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States and State-Preparing Procedures in Quantum Mechanics

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Summary

D'Espagnat and others have shown that different preparation procedures that mix systems prepared in inequivalent states and are objectively different, are nevertheless assigned the same

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state. This unpalatable result follows from the usual interpretative rules of quantum mechanics. It is shown here that this result is incompatible with the strengthened interpretative rules (requiring randomness of the measurement outcome sequence) recently proposed. Thus, the randomness requirement restores reasonableness.

I. Introduction

D'Espagant [1] and others [2] have recently discussed different types of procedures for preparing an unpolarized beam of spin 1/2 particles. They have shown that different preparation procedures which correspond to the same single particle density matrix nevertheless prepare ensembles of particles which have objectively different properties.

Among others, they consider two state-preparing procedures s_u^n and s_v^n for spin 1/2 particles. s_u^n is such that n repetitions prepare n/2 particles with spin along the +z axis and n/2 particles with sping along the -z axis. Thus m=kn+j repetitions of s_u^n prepare k runs of n particles with each run aligned as above and the remaining j (with $0 \leq j < n$) particles have spin alignments compatible with the above. s_v^n is similiar except that +x and -x replace +z and -z. D'Espagant shows that both s_v^n and s_v^n must be assigned the same density matrix $1/2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ yet they clearly prepare sequences of systems that have objectively different properties.

One of the authors [3] has recently proposed a new rule of interpretation for quantum mechanics which requires that the outcome sequence obtained from a repetition of preparing a system in some state followed by an observation be random. It will be shown here that the proposed new rules of interpretation eliminate as invalid some of the state preparing procedures proposed by D'Espagnat.

In more general terms, it will be shown here that quantum mechanics with the usual interpretative rules does not exclude certain physically unacceptable procedures which are ruled out by the strengthened rules. The method used will be to show by example that there exist procedures which 1) cannot be rejected as invalid by the usual interpretative rules, 2) are intuitively unsatisfactory, and 3) are rejected as invalid by the strengthened rules.

In particular, one can ask: if a state $\lambda_1\rho_1+\lambda_2\rho_2$ with $\lambda_1+\lambda_2=1$ is prepared by mixing the outputs of two state preparation procedures s_1 (for ρ_1) and s_2 (for ρ_2), what mixing procedures are allowed?

The strengthened interpretative rules give a precise meaning and rigorous proof for the intuitive answer: no mixing procedure which generates for some observable a nonrandom outcome sequence under repetition of the measurement, is allowed. This includes for example any programmable mixing of systems in ρ_1 and ρ_2.

In Section II, some background material is given. First, a reformulation is given of the usual scheme of quantum mechanics. This reformulation, proposed by one of us ([4]), distinguishes explicitly between the mathematical objects of the theory (states and observables) and their operational counterparts (state-preparing procedures and observation procedures). Then a brief discussion is given of why infinite repetitions and outcome sequences, rather than finite ones, are the relevant objects for quantum mechanics. This is followed by a discussion of why sequences of state preparing acts are used here rather than ensembles.

Section III gives some examples of state-preparing procedures in which the method of mixing is explicitly spelled out. Two of the examples, s_a and s_d, are of the type discussed by D'Espagnat as Methods I and II ([1]).

In Section IV the concept of randomness is discussed and defined preparatory to a statement of the strengthened interpretative rules of quantum mechanics.

In Section V, it is proved by means of the strengthened interpretative rule that some of the procedures given as examples in Section III are invalid preparation procedures. These examples include some of those discussed by D'Espagnat ([1]) as his methods I and II. It is also proved that any mixing procedure which mixes inequivalent states according to a definable mixing function is invalid.

Procedures which randomly mix state preparing procedures are considered in Section VI. It is proved that, under a natural extension of the strengthened interpretative rules, such procedures are valid state preparation procedures (Lemma 1). The paper concludes with a brief discussion of some remaining points.

The paper is self-contained and does not presuppose knowledge of esoteric mathematics or mathematical logic. The concepts are introduced and discussed in simple language.

II. Background

Since this paper is concerned with the relationship between different laboratory procedures that are assigned the same states, it is advisable to use a version of quantum mechanics which explicitly exhibits the correspondence between nonmathematical procedures and their mathematical counterparts ([4]).

In this scheme, state preparing procedures and observation procedures are identified with booklets of instructions and, as such, are considered to be repeatable. This means among other things, that instructions should not refer to the serial number of the experiment in a sequence. For instance, instructions such as: do f in the first 10 experiment's, g in the 11th etc., are excluded. One associates to each type of physical system, collections $\mathcal{S}$ and $\mathcal{O}$ of state-preparing procedures s and observation procedures o. Corresponding to $\mathcal{O}$ and $\mathcal{S}$ one has an algebra $\mathfrak{A}$ of observables and set S of states defined as positive, linear, norm-1 functionals from $\mathfrak{A}$ to the real numbers R. The correspondence between [4] $\mathcal{S}$ and S, and between $\mathcal{O}$ and $\mathfrak{A}$, is given by a pair of many-one maps $\Psi: \mathcal{S} \to S$ and $\Phi: \mathcal{O} \to \mathfrak{A}$. The algebra $\mathfrak{A}$ is considered to be a von Neumann algebra of operators on a Hilbert space and S the set of normal states, i.e., those implementable by density operators in $\mathfrak{A}$. Furthermore, the state prepared by some procedure is considered to be prepared at some instant of time. Throughout this paper we discuss only one pair (Φ, Ψ) of mappings, but the result holds for all maps (morphisms) compatible with the assumptions of Reference 4.

The connection between theory and experiment is made by carrying out n repetitions of doing some s in $\mathcal{S}$ followed by doing some α on the system so prepared and observing the outcome (a real number in the spectrum of $\Phi(\alpha)$). The mean $\bar{M}_n\psi_{s\alpha n}$ of the outcome sequence $\psi_{s\alpha n}$ so obtained, is close, in a probabilistic sense, to the expectation value $\Psi(s)(\Phi(\alpha))$. In the limit $n\to\infty$, $\bar{M}\psi_{s\alpha} = \Psi(s)(\Phi(\alpha))$ where $\bar{M}\psi_{s\alpha}$ is the limit mean of the infinite outcome sequence $\psi_{s\alpha}$.

