{\nabla}\phi - 2\pi\ F'(\psi)\overline{\psi_1{}^2}/2\ ]\ . \tag{58}$$

Using eqs.(56) through (58), we obtain

$$\Gamma = \int F(\overline{\psi})d\overline{\psi} + F'(\overline{\psi})\overline{\psi_1{}^2}/2 + r^2\overline{n}\overline{(\vec{Q}\times\dot{\vec{Q}})}\cdot\vec{\nabla}\phi/2\ . \tag{59}$$

This determines the 'poloidal' velocities, $\overline{\vec{v}}_r$ and $\overline{\vec{v}}_z$, which in turn determine the 'toroidal' magnetic field by the Ampère law:

$$\overline{B}_\phi = \frac{4\pi}{cr} \{ \sum_s q_s \Gamma_s + \frac{I}{2\pi} \} , \tag{60}$$

where I is the external current. The 'toroidal" velocity, $\overline{v}_\phi$, and the 'poloidal' magnetic field or the 'toroidal' vector potential, $\overline{A}_\phi$, are determined from the time average of eqs.(23) and (29):

$$\overline{v}_\phi = - (q/mc)(\overline{A}_\phi - \overline{\psi}/2\pi r) , \tag{61}$$

$$r\vec{\nabla}(\frac{1}{r^2}\vec{\nabla}(r\overline{A}_\phi)) = - (4\pi/c) \sum_s q_s [\overline{n}_s \overline{v}_{s\phi} + \overline{n_{s1} v_{s\phi 1}}] . \tag{62}$$

Our final task is to derive an equation which determines the flux function $\overline{\psi}$. It can be obtained by taking the force balance across the generalized magnetic surface,

$$\vec{\nabla}\Gamma \cdot \{\vec{\nabla}(\overline{\chi} + \overline{H}/q) - \overline{\vec{v}} \times \overline{\vec{\Omega}} - \overline{\vec{v}_1 \times \vec{\Omega}_1}\} = 0 . \tag{63}$$

Using eqs.(21) and (55), we have

$$\vec{\nabla}\Gamma \cdot (\overline{\vec{v}} \times \overline{\vec{\Omega}}) = \frac{\overline{v}_\phi}{2\pi r} \vec{\nabla}\Gamma \, \vec{\nabla}\overline{\psi} - \frac{\overline{\Omega}_\phi}{\overline{n}}[\frac{|\vec{\nabla}\Gamma|^2}{r} + (\vec{\nabla}\Gamma \times \overline{n_1\vec{v}_1})_\phi] , \tag{64}$$

where $\overline{\Omega}_\phi$ is determined from eqs.(17) and (19) as

$$\overline{\Omega}_\phi = \frac{mc}{q} [\frac{\partial \overline{v}_r}{\partial z} - \frac{\partial \overline{v}_z}{\partial r}] + \overline{B}_\phi . \tag{65}$$

Using these relations, one finally obtains the following equation to determine $\overline{\psi}$:

$$\vec{\nabla}\Gamma \cdot \{-\overline{n}\vec{\nabla}(\overline{\chi} + \overline{H}/q) + \overline{n}(\overline{\vec{v}_1 \times \vec{\Omega}_1}) + \overline{n} \, \overline{v}_\phi \vec{\nabla}\overline{\psi}/2\pi r\}$$
$$= \{ \frac{|\vec{\nabla}\Gamma|^2}{r} + (\vec{\nabla}\Gamma \times \overline{n_1\vec{v}_1})_\phi\}\{\overline{B}_\phi + \frac{mc}{q}(\frac{\partial \overline{v}_r}{\partial z} - \frac{\partial \overline{v}_z}{\partial r})\} , \tag{66}$$

where $(\overline{\chi} + \overline{H}/q)$ is given by eq.(53), $\overline{n}$ is determined from eqs.(53), (54) and $P = P(n, \psi, t)$, $\overline{v}_r$ and $\overline{v}_z$ are determined in terms of $\overline{n_1\vec{v}_1}$, $\bar{n}$ and Γ from eq.(55) and Γ is given by eq.(59).

