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IMPROVED EIGENVALUE EQUATIONS FOR THE COLLISIONLESS DRIFT INSTABILITY IN TOKAMAKS

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ABSTRACT

Previous work has considered the validity of the perturbation theory technique of solving the radial differential equation for the collisionless drift wave in a tokamak only near marginal stability for the most unstable radial eigenmode. The present work extends the previous result and determines more exact eigenvalue equations (for all even and odd spatial modes) that are valid for arbitrary growth rates. Away from and perhaps near marginal stability, these more complete eigenvalue equations are required in order to accurately calculate growth rates.
1. INTRODUCTION

In a tokamak the electrostatic potential, $\phi$, generated by the collisionless drift wave satisfies a second-order differential equation \[1,2\] in a radial (slab) variable, $x$, that measures the distance from the rational surface under consideration. The standard analytic technique for determining the eigenvalue equation associated with this rather involved differential equation employs classical perturbation theory. The complicated $x$-dependent nonadiabatic resonant electron response is neglected to lowest order and the lowest order eigenfunctions are obtained. The lowest order eigenfunctions are then employed to determine perturbatively the contribution of the resonant electrons to the eigenvalue equation. The difficulty with this procedure is that the destabilizing resonant electron term must dominate the stabilizing magnetic shear term in order for the instability to occur. As a result, the perturbatively determined correction to the eigenvalue equation due to the destabilizing resonant electron term must be larger than the shear stabilization term of lowest order, in contradiction to the assumption of the perturbation theory. Nonetheless, Rosenbluth and Catto [2] found that for the most unstable mode (the lowest even spatial mode for $\phi$) the perturbation theory is apparently a good approximation to the full eigenvalue equation near marginal stability.

The purpose of this paper is to extend the work of Rosenbluth and Catto [2] in order to derive the eigenvalue equation for all the even spatial modes for arbitrary growth rate. It is also pointed out that
the conclusion of Ref. [2] is only valid for \((m/m_i)^{1/2}(L_s/L_n)^{3/2} \ll 1\),
where \(m\) and \(M\) are the electron and ion masses and \(L_s\) and \(L_n\) are the shear and density scale lengths. It is shown that away from marginal stability the perturbation theory form is no longer valid and the more complete eigenvalue equation must be employed. Because the perturbation theory result can only be recovered by examining the more exact form in the vicinity of simple poles, the resulting sensitivity may require that the more exact eigenvalue equation be employed near marginal stability as well. Furthermore, the eigenvalue equation for all the odd spatial modes is determined, and it is found that it must be employed away from marginal stability.

In Sections 2 and 3 the differential equation to be solved is presented and the perturbation theory technique reviewed. In Sections 4 and 5 the work of Rosenbluth and Catto is employed and extended to obtain the eigenvalue equation for the even and odd spatial \(\phi(x)\) modes.

2. DIFFERENTIAL EQUATION

Including temperature gradient effects (not treated in Ref. [2]) as well as resonant electrons, magnetic shear, and finite ion gyration radius, the equation to be solved is [1,2]

\[
\frac{\partial^2 \phi}{\partial x^2} - \left[ \Lambda - \mu^2 x^2 + \frac{\sigma(|x|)}{|x|} \right] \phi = 0
\]  (1)
where

\[ B = B\left[\hat{z} + \left(x/L_s\right)\hat{\gamma}\right] \]

\[ \phi(\vec{r}, t) = \phi(x) \exp(-i\omega t + iky) \]

\[ \Lambda = \rho_i^{-2}D^{-1}\{\omega[1 + \tau(1 - r_0)] - \omega_\ast[r_0 - n_i b(r_0 - r_1)]\} \]

\[ \frac{\sigma(|x|)}{|x|} = \rho_i^{-2}D^{-1}\xi \left\{ \left[ \omega - \omega_\ast \left( 1 - \frac{1}{2} \eta_e \right) \right] Z(\xi) - \nu(e \omega_\ast \xi[1 + \xi Z(\xi)]) \right\} \]  

\[ Z(\xi) = \pi^{-1/2} \int_{-\infty}^{\infty} \frac{dt \exp(-t^2)}{t - \xi} \text{ for } \text{Im} \xi > 0 \]