Here, in agreement with Newton ([5]) the relevant objects are taken to be infinite sequences of state preparing and observation procedures even though one can, by any finite time, carry out only a finite number of procedures. One reason is that the comparison of a mean $\bar{M}_n\theta_{s\alpha}$ of a finite outcome sequence $\theta_{s\alpha}$ of length n with an expectation value $\Psi(s)(\Phi(\alpha))$ can only be given in terms of probability statements. For example, the law of large numbers states, "For each ε and δ there is an m such that for all $n > m$ the probability that $|\bar{M}_n\theta_{s\alpha}-\Psi(s)(\Phi(\alpha))|<\delta$ is greater than $1-\varepsilon$".

Now the meaning of such statements as "the probability that ..." can only be given either in terms of subjective probabilities ([6]), or it can be interpreted in terms of limit relative frequencies (of repetitions of runs of length n) which again brings in infinite repetitions and the associated infinite outcome sequences. Thus, if one excludes recourse to subjective probabilities, one cannot escape the infinite repetitions in a statistical theory such as quantum mechanics.

Let s and α denote, respectively, state-preparing and observation procedures and let X denote an infinite repetition of s and α. For example, X includes an infinite sequence of space-time regions, in each one of which it is possible to do exactly one ε and α.

An assumption of this paper is the following ([7]). With each infinite repetition X of each s in $\mathcal{S}$ and α in $\mathcal{O}$, there is associated a unique infinite outcome sequence $\psi^X_{s\alpha}$, such that for each n the actual carrying out of the first n repetitions in X of s and α gives as outcomes, the first n elements of $\psi^X_{s\alpha}$.

Note that this assumption requires that $\psi^X_{s\alpha}$ exist in some appropriate mathematical sense only; it does not require that $\psi^X_{s\alpha}$ be in hand at some finite time - something which is impossible. It also makes explicit the dependence of $\psi^X_{s\alpha}$ on X; if X and Y are two different sequences of space-time regions, then common experience tells us that in general $\psi^X_{s\alpha} \neq \psi^Y_{s\alpha}$. To simplify notation in what follows, the dependence of $\psi^X_{s\alpha}$ on X will be suppressed.

In this paper, the concept of sequence of state preparing acts is used instead of the concept of ensembles. One reason is that "ensemble" suggests a stored collection of noninteracting objects at one time. However, actual measurement sequences usually consist of repetitions of state-preparations and observations strung out in time so that an ensemble is not prepared. If an ensemble is prepared, then a sequence of selection and observation acts is performed which is equivalent to the former sequence.

Another reason is that it is impossible to assign any mean value whatsoever to an amorphous infinite ensemble of systems and an associated infinite set of single measurement outcomes (generated by each system interacting with a copy of a measurement apparatus) without a well-ordering of the ensemble ([8]). But this well-ordering which chooses elements of the ensemble and also well-orders the outcome set, converts it into a sequence ([9]).

III. Examples of Possible Preparation Procedures

In this section, some procedures, including some of the type considered by D'Espagnat, (Section I) are spelled out in apparently excessive detail. However, for mathematical proofs, explicitness is imperative. Also, it has to be shown that the shorthand instruction (Section I) for each run of n repetitions: "see to it that n/2 particles have their spin up," can be repeatedly implemented by a given set of instructions in agreement with the discussion given in Section II.

Consider the following six different proposed procedures for preparing unpolarized spin 1/2 particles.

a) Let K_0 and K_1 be procedures which compute {0,1} sequences θ_0 and θ_1 of length n such that θ_0 and θ_1 each contain n/2 0's and n/2 1's. s_a is the procedure: Read the number m on a screen and compute $j = m \bmod n$ ($m = kn+j$ for some largest $k = 0,1,\ldots$). If $j = 0$, flip a fair coin lying on a table, if $j > 0$, leave coin alone. Then read coin; if heads are up, input j to K_1, if tails are up, input j to K_0. Then, if the K-output is 1, emit particle with spin along +z axis, if output is 0, emit particle with spin along -z axis. Finally, add 1 to m.

b) s_b is the instruction: Examine a coin placed on a table. If heads are up, emit particle with spin along the +z axis, if tails are up, emit particle with spin along the -z axis. Then turn coin over.

c) s_c is the instruction: Flip a fair coin; if heads lands up, emit particle with spin along +z axis, if tails lands up, emit particle with spin along -z axis.

d,e,f) s_d, s_e and s_f are the same as s_a, s_b, and s_c, respectively, except that +x and -x replace +z and -z respectively.

Repetitions of s_a produce a sequence of +z, -z preparations which can be divided into runs of n repetitions of two types, K_0 and K_1 which are randomly selected by coin flips, and are such that both types give an equal number of +z and -z systems. Repetitions of s_b produce an alternating sequence of +z and -z particles and repetitions of s_c produce a randomly selected sequence of +z and -z particles. Repetitions of s_d, s_e and s_f are similar except that x replaces z everywhere. Note that part of each infinite repetition X of s_a is the initial value of m which can be chosen in any fashion whatever. Similarly for s_b the initial position of the coin on the table is part of X.

One should note that s_a is much less contrived than it appears to be. The essential reason is that, if one mixes beams of +z and -z particles in such a fashion that each run of n has exactly n/2 spins up and n/2 spins down, then the operator has to intervene in the mixing procedure — no random mixing will give such a spin up, spin down ratio in most sequences of n systems.

The random aspect of the mixing is that the orderings of the individual spin up, spin down particles in each sequence of n repetitions are randomly distributed among the sequences of successive n- repetitions. Now each such ordering corresponds to a 0-1 sequence θ of length n. Furthermore there are exactly $(n!/(n/2!)^2)$ possible orderings.

The procedure s_a describes such a random mixing of sequences of length n where, for simplicity, only two of the possible θ's are considered. This can be easily changed by rolling a $n!/(n/2!)^2$ -sided die instead of flipping a coin.

The only "really contrived" aspect of s_a, compared to the above, is that in s_a an experimenter knows before a sequence of n systems is produced what the ordering of the +z, -z systems will be. This can easily be fixed, say, by automating the procedure of choosing the ordering. In any case, it has no effect on the results. Also if desired the value of m for the first repetition of X can always be set equal to zero.

IV. Randomness

According to the usual interpretative rules of quantum mechanics ([10]), the mean value of a large but finite number n of repetitions of measuring an observable on a system prepared in some state converges to the expectation value for the observable and state as $n \to \infty$. Without further qualifications, this rule is not sufficient.

For example, an observer working on Mondays can find that the mean value of his observation results is very different from that obtained by his colleague working on Tuesdays on the same machine, no matter how many Mondays or Tuesdays they work. This can be done by slight modification of the procedure [5b] in Section III (substitute "m particles" for "particle"). Of course, the mean value of the two sequences combined agrees with the theory.

As another example, the observer may find that the mean of the first n repetitions, although converging to the expectation value, is greater than the expectation value. Furthermore, this remains the case, no matter how many repetitions he carries out.