5. Variation Principle for Nonaxisymmetric Torus

The analyses presented in the preceding two sections are restricted to the case where both the plasma and the oscillating field are axially symmetric. In the r.f. heating of the axially symmetric torus, however, the symmetry is destroyed by the r.f. field. The object of this section is to extend the general treatment of section 3 to nonaxisymmetric case by restricting ourselves to a toroidal system. We show below that by introducing an appropriate coordinate system we can indeed derive a similar variation principle. The relevant coordinate system is the one in which either the modified magnetic line of force or the plasma flow line becomes a straight line, an analogue of the Hamada coordinate. Here we only show the derivation for the case in which the modified magnetic line of force is made straight.

We start from the usual toroidal coordinate system, (ψ, θ, ϕ), where θ is the poloidal angle normalized to 2π:

$$\oint d\theta = 2\pi \quad . \tag{67}$$

We assume the existence of the generalized magnetic surface defined by eq.(22), which in the present coordinate system is written as

$$0 = g^{-1/2}[\frac{\partial}{\partial\theta}(g^{1/2}\vec{\Omega}\cdot\vec{\nabla}\theta) + \frac{\partial}{\partial\theta}(g^{1/2}\vec{\Omega}\cdot\vec{\nabla}\phi] \ , \tag{68}$$

where

$$g^{-1/2} \equiv (\vec{\nabla}\psi \times \vec{\nabla}\theta)\cdot\vec{\nabla}\phi \quad . \tag{69}$$

From eq. (68), we find

$$g^{1/2}\ \vec{\Omega}\cdot\vec{\nabla}\theta = -\frac{1}{2\pi}\frac{\partial}{\partial\phi}\ \Phi(\psi,\ \theta,\ \psi,\ t) \tag{70}$$

$$g^{1/2}\ \vec{\Omega}\cdot\vec{\nabla}\phi \doteq \frac{1}{2\pi}\frac{\partial}{\partial\theta}\ \Phi(\ \psi,\ \theta,\ \phi,\ t)\quad , \tag{71}$$

where the generating function Φ can be written as

$$\Phi = G(\psi,\ t)\phi\ + K(\psi,\ t)\theta\ +\ \Xi\ (\psi,\theta,\phi,\ t) \tag{72}$$

with Ξ being a periodic function of θ and ϕ , because the left-hand sides of eqs.(70) and (71) are to be single-valued functions of the position. Then by introducing the transformation

$$\tilde{\psi} = \int^{\psi} G(\psi,\ t)d\psi\quad , \tag{73}$$

$$\tilde{\theta} = \theta\ +\ \Xi/K\quad ,\qquad \oint d\tilde{\theta}\ = 2\pi\quad , \tag{74}$$

we can derive the following formula (see Appendix I):

$$\vec{\Omega} = \frac{1}{2\pi}\ \vec{\nabla}\tilde{\psi}\times[\vec{\nabla}\phi\ +\ \tilde{K}(\tilde{\psi},t)\vec{\nabla}\tilde{\theta}]\ , \tag{75}$$

$$\tilde{K}(\tilde{\psi},\ t)\ \equiv K(\psi,\ t)/G(\psi,t)\ , \tag{76}$$

where for simplicity we assumed that $G(\psi,\ t)$ is nonvanishing. For G=0, one can derive a relation of the form

$$\vec{\Omega} = \frac{1}{2\pi}\ \vec{\nabla}\tilde{\psi}\ \times\ [\tilde{G}(\tilde{\psi},\ t)\vec{\nabla}\phi\ \ +\ \vec{\nabla}\tilde{\theta}\ ]\quad . \tag{77}$$

which case we shall not discuss here. Note that the straight line, $\xi = \phi + \tilde{K}\tilde{\theta}$ = const., gives the equation for the modified magnetic line of force. Note also that $-\tilde{K}(\tilde{\psi},\ t)$ is nothing but the safety factor of the torus.