\[ D = (\omega + \omega_\ast)(r_0 - r_1) + n_i \omega_\ast(r_0 - 2br_0 + 2br_1) \]

and \( u = (\omega_\ast L_n/\omega T_s r_i^2)S^{1/2} \), with \( x_t = |\mu^{-1/2}| \), \( \omega_\ast = kcT_e/eBL_\perp \), \( L_n^{-1} = -\partial \ln N/\partial x \), \( L_s \) is the shear length, \( \tau = T_e/T_i \), \( \eta_j = d \ln T_j/d \ln N \),

\[ b = (kr_i)^2, \rho_i^2 = Mc^2T_i/e^2B^2, S = \Phi^{-1}[(\omega + \omega_\ast)r_0 + n_i \omega_\ast(r_0 - 2br_0 + 2br_1)], \]

\[ \xi = \omega/|k_\parallel|v_e, k_\parallel = kx/L_s, v_e^2 = 2T_e/m, \Gamma_n = I_n(b) \exp(-b), \text{and } I_n(b) = \text{the modified Bessel function.} \]

In Eq. (1), \( \Lambda \) contains the local dispersion relation for drift waves with finite ion gyration corrections included, the \( \partial^2 \phi/\partial x^2 \)
term occurs because of the finite ion gyration radius, the ion inertia gives \( \mu^2 \chi^2 \) because of the magnetic shear and tokamak geometry, and the nonadiabatic electron response \( \sigma \) is the only destabilizing term in the equation.

The expressions for the growth rates associated with the collisionless drift instability are found from the eigenvalue equation associated
with the (even and odd) eigenfunctions of Eq. (1) that are spatially
decaying for unstable modes; namely, \( \phi (|x| \to \infty) \to 0 \) for \( \text{Im } \omega > 0 \).
These boundary conditions correspond to the outgoing wave boundary
conditions of Ref. [3].

3. PERTURBATION THEORY

The perturbation theory approach to solving Eq. (1) seeks unstable
eigenfunctions (\( \text{Im } \omega > 0 \)) by neglecting the \( \sigma \) term to lowest order.
Because \( \sigma \) is the only source of instability, this leads to a lowest
order eigenvalue equation \( \Lambda + i(2n + 1)\mu = 0 \), with \( n = 0, 1, 2, 3, \ldots \)
which permits only stable solutions and therefore is not consistent with
\( \text{Im } \omega > 0 \). This inconsistency is best removed by solving Eq. (1) with \( \sigma \)
retained to lowest order near the rational surface [2], the approach to
be followed in the next two sections. Near marginal stability, however,
the inconsistency is assumed to be removed by including the effect of \( \sigma \)
in the eigenvalue equation via classical perturbation theory, never
solving the lowest order eigenvalue equation.

The solutions to the lowest order equation, 
\[ \frac{d^2\phi}{dx^2} - (\Lambda - \mu^2x^2)\phi = 0, \]
which go to zero as \( |x| \to \infty \) for \( \text{Im } \omega > 0 \), are simply
\[ \phi_n = H_n [(i\mu)^{1/2}x] \exp (-i\mu x^2/2) \quad (3) \]
where \( \phi_n \) is the eigenfunction associated with the \( n \)th eigenvalue, \( H_n \) is
the Hermite polynomial of order \( n \), and \( n = 0, 1, 2, 3, \ldots \).
Employing the lowest order eigenfunctions of Eq. (3) in a classical perturbation theory to determine the correction to the lowest order eigenvalue equation, \( \Lambda + i(2n + 1)\mu = 0 \), gives the relevant eigenvalue equation, namely

\[
\Lambda + i(2n + 1)\mu = \frac{-\int_0^\infty dx \, \phi_n^2/\phi_n}{\int_0^\infty dx \, \phi_n^2} \tag{4}
\]

with

\[
\int_0^\infty dx \, \phi_n^2 = 2^{n-1}n!(\pi/\mu)^{1/2} \tag{5}
\]