All experimenters assume implicitly that these possibilities will not materialize. In essence, they assume that the outcome sequence of a large number n of repetitions is sufficiently random so that the probability of numerous Monday results deviating appreciably from numerous Tuesday results, is small. Similarly, they assume that the probability of an appreciable deviation of the mean of the first n outcomes from the limit mean (and the theoretical expectation value) is small, and that it goes to 0 as n increases indefinitely.

The strengthened interpretative rules recently proposed [3] make this implicit assumption explicit by requiring that the outcome sequence associated with an infinite repetition of carrying out a state preparing procedure and an observation procedure, be random.

In order to understand the precise definition of randomness used here, it is good to note that, heuristically, a random sequence is in essence a 'typical' sequence and that 'most' sequences are random. This statement seems at first counter-intuitive because all sequences that one can explicitly define, exhibit, point at, are not random, so that none of the familiar sequences is accepted as typical. The paradox is somewhat similar to the surprise the student of elementary analysis feels when he is told that <u>most</u> real-valued functions are not differentiable while he is only familiar with differentiable ones. Yet a little thinking shows that there is a real sense in which the existence of an algorithm for constructing a sequence is something very special. Events for the infinite future can be predicted from a finite algorithm only for especially simple systems. An infinite sequence that can be specified by a finite number of words is not typical.

Continuing our preliminary inquiry in loose words, we give a more explicit meaning for a typical element of a class. For a particular cat, Kitty, to be typical, it has to have one head and four legs, it must jump, purr, etc., i.e, it must share relevant properties with most cats. It cannot share all properties that almost all cats have, because almost all cats have the property of not being my Kitty, but Kitty cannot both Kitty and non-Kitty. More generally, a typical member of a class must share all relevant properties with almost all members of the class. It remains to find natural precise definitions for almost all and for relevant.

A suitable definition of randomness must capture this. It must use a precise definition of a property possessed by 'almost all' sequences. It must limit the class of such properties so that all properties which one intuitively feels to be necessary, are included and it must also take account of the fact that properties which are relevant or typical in some experimental situations are not relevant to others. Finally, it must say nothing about irrelevant properties which only some typical sequences have (e.g. the outcome of the third repetition is zero).

A precise definition of 'almost all' which captures much of the above is that of a property being true almost everywhere with respect to a product probability measure determined by the experiment. Consider an infinite repetition of doing s following by doing α on the system so prepared. To avoid unnecessary details, assume that the self-adjoint operator $\Phi(\alpha)$ has only a discrete spectrum, i.e.

$$\Phi(\alpha) = \sum_{r\epsilon\sigma} r\, P_r^{\phi(\alpha)} , \qquad (1)$$

where σ is the spectrum of $\Phi(\alpha)$ and $P_r^{\Phi(\alpha)}$ are projection operators. Let σ^N denote the set of all infinite sequences of elements of σ and $\mathcal{B}(\sigma^N)$ the set of all Borel subsets of σ^N. Clearly, any outcome sequence associated with an infinite repetition of doing s and α lies in σ^N. N is the set of positive integers.

Define a probability measure $p_{s\alpha}$ on the set of all subsets E of σ as follows:

$$p_{s\alpha}(\{r\}) = \Psi(s)(P_r^{\phi(\alpha)}) \tag{2}$$

for each singleton subset $\{r\}$ and

$$p_{s\alpha}(E) = \sum_{r\in E} p_{s\alpha}(\{r\}). \tag{3}$$

If, as usual, the state $\Psi(s)$ is implemented by a density operator ρ_s, Eq. (2) reads: $p_{s\alpha}(\{r\}) = Tr\rho_s P_r^{\phi(\alpha)}$.

Now define $P_{s\alpha}$ by $P_{s\alpha} = \otimes_{j=1}^{\infty} p_{s\alpha j}$ with $p_{s\alpha j} = p_{s\alpha}$ for each j. Thus, $P_{s\alpha}$, a probability measure on $\mathcal{B}(\sigma^N)$, is the product of the measure $p_{s\alpha}$. This measure $P_{s\alpha}$ has the property that for each finite sequence θ of length ℓ,

$$P_{s\alpha}B_\theta = \prod_{j=1}^{\ell} p_{s\alpha}(\{\theta(j)\}) \tag{4}$$

where $B_\theta = \{\psi|\psi\epsilon\sigma^N$ and for each $j\leq\ell$, $\theta(j) = \psi(j)\}$, i.e. the set of all infinite sequences whose first j members agree with θ.

Substantially every property $Q(-)$ of outcome sequences corresponds to a unique Borel subset of σ^N, namely, $\{\psi|Q(\psi)\}$, the set of all sequences which have the property Q. By definition, Q is true $P_{s\alpha}$ almost everywhere if '$P_{s\alpha}$ almost all' sequences have Q, i.e. $P_{s\alpha}\{\psi|Q(\psi)\} = 1$.

Note that for the interesting cases ($P_{s\alpha} = \otimes p_{s\alpha}$ with $0 < p_{s\alpha}(\{r\}) < 1$ for at least one r) no sequence has all such properties. For any ψ, let Q be the property, "- is not equal to ψ". Since $P_{s\alpha}(\{\psi\}) = \Pi_j p_{s\alpha}(\{\psi(j)\}) = 0$, (being the infinite product of positive numbers smaller than one), $P_{s\alpha}\{\psi' | Q(\psi')\} = P_{s\alpha}(\sigma^N - \{\psi\}) = 1$. Hence, Q is true for $P_{s\alpha}$ almost all sequences. But ψ cannot be "not equal to ψ", so that some restriction is needed.

There are several definitions of randomness for infinite sequences in the literature ([11-15]) which differ in the definition of the set of relevant properties, which is a subset of the set of all properties that are true almost everywhere. These will be further discussed in Section VII. Suffice it to say here that all these definitions include all common-sense relevant properties. Other definitions are given in terms of finite sequences ([12,16]). Definitions in terms of subsequence selection procedures only ([17]) have been shown to be unsatisfactory ([18]).

For reasons not be be discussed here we use a strong definition of randomness consistent with the earlier work of one of us ([3,19]). A property Q(-) is ZF-definable if there is a formula F(-) in the language of Zermelo Frankel set theory (*) such that a sequence ψ has Q if and only if $F(\psi)$ is true.

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*This is the most widely used axiomatization of the common intuitive set theory.

Let $P = \otimes_j p_j$ with $p_j = p$. A sequence ψ is said to be ZF-random for P if all properties Q which are ZF definable and are true P almost everywhere, are properties of ψ. A sequence ψ is ZF-random if there exists a product measure P such that ψ is ZF random for P (**).