Substituting eq.(77) into eq.(19) and using the fact that $\vec{\alpha}$ is a single-valued function of the position, we can derive the following expression for the modified vector potential,

$$\vec{\alpha} = \frac{1}{2\pi} \{ \tilde{\psi} \nabla\phi + \int^{\tilde{\psi}} \tilde{K}(\tilde{\psi}, t) \, d\tilde{\psi} \, \vec{\nabla}\tilde{\theta} + \vec{\nabla}\mu \} , \qquad (78)$$

where μ is a single-valued function of the position. Equation of motion (18) then becomes

$$\frac{1}{2\pi c} \frac{\partial}{\partial t} \{\tilde{\psi}\vec{\nabla}\phi + \int \tilde{K} \, d\tilde{\psi} \, \vec{\nabla}\theta + \vec{\nabla}\mu\}$$

$$= - \vec{\nabla}\chi - \frac{1}{nq} \vec{\nabla}P + \frac{1}{2\pi c} \vec{v} \times [\vec{\nabla}\tilde{\psi} \times (\vec{\nabla}\phi + \tilde{K}\vec{\nabla}\theta] . \qquad (79)$$

We now assume an equation of state in the form

$$n = n(P, \tilde{\psi}, t) . \qquad (80)$$

In general, n can be a function of ξ as well. This dependence, however, disappears if the modified magnetic surface is covered quasi-ergodically by a single line of force. The quasi-ergodicity breaks down when the safety factor $\tilde{K}$ takes a rational value. Thus the assumtion (80) is equivalent to excluding the existence of rational surfaces. With this assumption, we can write

$$\frac{1}{n} \vec{\nabla}P = \vec{\nabla} \int \frac{dP}{n} - \frac{\partial}{\partial\tilde{\psi}}(\int \frac{dP}{n}) \vec{\nabla}\tilde{\psi} . \qquad (81)$$

Substituting this expression into eq.(79) and equating the coefficients of $\vec{\nabla}\tilde{\psi}$, $\vec{\nabla}\phi$ and $\vec{\nabla}\theta$ separately, we obtain after some calculations presented in Appendix II the following three euqations:

$$\chi + \frac{1}{q}\int \frac{dP}{n} + \frac{1}{2\pi c}\,[\dot{\mu} + \int \tilde{K}\, d\tilde{\psi}\, \dot{\tilde{\theta}} + \int \dot{\tilde{K}}\, d\tilde{\psi}\, \tilde{\theta}] = f(\tilde{\psi},\xi,\ t), \tag{82}$$

$$\dot{\tilde{\psi}} + \vec{v}\cdot\vec{\nabla}\tilde{\psi} = -\,2\pi c\ \frac{\partial}{\partial \xi}\, f\ , \tag{83}$$

$$\frac{\partial}{\partial t}(\tilde{K}\tilde{\theta}) + \tilde{K}\vec{v}\cdot\vec{\nabla}\tilde{\theta} = -\ \vec{v}\cdot\vec{\nabla}\phi - 2\pi c\,\frac{\partial}{\partial \tilde{\psi}}\,[\ \frac{1}{q}\int \frac{dP}{n} - f]\ , \tag{84}$$

where f is an arbitrary function of $\tilde{\psi}$, ξ and t. These equations combined with the equation of continuity (28) and the Maxwell-Poisson system of equations determine the steady state propagation in non-axisymmetric torus In particular, eq.(82) with eq.(80) determines the pressure profile as before and μ is determined from the equation of continuity.

A straightforward calculation now shows that the following variation principle derives the above set of equations:

$$\delta L\ [\tilde{\psi}_s,\ \tilde{\theta}_s,\ \mu_s,\ \vec{A},\ \Psi] = 0 \tag{85}$$

with

$$L = \int dt \int d^3\vec{x}\ \ [\ \frac{|\vec{E}|^2 - |\vec{B}|^2}{8\ \pi} + \sum_s P_s\ ]\ . \tag{86}$$

Namely, $\delta L/\delta\tilde{\psi}$ derives eq.(83), $\delta L/\delta\tilde{\theta}$ derives eq.(84), $\delta L/\delta\mu$ derives the equation of continuity, $\delta L/\delta\vec{A}$ derives the Ampère equations and $\delta L/\delta\Psi$ derives the Poisson equation. Note that the above derivation is valid for arbitrary amplitude of the r.f. field.

6. Summary

We have derived a simple variation principle for the following three cases:

i) Weak oscillation field without static flow, $\overline{\vec{v}}=0$, whence $\overline{\vec{B}} = B_0$, for an arbitrary magnetic field configuration;

ii) Arbitrary field strength in axisymmetric system using the Clebsch representation for the 'poloidal' velocity components; and

iii) Arbitrary field strength in nonaxisymmetric torus using the Hamada-like coordinate in which the modified magnetic line of force becomes straight.