Equation (4) is most easily found by multiplying Eq. (1) for \( \phi_n \) by \( \phi_n \), employing Eq. (3) to obtain \( \phi_n'' = -[u^2x^2 + i\mu(2n + 1)]\phi_n \) and integrating from \( x = 0 \) to \( \infty \). In the integral in the numerator of Eq. (4), the convergence as \( x \to \infty \) is provided by \( \exp(-i\mu x^2) \) because \( \text{Im } \omega > 0 \) gives \( \text{Im } \mu < 0 \). As \( x \to 0 \), the convergence is a result of the \( Z \) functions changing from their large \( x \) form \( (|\xi| < 1) \) to their small \( x \) limit \( (|\xi| > 1) \). This change in form occurs for \( x \sim x_e \), where \( x_e \) is defined as the position at which \( \omega/k_{\|}(x = x_e) = \omega L_s/kx_e = v_e \). Because \( x_e \ll x_t \), only the dominant large \( x \) \( (|\xi| < 1) \) contribution to \( \sigma \) is significant, so that
\[ \sigma \approx \imath \pi^{1/2} x_e \exp \{-2D^{-1} \left[ \omega - \omega_* \left( \imath - \frac{1}{2} \eta_e \right) \right] \exp \left( -x_e^2 / x^2 \right) \} \tag{6} \]

where the \( \exp \left( -x_e^2 / x^2 \right) \) has been retained to provide convergence as \( x \to 0 \) in \( \int dx \phi_n^2 \sigma / \chi \). The additional terms in \( \sigma \) can result in order unity corrections to \( \ln \left( x_t / x_e \right) \) at most. Consequently, the integrals to be evaluated in order to simplify the eigenvalue equation, Eq. (4), are of the form

\[ L_n = \int_0^\infty \frac{dx}{x} H_n^2 \left[ \left( \imath \mu \right)^{1/2} x \right] \exp \left[ -\imath \mu x^2 - \left( x_e / x \right)^2 \right] \] \tag{7}

Because of the \( 1/x \) in Eq. (7), only the \( x \ll x_t \) form of \( H_n \) is required for \( n = 0 \) or even; the corrections due to the remaining terms will be of order unity compared to \( \ln \left( x_t / x_e \right) \gg 1 \). For \( n \) odd the full \( H_n \) is required. As a result, Eq. (7) is integrated using

\[ H_n(t) \approx \frac{\left( -1 \right)^{n/2} n!}{(n/2)!} \left[ 1 + O \left( \frac{x^2}{x_t^2} \right) \right] \quad n = 0 \text{ or even} \tag{8a} \]

\[ H_n(t) = n! \sum_{\ell=0}^{(n-1)/2} \frac{\left( -1 \right)^{\ell} (2\ell)!}{\ell! \Gamma(n - 2\ell + 1)} \quad n \text{ odd} \tag{8b} \]

One may note that the lower cutoff at \( x_e \) is only significant for \( n = 0 \) or even and employ \([4]\)

\[ \int_0^{\infty} dt \ t^{n-2\ell-1} H_n(t) \exp \left( -t^2 \right) = \frac{\pi^{1/2} 2^{2\ell} \Gamma(n - 2\ell)}{\Gamma \left( \frac{1}{2} - \ell \right)} \]
Then Eq. (7) yields

\[
L_n = \begin{cases} 
\left[ \frac{n!}{(n/2)!} \right]^2 \ln \left( \frac{x_t}{x_e} \right) & n = 0 \text{ or even} \\
2^n \sum_{\ell=0}^{(n-1)/2} \frac{(2\ell)!}{2^{2\ell}(\ell!)^2(n - 2\ell)} & n \text{ odd}
\end{cases}
\]

(9)

Inserting Eqs. (5) and (9) into Eq. (4) yields the perturbation theory form of the eigenvalue equation,

\[
\Lambda + i (2n + 1) \mu = -2\sigma_0 \left( \frac{i\mu}{\pi} \right)^{1/2} \begin{cases} 
n! \ln \left( \frac{x_t}{x_e} \right) & n = 0 \text{ or even} \\
\frac{2^n [(n/2)!]^2}{\sum_{\ell=0}^{(n-1)/2} \frac{(2\ell)!}{2^{2\ell}(\ell!)^2(n - 2\ell)}} & n \text{ odd}
\end{cases}
\]