One must now show that the definition of ZF-randomness is intuitively satisfactory. First, one notes that since ZF formulas are certain finite strings of symbols ([20]) of the formal language of ZF set theory, there is a countable infinity of them. Thus there is at most a countable infinity of ZF-definable properties, thus the set of sequences which are ZF-random for P, is the intersection of a countable infinity of sets of P measure 1. Thus, it is also a set of P measure 1: hence 'P almost all' sequences are ZF-random for P.

Let s and α be such that $p_{s\alpha}$, Eqs. (2) and (3), is ZF definable. Then so is the property $Q(\psi)$ defined by $\bar{M}\psi = \Sigma_{r \in \sigma} r p_{s\alpha}(\{r\})$. That is, $Q(\psi)$ is the property "the limit mean of ψ equals the expectation value $\Sigma r p_{s\alpha}(\{r\}) (=\Psi(s)(\Phi(\alpha)))$. Since $P_{s\alpha}$ is a product measure and is a product of the same measure, it follows ([20]) that $P_{s\alpha}(\{\psi | Q(\psi)\}) = 1$. Thus, if $\psi_{s\alpha}$ is ZF-random for $P_{s\alpha}$, one has the desirable result that $\bar{M}\psi_{s\alpha} = \Psi(s)(\Phi(\alpha))$.

**These definitions are slightly different from those used in references 3 and 14, in that the latter are given in terms of properties ZF-definable from P. This minor change, which is made here to keep the discussion technically simple, means that consideration is here limited to ZF-definable probability measures.

Let $g:N \to N$ be a one-one ZF-definable function from N into N. Then, the property $\bar{M}\psi = \bar{M}\psi_g$ is ZF-definable, if ψ_g is the subsequence of ψ defined by $\psi_g(j) = \psi(g(j))$ for each j in N. If P is a product measure, i.e. $P = \bigotimes_j p_j$ with $p_j = p$ then [22] $P\{\psi|\bar{M}\psi = \bar{M}\psi_g = 1\}$. Thus, if $\psi_{s\alpha}$ is ZF-random, one has the result that $\bar{M}\psi_{s\alpha} = \bar{M}\psi_{s\alpha g}$, or the limit mean of $\psi_{s\alpha}$ is invariant under ZF definable subsequence selections.

The definition of randomness also includes subsequence selection procedures which examine elements of a sequence before making a selection [17]. However, for such procedures, the selection or rejection of the outcome of the jth repetition cannot depend on the value of $\psi_{s\alpha}(j)$.

On the other hand, for the property "$\psi(3) = r$" with $0 < p_{s\alpha}(\{r\}) < 1$, one has $0 < P_{s\alpha}B_{3r} < 1$ where $B_{3r} = \{\psi|\psi(3) = r\}$. For this property the fact that $\psi_{s\alpha}$ is ZF-random says nothing about whether $\psi_{s\alpha}(3) = r$ or not.

The definition of ZF-randomness is given so as to include constant r sequences for each r in σ. This is the case, for example, if $\Psi(s)$ is dispersion-free for $\Phi(\alpha)$ at r. These cases can be easily excluded if desired, by requiring P to be a nonatomic product measure ($0 < p(\{r\}) < 1$ for at least two values of r).

To keep the discussion as simple as possible, from now on we limit the discussion to question measuring procedures (those α which have "yes" (1) or "no" (0) outcomes only and for which $(\Phi(\alpha))^2 = \Phi(\alpha))$ and to state-preparing procedures such that for each question procedure α, $p_{s\alpha}(\{1\})$ is a rational number in the unit interval. This has the result that all $P_{s\alpha} = \bigotimes_j p_{s\alpha j}$ with $p_{s\alpha j} = p_{s\alpha}$ for each j, are ZF-definable.

With these definitions, the strengthened rules of interpretation ([3]) for quantum mechanics become: For each state-preparing procedure s and question procedure α and each infinite repetition, the associated outcome sequence $\psi_{s\alpha}$ must be ZF-random for $P_{s\alpha}$.

It can be shown ([3]) that the usual interpretative rules, i.e. the spectrum rule and the expectation value rule, follow from this strengthened rule.

V. Exclusion of Preparation Procedures

Consider the preparation procedures $s_a \ldots s_f$, defined in Sect. III. It is clear from their construction that for each observation procedure α and each infinite repetition, $\bar{M}\psi_{s_a\alpha} = \bar{M}\psi_{s_b\alpha} = \cdots, = \bar{M}\psi_{s_f\alpha}$. Thus, under the usual interpretative rules of quantum mechanics, $s_a, \ldots, s_f$ are all valid preparation procedures for the same density matrix $1/2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

However, the strengthened interpretative rules exclude s_a, s_b, s_d, and s_e as invalid preparation procedures. For, let α be a question measuring procedure corresponding to the projection operator for the spin along the +z axis. Then, $\psi_{s_a\alpha}$ will be a sequence of 0's and 1's such that $\Sigma_{j=kn+1}^{(k+1)n}\psi_{s_a\alpha}(j) = n/2$ for each $k = 0, 1, \ldots$ (with no loss of generality, we assume that for the first repetition the value of m in s_a is 0). For the same α, $\psi_{s_b\alpha}$ will be an alternating 0-1 sequence. Both of these sequences are clearly not random, so that s_a and s_b are not acceptable procedures.

More exactly, the proof goes as follows: If s_a is a valid state-preparing procedure, then by the strengthened interpretative rule for each α and each infinite repetition, the associated outcome sequence $\psi_{s_a\alpha}$ must be ZF random for some product measure $Q_{s_a\alpha}(= \otimes_j q_{s_a\alpha j}$ with $q_{s_a\alpha j} = q_{s_a\alpha})$ on $\mathcal{B}(\{0,1\}^N)$.

Assume that s_a is valid. Suppose that one can find an α and a property which $\psi_{s_a\alpha}$ does not have but which is ZF-definable and is true $Q_{s_a\alpha}$ almost everywhere. Then by hypothesis and the definition of ZF-randomness, $\psi_{s_a\alpha}$ must have that property. But this is a contradiction; thus s_a is not a valid state-preparing procedure.

We now exhibit such a property. Let α be the question procedure above. Then, by the construction of s_a, $q_{s_a\alpha}(\{1\}) = 1/2$. Define the sequence V_{0n}, V_{1n}, ... V_{kn}, ... of random variables as functions from $\{0,1\}^N$ to the real numbers by

$$V_{kn}(\psi) = \left(\sum_{j=kn+1}^{(k+1)n} \psi(j) - \frac{n}{2}\right)^2. \qquad (5)$$

$V_{kn}(\psi)$ gives the square of the deviation from the mean of $\Sigma\psi(j)$ in the kth run of n repetitions.