In case i) and in the weak-field limit of case ii), we have derived expressions or equations for the quasilinear modification of various physical quantities, i.e. $\overline{n}$ and $\overline{\Psi}$ in case i) and $\overline{n}$, $\overline{\Psi}$, $\overline{\vec{v}}$ and $\overline{\vec{B}}$ in case ii).

In all cases, we have assumed the collisionless two-fluid model with an equation of state ($P=P(n)$ in case i) and $P=P(n, \psi, t)$ in cases ii) and iii)), steady state propagation of a periodic wave and existence of the modified magnetic surface defined by eq.(22).

Although the general variation principle obtained in cases ii) and iii) is quite complicated and is nothing but an alternate representation of the basic set of differential equations, the one derived in section 2 for case i) is extremely simple since the variation is taken only with respect to the oscillating electric feild. This variation principle can be generalized to include some of the finite Larmour radius effects [2] and has also been applied to investigate the steady state in r.f. plugging of plasma at a line cusp [3].

Finally, we are extending the above formulation to the following problems:

i) Quasilinear calculations for nonaxisymmetric torus;

ii) Derivation of the variation principle for general nonaxisymmetric system;

iii) Problems associated with the existence of rational surface; and

iv) Stability analysis of the steady state.

As applications of the above formulation, we are planning several numerical calculations, including the r.f. plugging, lower-hybrid propagation, etc.

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Appendix I, Proof of Equation (75)

This Appendix provides a proof that the coordinate stretching given by eqs.(73) and (74) transforms the generalized magnetic line of force to a straight line.

Noting the relation

$$\vec{\Omega}\cdot\vec{\nabla}\tilde{\psi} = G\,\vec{\Omega}\cdot\vec{\nabla}\psi = 0 \quad ,$$

we first write $\vec{\Omega}$ in the form

$$\vec{\Omega} = \frac{1}{2\pi}\vec{\nabla}\tilde{\psi} \times [a\,\vec{\nabla}\phi + b\,\vec{\nabla}\tilde{\theta}] \quad . \tag{A-1}$$

Obviously, we have

$$\vec{\Omega}\cdot\vec{\nabla}\phi = \frac{b}{2\pi}[\vec{\nabla}\tilde{\psi} \times \vec{\nabla}\tilde{\theta}]\cdot\vec{\nabla}\phi = \frac{b}{2\pi}\tilde{g}^{-1/2} \tag{A-2}$$

$$\vec{\Omega}\cdot\vec{\nabla}\tilde{\theta} = \frac{a}{2\pi}[\vec{\nabla}\tilde{\psi} \times \vec{\nabla}\phi]\cdot\vec{\nabla}\tilde{\theta} = -\frac{a}{2\pi}\tilde{g}^{-1/2} \quad , \tag{A-3}$$

where

$$\tilde{g}^{-1/2} = (\vec{\nabla}\tilde{\psi} \times \vec{\nabla}\tilde{\theta})\cdot\vec{\nabla}\psi \quad , \tag{A-4}$$

which can be calculated from eqs.(73) and (74) as follows:

$$\begin{aligned}\tilde{g}^{-1/2} &= [G\vec{\nabla}\psi \times (1 + \frac{1}{K}\frac{\partial\Xi}{\partial\theta})\vec{\nabla}\theta]\cdot\vec{\nabla}\phi \\ &= g^{-1/2}\,G[1 + \frac{1}{K}\frac{\partial\Xi}{\partial\theta}] \quad .\end{aligned} \tag{A-5}$$

On the other hand, use of eqs. (70) through (72) yields

$$\vec{\Omega}\cdot\vec{\nabla}\phi = g^{-1/2}\frac{1}{2\pi}[K + \frac{\partial\Xi}{\partial\theta}] \quad , \tag{A-6}$$

$$\begin{aligned}\vec{\Omega}\cdot\vec{\nabla}\tilde{\theta} &= [1 + \frac{1}{K}\frac{\partial\Xi}{\partial\theta}]\vec{\Omega}\cdot\vec{\nabla}\theta + \frac{1}{K}\frac{\partial\Xi}{\partial\theta}\vec{\Omega}\cdot\vec{\nabla}\phi \\ &= -g^{-1/2}\frac{1}{2\pi}G[1 + \frac{1}{K}\frac{\partial\Xi}{\partial\theta}] \quad .\end{aligned} \tag{A-7}$$

Comparing eqs.(A-2) and (A-3) with eqs.(A-6) and (A-7), we obtain

$$\text{ↄ} = K/G = \tilde{K}(\tilde{\psi},\ t), \quad a=1 \ ,$$

which proves eq.(75).