(10)

with

\[
\sigma_0 = i\pi^{1/2} x_e^{2D^{-1}} \left[ \omega - \omega_* \left( 1 - \frac{1}{2} \eta_e \right) \right]
\]

(11)

From Eq. (10), it can be seen that \( n = 0 \) is the most difficult eigenmode to stabilize, \( n = 1 \) is the most difficult odd mode to stabilize, and
the destabilizing term for even modes is of order $\ln \left( x_t/x_e \right)$ larger than the destabilizing term of odd modes for comparable $n$. The terms in Eq. (2) not retained in Eq. (6) would result in corrections of order unity compared to the $\ln \left( x_t/x_e \right) \gg 1$ for the $n = 0$ or even modes and $O(x_e/x_t)$ versus unity for the odd $n$ modes.

4. EVEN MODES

In this section the eigenvalues associated with Eq. (1) for the even ($n = 0$ or even) spatial eigenmodes are found by employing and extending the matched asymptotic solution of Rosenbluth and Catto [2]. (In Section 5 we find the eigenvalues for the odd eigenmodes.) In order to take full advantage of Ref. [2],

$$\sigma \approx \rho_i^{-2} D^{-1} x_e \left[ \omega - \omega^* \left( 1 - \frac{1}{2} \eta_e \right) \right] Z(x_e/x)$$  \hspace{1cm} (12)

is employed rather than Eq. (6). The additional $\eta_e$ terms of Eq. (2) can only result in order unity corrections to $\ln \left( x_t/x_e \right) \gg 1$, at most, and so are neglected.

To obtain the full eigenvalue equation for the even spatial modes, two forms of Eq. (1) are employed. The first,

$$\frac{\partial^2 \phi}{\partial x^2} - \left( \Lambda - \mu^2 x^2 + \frac{\sigma_0}{x} \right) \phi = 0$$  \hspace{1cm} (13)
is valid in the outer region where \( x > x_e \), while the other,

\[
\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\sigma(|x|)}{x} \phi
\]  

(14)

is appropriate in the inner region \( 0 \leq x \leq x_e \).

Reference [2] solves Eq. (13) iteratively by neglecting the \( \sigma_0/x \) term in lowest order. The homogeneous solutions of Eq. (13) are found to within the unknown relative constant \( C \) between the two series solutions. The constant \( C \) will be determined from Eq. (14). In terms of the constant \( C \), an eigenvalue equation is obtained by demanding that \( \phi \) decay spatially as \( |x| \to \infty \) for unstable modes. Retaining only the dominant terms, this eigenvalue equation is

\[
1 + \frac{C \Gamma \left( \frac{1}{4} - \frac{i \Lambda}{4 \mu} \right)}{2^{1/2} \Gamma \left( \frac{3}{4} - \frac{i \Lambda}{4 \mu} \right)} = 0
\]  

(15)

In Ref. [2] further corrections to Eq. (15) are evaluated, but such terms may be neglected for the \( n = 0, 2, 4, \ldots \) modes when

\[
(m/M)^{1/2} \left( L_\perp/L_n \right)^{3/2} \ll 1
\]

The solution of Eq. (13) in the outer region can also be used to form an expression for the logarithmic derivative of \( \phi \) as the inner
region is approached \((x \to x_e)\). In terms of the relative constant \(C\), the result is

\[
\frac{1}{\phi} \frac{\partial \phi}{\partial x} = C(i2\mu)^{1/2} + \sigma_0 \ln \left(x/x_t\right) \tag{16}
\]

The constant \(C\) can then be evaluated by integrating Eq. (14) across the inner region as in Ref. [2] to obtain

\[
\frac{1}{\phi} \frac{\partial \phi}{\partial x} = \sigma_0 \ln \left(x/x_e\right) \tag{17}
\]

In Eq. (21) and the preceding equation of Ref. [2], \(\exp (i\pi/4)\) should be \(\exp (-i\pi/4)\), as can be seen from Eq. (6) of Ref. [2]. Combining Eqs (16) and (17) gives

\[
C = \sigma_0(i2\mu)^{-1/2} \ln \left(x_t/x_e\right) \tag{18}
\]