The average deviation or variance is given by

$$\overline{V}_n(\psi) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} V_{jn}(\psi), \qquad (6)$$

if the limit exists. Now $Q_{s_a\alpha}$, as a product of the same measure, is such that [21] $Q_{s_a\alpha}\{\psi|\overline{V}_n(\psi) \text{ exists}\} = 1$, and

$$Q_{s_a\alpha}\{\psi|\overline{V}_n(\psi) = \int V_{0n} dQ_{s_a\alpha}\} = 1.$$

$\int V_{0n} dQ_{s_a\alpha}$ is the expectation value of V_{0n} with respect to $Q_{s_a\alpha}$ and $\int V_{0n} dQ_{s_a\alpha} = n \cdot q_{s_a\alpha}(\{1\}) \cdot q_{s_a\alpha}(\{0\}) = n/4$.

By the above, $\bar{V}_n(-) = n/4$ is a property which is true $Q_{s_a\alpha}$ almost everywhere. It is also clearly (Eqs. (5) and (6)) ZF-definable. Also, by the construction of s_a, $V_{kn}(\psi_{s_a\alpha}) = n/2 - n/2 = 0$, Eq. (5), for each k. Thus, $\bar{V}_n(\psi_{s_a\alpha}) = 0$ and $\psi_{s_a\alpha}$ does not have the property $\bar{V}_n(-) = n/4$. Hence, $\bar{V}_n(-) = n/4$ is the desired property and the proof is complete.

For s_b, the desired property is $\bar{M}\psi^e = \bar{M}\psi^o$, where ψ^e and ψ^o denote the respective even-numbered and odd-numbered elements of ψ. By construction of s_b, for the α given above, $\bar{M}\psi^e_{s_b\alpha} \neq \bar{M}\psi^o_{s_b\alpha}$ as $\psi_{s_b\alpha}$ is an alternating 0-1 sequence, so that $\psi_{s_b\alpha}$ does not have the desired property.

Entirely similar proofs hold for the procedures s_d and s_e when α is the question procedure for 'the spin lies along the +x axis'.

s_b and s_e are especially simple examples of procedures for mixing two inequivalent state-preparing procedures s_0 and s_1. (s_0 and s_1 are inequivalent if for some α, $\Psi(s_0)(\phi(\alpha)) \neq \Psi(s_1)(\phi(\alpha))$. The results obtained can be extended as follows: Let s_K be a procedure which mixes s_0 and s_1 according to some procedure K which, given j = 1,2,..., outputs a 0 or 1. For example, let s_K be the procedure "input to K the number on a screen, if K outputs 0, do s_0, if K outputs 1, do s_1, add 1 to the number on the screen" Let ϕ_K be the function associated with K during an infinite repetition of s_K. Then, one can prove ([19]) from the strengthened interpretative rules that, if K is such that ϕ_K is ZF-definable, then s_K is an invalid state-preparing procedure.

For this proof, consider the property $\bar{M}g^1_{\phi_K}\psi = \bar{M}g^0_{\phi_K}\psi$, where $g^1_{\phi_K}\psi$ is the subsequence of ψ which selects in the natural order, all and only those elements $\psi(j)$ of ψ for which $\phi_K(j) = 1$. $g^0_{\phi_K}$ does a similar selection for $\phi_K(j) = 0$. Then $\bar{M}g^1_{\phi_K}\psi = \bar{M}g^0_{\phi_K}\psi$ is ZF-definable if ϕ_K is, and is true almost everywhere for an arbitrary product measure. Let α be such that $\Psi(s_0)(\Phi(\alpha)) \neq \Psi(s_1)(\Phi(\alpha))$. Then by the definition of s_K $\bar{M}g^1_{\phi_K}\psi_{s_K\alpha} = \Psi(s_1)(\Phi(\alpha)) \neq \Psi(s_0)(\Phi(\alpha)) = \bar{M}g^0_{\phi_K}\psi_{s_K\alpha}$, so $\bar{M}g^1_{\phi_K}\psi = \bar{M}g^0_{\phi_K}\psi$ is the desired property for the proof.

Thus, one sees that by the strengthened interpretative rules of quantum mechanics, where randomness is defined to be ZF-randomness, no procedure which mixes systems in inequivalent states by a ZF-definable choice function is a valid state preparation procedure.

If the procedure K is such that an infinite repetition gives an output choice function ϕ which is not ZF-definable, then the above proof fails and, in general, it is an open question whether or not s_K is a valid state preparation procedure. However, if ϕ is ZF-random, as in the case of s_c or s_f, then one has a stronger result to be discussed in the next section.

VI. Random Mixing of Preparation Procedures

The discussion of random mixing is best begun by noting that an infinite repetition of identical measurements is a special type of a general stochastic process. In a general process, successive single step outcomes can have a quite complex statistical dependence on the previous outcomes and the probabilities of the single step outcomes can depend on the step number. However, even here one requires that the outcome sequence associated with the carrying out of such a process be 'typical' for the process.

This is expressed in precise terms as follows: Let P be the probability measure assigned (by the theory of such processes) to the given stochastic process on $\{0,1\}^N$. The measure P is said to be *ZF-correct for the outcome sequence* ψ if all properties which are ZF-definable and define Borel subsets of $\{0,1\}^N$ and are true P almost everywhere, are true for ψ (†).

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†This concept of correctness of a measure for an outcome sequence has been used elsewhere [P. A. Benioff, J. Math. Phys., 11, 2553 (1970); 12, 360 (1971); Foundations of Physics, 3, 359 (1973)] as a basis for a definition of agreement between a statistical theory and experiment.

From this definition, one has the immediate result that a sequence ψ is ZF-random for P if P is a product measure and is a product of the same measure p (i.e. $P = \otimes_j p_j$ with $p_j = p$) where p is ZF-definable and P is ZF-correct for ψ. Thus, the proposed strengthened interpretative rules of Quantum Mechanics are equivalent to the requirement that for each state-preparing procedure s and question procedure α for which $p_{s\alpha}$ is ZF definable, the measure $P_{s\alpha} = \otimes_j p_{s\alpha j}$ with $p_{s\alpha j} = p_{s\alpha}$ is ZF-correct for $\psi_{s\alpha}$.