Appendix II, Derivation of Equations (82), (83) and (84).

We fist substitute eq.(81) into eq.(79) to obtain

$$\frac{1}{2\pi c}\{\dot{\tilde{\psi}}\vec{\nabla}\phi + [\tilde{K}\dot{\tilde{\psi}} + \int\dot{\tilde{K}}\,d\tilde{\psi}]\vec{\nabla}\tilde{\theta} + \int\tilde{K}\,d\tilde{\psi}\,\vec{\nabla}\dot{\tilde{\theta}} + \vec{\nabla}\dot{\tilde{\mu}}\}$$

$$= -\vec{\nabla}\,[\chi + \frac{1}{q}\int\frac{dP}{n}] + \frac{1}{q}\frac{\partial}{\partial\tilde{\psi}}\int\frac{dP}{n}\,\vec{\nabla}\tilde{\psi}$$

$$+ \frac{1}{2\pi c}\{\ \vec{v}\cdot(\vec{\nabla}\phi + \tilde{K}\vec{\nabla}\tilde{\theta})\vec{\nabla}\tilde{\psi} - \vec{v}\cdot\vec{\nabla}\tilde{\psi}(\vec{\nabla}\psi + \tilde{K}\vec{\nabla}\tilde{\theta})\}\ . \qquad \text{(A-8)}$$

Equating the coefficients of $\vec{\nabla}\tilde{\psi}$, $\vec{\nabla}\tilde{\theta}$ and $\vec{\nabla}\phi$, and using the relation

$$\int\tilde{K}\,d\tilde{\psi}\,\vec{\nabla}\dot{\tilde{\theta}} = \vec{\nabla}\int\tilde{K}\,d\tilde{\psi}\,\dot{\tilde{\theta}} - \tilde{K}\,\dot{\tilde{\theta}}\,\vec{\nabla}\tilde{\psi} \quad ,$$

we obtain

$$\frac{1}{2\pi c}[\tilde{K}\dot{\tilde{\theta}} + \vec{v}\cdot(\vec{\nabla}\phi + \tilde{K}\vec{\nabla}\tilde{\theta})] + \frac{1}{q}\frac{\partial}{\partial\tilde{\psi}}\int\frac{dP}{n}$$

$$= \partial J/\partial\tilde{\psi} \quad , \qquad \text{(A-9)}$$

$$\frac{1}{2\pi c}[\tilde{K}\ \dot{\tilde{\psi}} + \tilde{K}\ d\tilde{\psi} + \tilde{K}\ \vec{v}\cdot\vec{\nabla}\tilde{\psi}] = -\ \partial J/\partial\tilde{\theta} \qquad \text{(A-10)}$$

$$\frac{1}{2\pi c}[\dot{\tilde{\psi}} + \vec{v}\cdot\vec{\nabla}\tilde{\psi}] = -\ \partial J/\partial\phi\ , \qquad \text{(A-11)}$$

where

$$J = \chi + \frac{1}{q} \int \frac{dP}{n} + \frac{1}{2\pi c} [\dot{\mu} + \int \tilde{K}\, d\tilde{\psi}\, \dot{\tilde{\theta}}] \quad . \qquad \text{(A-12)}$$

Combining eq.(A-10) with eq.(A-11), we have

$$\frac{1}{2\pi c} \int \dot{\tilde{K}}\, d\tilde{\psi} = \tilde{K} \quad \partial J/\partial\phi \quad - \quad \partial J/\partial\theta$$

from which we obtain

$$J = - \frac{1}{2\pi c} \int \dot{\tilde{K}}\, d\tilde{\psi}\, \tilde{\theta} + f(\tilde{\psi}, \xi , t) \qquad \text{(A-13)}$$

where f is an arbitrary function of $\tilde{\psi}$, $\xi \equiv \phi + \tilde{K}\tilde{\theta}$ and t. Equating eq.(A-12) to eq.(A-13) immediately yields eq.(82). Equation (83) can be derived from eqs.(A-11) and (A-13), while eq.(84) obtains by substitution of eq.(A-13) into eq. (A-9).

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