The constant \(C \neq 0\) because the integration across the inner region generates a contribution from both of the series solutions of Eq. (13). Inserting \(C\) into Eq. (15), the full eigenvalue equation for the even spatial modes may be written as

\[
1 + \frac{\sigma_0}{2\pi(i\mu)^{1/2}} \Gamma \left(\frac{1 - i\Lambda}{4} \right) \Gamma \left(\frac{1 + i\Lambda}{4} \right) \sin \left[\frac{\pi}{4} \left(1 + \frac{i\Lambda}{\mu}\right)\right] \ln \left(x_t/x_e\right) = 0
\]

\[
\tag{19}
\]
To verify that Eq. (19) does indeed reduce to Eq. (10) near marginal stability, let \( \lambda + i(2n + 1)\mu = \delta \) and solve Eq. (19) for \( \delta \) assuming \( |\delta/\mu| < 1 \) to obtain the \( n = 0 \) or even form of Eq. (10). This reduction occurs because the gamma function has simple poles at the points at which its argument is zero or a negative integer. Because \((2n + 1)\mu \approx \text{Im} \delta \) at marginal stability, Eq. (10) can only be qualitatively correct near and away from \( \text{Im} \omega = 0 \). In addition, because Eq. (10) is recovered by examining Eq. (19) near the simple poles of \( r[(1 - i\lambda/\mu)/4] \), it is possible that the perturbation theory form of Eq. (10) does not properly predict the growth rate even near marginal stability. For \( \text{Im} \delta \gg (2n + 1)\mu \), the full eigenvalue equation, Eq. (19), must be employed. In fact, for strongly unstable modes, \( |\lambda/\mu| \gg 1 \) and the asymptotic forms of the gamma functions may be employed to reduce Eq. (19) to a form completely different from Eq. (10).

5. ODD MODES

To obtain the full eigenvalue equation for the odd spatial modes, the work of Rosenbluth and Catto [2] is again employed and extended, using the value of \( \sigma \) as given by Eq. (12). Then for the odd modes the dominant term of \( \phi \) must be \( \phi \approx Ax \) for \( 0 \leq x \lesssim x_e \), where \( A \) is a constant. Integrating Eq. (14) from \( x = 0 \) to \( x \gg x_e \) (but \( x \ll x_t \)) and employing the techniques of Ref. [2] give
\[
\frac{1}{\phi} \frac{\partial \phi}{\partial x} = \frac{1}{x} + \frac{\sigma_0}{\pi x} \int_0^x dx \int_{-\infty}^\infty dt \exp (-t^2) \frac{\exp (-x_e/x)}{t - (x_e/x)}
\]

\[
= \frac{1}{x} + \frac{\sigma_0}{\pi x} \int_0^x dx \int_{-\infty}^\infty dt \exp (-t^2) \int_0^\infty d\eta \exp \left[ -i \eta \left(t - \frac{x_e}{x}\right) \right]
\]

\[
= \frac{1}{x} + \frac{\sigma_0}{\pi^{1/2} x} \int_0^x dx \exp (-x_e^2/x^2) \int_0^\infty d\eta \exp \left[ -\frac{1}{4} \left( \eta - i \frac{2x_e}{x} \right)^2 \right]
\]

\[
\approx \frac{1}{x} + \frac{\sigma_0}{x} \int_0^x dx \exp (-x_e^2/x^2)
\]

\[
\approx \frac{1}{x} + \sigma_0
\]

(20)

In obtaining Eq. (20), \( \text{Im} \; x_e > 0 \) (\( \text{Im} \; \omega > 0 \)) is employed to get from the first to the second line, while to get from the third to the fourth line, it is noted that because of \( \exp (-x_e^2/x^2) \), the region \( x > x_e \) dominates, so that to lowest order the \( \eta \) integral is \( \pi^{1/2} \).