We now return to the question of whether a procedure which mixes two inequivalent state-preparing procedures in a "random manner" is an acceptable state-preparing procedure. More exactly, let s_2 be a state-preparing procedure and β a question procedure. (The possible outcomes of β are 0 and 1 with $(\Phi(\beta))^2 = \Phi(\beta)$.) Let s_0 and s_1 be the two inequivalent state-preparing procedures. Define the procedure $s_{s_2\beta}$ by the instruction: do s_2 and β, if outcome is 1, do s_1, if outcome is 0, do s_0.

The procedures s_c and s_f given earlier are special cases of $s_{s_2\beta}$, where s_2 prepares and flips a coin, β is the procedure for observing which side is up, and s_1 and s_0 prepare a spin 1/2 system with spin along the respective +z and -z axes for s_b and the +x and -x axes for s_d.

An infinite repetition of doing $s_{s_2\beta}$ clearly prepares s_0 and s_1 systems in the order given by $\psi_{s_2\beta}$, the outcome sequence obtained by the corresponding infinite repetition of doing s_2 and β, and mixes the s_0 and s_1 systems in the ratio $\bar{M}\psi_{s_2\beta}/(1-\bar{M}\psi_{s_2\beta})$. The question then arises as to whether $s_{s_2\beta}$ is a valid preparation procedure for systems in the state γ defined by

$$\gamma = (\bar{M}\psi_{s_2\beta})\Psi(s_1) + (1-\bar{M}\psi_{s_2\beta})(\Psi(s_0)) \tag{7}$$

Let L denote the procedures s_2 and β and define the probability measures $P_{s_L\alpha}$ and $Q_{s_L\alpha}$ on $\mathcal{B}(\{0,1\}^N)$ by $P_{s_L\alpha} = \otimes_{j=1}^{\infty} p_{s\alpha j}$ and $Q_{s_L\alpha} = \otimes_{j=1}^{\infty} q_{s\alpha j}$ where for each j and question procedure α,

$$p_{s\alpha j}(\{1\}) = \Psi(s_{\psi_{s_2\beta}(j)})(\Phi(\alpha)) \tag{8}$$

and

$$q_{s\alpha j}(\{1\}) = \gamma(\Phi(\alpha)). \tag{9}$$

with γ given by Eq. (7). Both $P_{s_L\alpha}$ and $Q_{s_L\alpha}$ are product measures; however, only for the latter are the components independent of j.

Since s_2 and β are valid procedures, the strengthened interpretative rules give the results that $\psi_{s_2\beta}$ is ZF random for $P_{s_2\beta} = \otimes_j p_{s_2\beta}$ and thus that $\bar{M}\psi_{s_2\beta} = \Psi(s_2)(\Phi(\beta))$.

A reasonable extension of the strengthened interpretative rule to cover infinite repetitions of $s_{s_2\beta}$ and α requires that $P_{s_L\alpha}$ be ZF correct for $\psi_{s_{s_2\beta}\alpha}$, the outcome sequence associated with an infinite repetition of doing $s_{s_2\beta}$ and α.

To see that this is so, consider the case in which s_0, s_1, and α are such that $\Psi(s_0)$ and $\Psi(s_1)$ are dispersion-free for $\Phi(\alpha)$ at 0 and 1, respectively (i.e. $\Psi(s_1)(\Phi(\alpha)) = 1$ and $\Psi(s_0)(\Phi(\alpha)) = 0$). Then, on intuitive grounds, if in the jth repetition of $s_{s_2\beta}$ and α, $\psi_{s_2\beta}(j) = 1$, one would expect $\psi_{s_L\alpha}(j) = 1$. If $\psi_{s_2\beta}(j) = 0$, one would expect $\psi_{s_L\alpha}(j) = 0$. This result follows from the ZF-correctness of $P_{s_L\alpha}$ for $\psi_{s_L\alpha}$. It does <u>not</u> follow from the ZF-correctness of $Q_{s_L\alpha}$ for $\psi_{s_L\alpha}$.

However, one has ([19])

Lemma 1

If $P_{s_L\alpha}$ is ZF-correct for $\psi_{s_L\alpha}$, and s_2 and β are valid procedures, then $\psi_{s_L\alpha}$ is ZF-random for $Q_{s_L\alpha}$.

The proof of this Lemma is given in the Appendix as it is long and not directly relevant to the considerations of this paper.

It follows from this Lemma and the strengthened interpretative rules that, if $P_{s_L\alpha}$ is ZF-correct for $\psi_{s_L\alpha}$, and s_2 and β are valid procedures, then $s_{s_2\beta}$ is a valid state-preparing procedure and prepares the state γ given by Eq. (7), or

$$\Psi(s_{s_2\beta}) = \lambda_0\Psi(s_0) + \lambda_1\Psi(s_1)$$

with $\lambda_1 = \Psi(s_2)(\Phi(\beta))$ and $\lambda_0 = 1 - \lambda_1$.

It follows similarly that if the coin-tossing in an infinite repetition of s_e gives a ZF-random sequence ψ_0, and $P_{s_e\psi_0\alpha} = \otimes_j P_{s\alpha j}$, with $P_{s\alpha j}$ defined by Eq. (8) with $\psi_{s_2\beta}$ being replaced by ψ_0, is ZF-correct for $\psi_{s_c\alpha}$, then s_c is a valid state preparation procedure. A similar argument holds for s_f. Note that in this case both s_c and s_f correspond to the same density matrix, $1/2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

VII. Discussion

We begin by noting that the requirement that an outcome sequence $\psi_{s\alpha}$ be random is more important for quantum mechanics than for classical mechanics. The reason is that in classical mechanics, if s prepares a pure state, then, for any observation procedure α, $\psi_{s\alpha}$ is a constant sequence. This is not so in quantum mechanics. There, for any s which prepares a pure state, there exist α such that $\Psi(s)$ is not dispersion-free for $\Phi(\alpha)$, and $\psi_{s\alpha}$ is random. This is important because the pure states are the generators of the convex set of all states.

All of the results of this paper were obtained using a strong definition of randomness and of correctness. There are however, many proposed definitions of randomness in the literature (11-15) which are, so far at least, just as suitable as the one used here. For example, a definition of Martin Löf, (12) although given in terms of effectively enumerable sequential tests, is provably equivalent (††) to the definition given here if one

††Each Π_2^0 Borel set of P measure 0 with $P = \otimes p$ and p a computable probability measure on {0,1} is defined by an effectively enumerable P-sequential test and conversely. The complement of a Π_2^0 set of measure 0 is a Σ_2^0 set of measure 1 which gives the desired result.

replaces ZF-definable Borel sets by "Σ_2^0 Borel sets" (the definition of Σ_2^0 need not concern us here) and requires the product measures to be computable. For this definition (which is weaker than the one used here) one has results similar to those obtained here. For example, the results proved in Section V hold provided that $p_{s_0\alpha}$ and $p_{s_1\alpha}$ are computable and the condition that the choice function ϕ_K given by K be "ZF-definable" is replaced by the condition that ϕ_K be effectively enumerable. Since there exist sequences which are ZF-definable but not effectively enumerable, weaker definitions of randomness may exclude fewer state preparation procedures as invalid.

That definition of randomness which is the weakest possible without leading to contradiction, may be considered correct. Its nature is an important open question ([7]). Until a decision by proof is given, one is free to use whichever definition of randomness he likes best.

In conclusion, it is noted that the strengthend interpretative rules have several consequences for procedures ([19]) other than those considered here. For example, one can show that if $\psi_{s\alpha}$ is ZF-random, it is possible to construct from $\psi_{s\alpha}$, in a canonical way, a product probability measure $U(\psi_{s\alpha})$ such that $\psi_{s\alpha}$ is ZF-random for $U(\psi_{s\alpha})$. This has the important consequence that it is not necessary, given s and α, to know at the outset which probability measure is ZF correct for $\psi_{s\alpha}$.

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Appendix

The proof of Lemma 1) has two main parts. First one proves 1) For each Borel subset B of $\{0,1\}^N$, $Q_{s_L\alpha}(B) = \bar{P}^T_{s_L\alpha}(B)$ with $\bar{P}^T_{s_L\alpha}$ defined by

$$\bar{P}^T_{s_L\alpha}\, B = \lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} P_{s_L\alpha}(\tau^j B).$$

Here $\tau: \mathcal{B}(\{0,1\}^N) \to \mathcal{B}(\{0,1\}^N)$ maps subsets of $\{0,1\}^N$ to subsets of $\{0,1\}^N$ and is defined from T by $\tau B = \{\phi \mid T\phi \in B\}$. The one sided shift operator $T:\{0,1\}^N \to \{0,1\}^N$ is defined by $(T\phi)(j) = \phi(j+1)$ for each $j = 1,2 \ldots$. $P_{s_L\alpha}$ and $Q_{s_L\alpha}$ are given by Eqs. (7), (8), and (9).

From this one proves the second part:

2) For each ZF definable Borel set B, if $Q_{s_L\alpha}(B) = 1$, then $P_{s_L\alpha}(B) = 1$. Lemma 1) is proved from the above as follows: Assume the hypothesis of the Lemma, i.e. that $P_{s_L\alpha}$ is ZF-correct for $\psi_{s_L\alpha}$ and let the Borel set B be ZF-definable with $Q_{s_L\alpha}B = 1$. By 2) above, $P_{s_L\alpha}(B) = 1$. Thus, from the definition of ZF-correctness $\psi_{s_L\alpha} \in B$ which gives the result that $\psi_{s_L\alpha}$ is ZF random for $Q_{s_L\alpha}$.

Proof of 1)

Define $\mathcal{F}$ to be the smallest set of subsets of $\{0,1\}^N$ which contains all sets of the form $B_{n1} = \{\phi \mid \phi(n) = 1\}$ and is closed under complementation and finite unions. Then $\mathcal{F} \subset \mathcal{B}(\{0,1\}^N)$. We now show $\bar{P}^T_{s_L\alpha}$ exists on $\mathfrak{J}$ and equals $Q_{s_L\alpha}$ on $\mathfrak{J}$. The proof is by induction.

a. $\bar{P}^T_{s_L\alpha}(B_{n1}) = Q_{s_L\alpha}(B_{n1})$

Proof: By the definitions of B_{nl}, $\bar{P}^T_{s_L\alpha}$, $Q_{s_L\alpha}$, and τ

$$\bar{P}^T_{s_L\alpha}(B_{n1}) = \lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} P_{s_L\alpha}(\tau^j B_{n1}) \lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} P_{s_L\alpha}(B_{n+j,1})$$

$$= \lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} [\psi_{s_2\beta}(n+\ )\Psi(s_1)(\Phi(\alpha)) + (1-\psi_{s_2\beta}(n+j))\Psi(s_0)(\Phi(\alpha))] =$$

$$\lambda_1\Psi(s_1)(\Phi(\alpha)) + \lambda_0\Psi(s_0)(\Phi(\alpha)) = Q_{s_L\alpha}B_{n1}.$$

Here, $\overline{M}\psi_{s_2\beta} = \Psi(s_2)(\Phi(\beta)) = \lambda_1$ $(\lambda_0 = 1-\lambda_1)$ follows from the strengthened interpretative rules and the assumed validity of s_2 and β.

b. Let $B = \bigcup_{j=0}^{n} B_j$ with B_j pairwise disjoint and assum $\bar{P}^T_{s_L\alpha}(B_j) = Q_{s_L\alpha}(B_j)$ for each $j = 0,1\cdot\cdot n$. Then $\bar{P}^T_{s_L\alpha}(B) = Q_{s_L\alpha}(B)$.

Proof: Since $P_{s_L\alpha}$ and $Q_{s_L\alpha}$ are probability measures, one has from the fact that $\tau \bigcup_{k=0}^{n} B_k = \bigcup_{k=0}^{n} \tau B_k$,

$$\bar{P}^T_{s_L\alpha}(B) = \lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} P_{s_L\alpha}(\tau^j \bigcup_{k=0}^{n} B_k) = \lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{n} P_{s_L\alpha}(\tau^j B_k)$$

$$= \sum_{k=0}^{n} \bar{P}^T_{s_L\alpha}(B_k) = \sum_{k=0}^{n} Q_{s_L\alpha}(B_k) = Q_{s_L\alpha}(B).$$

c. Let $B = \{0,1\}^N - B'$ and $\bar{P}^T_{s_L\alpha}(B') = Q_{s_L\alpha}(B')$. Then, by an argument similar to that given above and left to the reader, one proves that $\bar{P}^T_{s_L\alpha}(B) = Q^T_{s_L\alpha}(B)$.

Thus, $\bar{P}^T_{s_L\alpha} = Q_{s_L\alpha}$ on $\mathcal{F}$ and is a probability measure on $\mathcal{F}$. By the Hahn extension theorem [23], $\bar{P}^T_{s_L\alpha}$ exists and equals $Q_{s_L\alpha}$ on $\mathcal{B}(\{0,1\}^N)$ and the first part is proved.