Integrating Eq. (20) gives the next correction to \( \phi \) for \( x_e \ll x \ll x_\perp \), namely \( \phi \approx q \left( x + \frac{1}{2} \sigma_0 x^2 \right) \). However, unlike the even mode case, the integration across the inner region does not generate the leading term of the second homogeneous solution of Eq. (13), which starts off like \( 1 + \sigma_0 x \ln (x/x_\perp) \). As a result, in the outer region where Eq. (13) is appropriate, only the homogeneous solution having a leading term proportional to \( x \) is permitted (the \( x_1^{(1)} \) of Ref. [2]), and the coefficient of the other homogeneous solution \( (x_0^{(0)} + x_1^{(0)}) \) in Ref. [2])
must vanish ($C \to \infty$ in Ref. [2]). Consequently, the eigenvalue equation for the odd modes is recovered from Eq. (18) of Ref. [2] by retaining only the terms containing $C$ ($C \to \infty$). In our notation, Eq. (18) of Ref. [2] becomes

$$1 + \frac{\sigma_0}{4(i\mu)^{1/2}} \sum_{s=0}^{\infty} \frac{\Gamma(s - \frac{i\Lambda}{4\mu} + \frac{3}{4})}{\Gamma(s + \frac{1}{2}) \Gamma(s - \frac{i\Lambda}{4\mu} + \frac{5}{4})} = 0 \tag{21}$$

To obtain the perturbation theory result from Eq. (21), let $\Lambda + i(2n + 1)\mu = \delta$ with $n = 1, 3, 5, \ldots$ and solve for $\delta$ by assuming $|\delta/\mu| < 1$. Because $\Gamma[s - (i\Lambda/4\mu) + 3/4]$ behaves like a simple pole when its argument is near zero or a negative integer, only the $0 \leq s \leq (n - 1)/2$ terms in the sum of Eq. (21) are significant in this $|\delta/\mu| < 1$ limit. For $|\delta/\mu| < 1$, Eq. (21) therefore reduces to the perturbation theory result, Eq. (10). Again, the perturbation theory result is valid at most for $\text{Im } \omega \to 0$; away from marginal stability, the more complete eigenvalue equation, Eq. (21), must be employed.

6. DISCUSSION

In the preceding sections, the eigenvalue equations associated with the collisionless drift wave in tokamaks have been investigated. Both the perturbation theory and the method of matched asymptotic expansion forms for the eigenvalue equations have been obtained for all even and
odd radial eigenmodes. In addition, the condition under which the perturbation theory forms can be recovered from the more exact matched asymptotic expansion forms is presented. In particular, the perturbation theory forms are found to be qualitatively correct only near marginal stability; away from marginal stability the more exact forms must be employed. However, because the perturbation theory form is obtained from the more exact form by expanding about simple poles (of the gamma function) it appears possible that the more exact form may be required even near marginal stability. This possibility is currently being investigated by numerical solution of the perturbation theory and more exact forms and comparison of them against one another and against a numerical solution of the differential equation, Eq. (1).

The solution technique from Ref. [2] that is employed in this paper assumes that the ratio of the resonant electron term divided by the shear term, both evaluated at \( \sigma_0 \), is less than or on the order of unity. This inequality reduces to \( \sigma_0 \approx 1 \). Taking \( b \approx 1 \) and \( \omega \approx \omega_\star \), this inequality becomes roughly \( (m/M)^{1/2} (L_s/L_n)^{3/2} < 1 \), which is more restrictive than \( |x_e/x_t| \approx (mL_s/ML_n)^{1/2} < 1 \). These \( \sigma_0 \approx 1 \) corrections occur because the resonant electron term results in modifications of the eigenfunctions so that they are no longer simply Hermite polynomials times exponentials.

The results obtained herein are expected to be valid as long as the outgoing waves are not significantly reflected at adjacent rational surfaces. This restriction will be satisfied for inverse aspect ratios
\[ \varepsilon < (k\rho_1)^2(qR/L_s)^2, \]

where \( qR \) is the connection length [5]. For the collisionless drift wave, the perturbation theory [6] results indicate that the maximum growth rates occur for \( (k\rho_1)^2 > 1 \), while typically \( qR/L_s \sim 1 \). Consequently, the defeat of shear stabilization by the reflection mechanism of Taylor [5] is not expected to be a significant effect.

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