Proof of 2):

Let ϕ be any sequence in $\{0,1\}^N$ and define the product measure $P_{s_\phi\alpha} = \otimes_j p_{s_{\phi(j)}\alpha}$, where $p_{s_{\phi(j)}\alpha}(\{1\}) = \Psi(s_{\phi(j)})(\Phi(\alpha))$. Thus $P_{s_\phi\alpha} = P_{s_L\alpha}$ if $\phi = \psi_{s_L\beta}$. Let τ and T be as above. Then, for each Borel set B, $P_{s_{T\phi}\alpha}(B) = P_{s_\phi\alpha}(\tau B)$.

The proof by induction is briefly as follows:

If $B = B_{n1}$, then $P_{s_{T\phi}\alpha}(B_{n1}) = p_{s_{(T\phi)(n)}\alpha}(\{1\}) = p_{s_{\phi(n+1)}\alpha}(\{1\}) = P_{s_\phi\alpha}(\tau B_{n1})$. The rest of the proof is quite similar to that given in part 1) and is left to the reader.

Now let B be such that $Q_{s_L\alpha}(B) = 1$ and define the [0,1] valued function f_B on $\{0,1\}^N$ by

$$f_B(\phi) = P_{s_\phi\alpha}(B).$$

Then for $\bar{f}^T_{Bn}$ defined by $\bar{f}^T_{Bn} =$

$\frac{1}{n}\sum_{j=0}^{n-1} f_B(T^j\phi)$, one has from part 1), $\bar{f}^T_B(\phi) = \lim_{n\to\infty} \bar{f}^T_{Bn}(\phi) = \bar{P}^T_{s_\phi\alpha}(B) = \bar{P}^T_{s_L\alpha}(B) = Q_{s_L\alpha}(B) = 1$, provided that ϕ is ZP-random for the product measure $P_{s_2\beta} = \otimes_j p_{s_2\beta}$ with $p_{s_2\beta}(\{1\}) = \Psi(s_2)(\Phi(\beta)) = \lambda_1$. The existence of $\bar{F}^T_{s\phi\alpha}$ and the second equality above follow from this proviso.

Now f_B and $\bar{f}^T_B$ are both random variables [21] defined on the probability measure space $(\{0,1\}^N, \mathcal{B}(\{0,1\}^N), P_{s_2\beta})$. Also since $P_{s_2\beta}$ is a product of the same measure $p_{s_2\beta}$, the ergodic theorem and ergodic hypothesis give the result that [21]

$$\bar{f}^T_B(\phi) = \int f_B dP_{s_2\beta}$$

for $P_{s_2\beta}$ almost all sequences ϕ. It then follows that, since $P_{s_2\beta}$ almost all sequences are ZF-random for $P_{s_2\beta}$, $\int f_B dP_{s_2\beta} = 1$. From $0 \leq f_{B(\phi)} \leq 1$, one has that $f_B(\phi) = 1$ for $P_{s_2\beta}$ almost all ϕ.

Assume that B is ZF-definable. Then since $P_{s_2\beta}$, $P_{s_1\alpha}$, $P_{s_0\alpha}$, and T are ZF definable, so are f_B and $\bar{f}^T_B$ ZF-definable. By hypothesis, $\psi_{s_2\beta}$ is ZF-random for $P_{s_2\beta}$; thus,

$$1 = \bar{f}^T_B(\psi_{s_2\beta}) = \int f_B dP_{s_2\beta},$$

and $1 = f_B(\psi_{s_2\beta})$. By the definitions of f_B and $P_{s_L\alpha}$, $1 = P_{s_L\alpha}(B)$ and Lemma 1) is proved.

References

(1) B. D'Espagnat, Conceptual Foundations of Quantum Mechanics, W. A. Benjamin, Inc., Menlo Park, Calif., 1971, Chap. 10.

(2) G. C. Ghirardi, A. Rimini and T. Weber, Nuovo Cim 29B, 135 (1975); 33B, 457 (1976).

(3) P. A. Benioff, Phys. Rev. D7, 3603 (1973).

(4) H. Ekstein, Phys. Rev. 184, 1315 (1969).

(5) R. Newton, Nuovo Cim. 33B, 454 (1976).

(6) R. Giles, J. Math. Phys. 11, 2139 (1970).

(7) P. A. Benioff, J. Math. Phys. 17, 618,629 (1976).

(8) B. Russell, Human Knowledge, Its Scope and Limits, Simon and Schuster, Inc., New York, 1948, Part V, Chap. IV.

(9) See however, D. Blokhintsev, The Philosophy of Quantum Mechanics, D. Reidel Publ. Co., Inc., Dordrecht, 1968, Chap. IV.

(10) J. Von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton University Press, 1955, Chap. IV, Translated by R. T. Beyer.

(11) C. P. Schnorr, Zufälligkeit und Wahrscheinlichkeit, Math Lecture notes #218, Springer Verlag, Berlin, New York, 1971.

(12) P. Martin Löf, Information and Control, 9, 603 (1966).

(13) P. Martin Löf, On the Notion of Randomness. Proceedings Summer Institute on Proof Theory and Intuitionism, Buffalo, Ed. by J. Myhill, R. Kino and A. Vesley, North Holland, Amsterdam, 1970.

(14) R. Solovay, Ann. Math. 92, 1 (1970).

(15) A. H. Kruse, Zeit. Math. Logik Grundlagen d. Math. 13, 299 (1967).

(16) T. Fine, Theories of Probability, Academic Press, New York, 1973, Chap. V, provides a good discussion and comparison of the literature definitions.

(17) R. Von Mises, Mathematical Theory of Probability and Statistics, H. Geiringer, Ed. Academic Press, New York, 196[illegible] Chap. I; A. Church, Bull. Am. Math. Soc. 46, 130 (1940); A. Wald, Mathematischen Kolloquiums, Vienna 1973, Vol. 8; Selected Papers in Statistics and Probability of A. Wald, McGraw Hill, 1955, pp. 79-99. D. Loveland, Trans. Amer. Math. Soc. 125, 497 (1966); Zeit. Math. Logik Grundlagen d. Math. 12, 279 (1966).

(18) J. Ville, Etude Critique de la Notion de Collectif. Thesis University of Paris, Gauthier Villars, Paris, 1939.

(19) P. A. Benioff, J. Math. Phys. 15, 522 (1974). In this reference the proof of Eq. (21), outlined in footnote 20, is incorrect. Instead the proof outlined in footnote 22 can be used.

(20) G. Takeuti and W. Zaring, Introduction to Axiomatic Set Theory, Graduate texts in Mathematics #1, Springer-Verlag, Inc., New York, 1970 or any other text on axiomatic set theory.

(21) M. Loeve, Probability Theory, D. Van Nostrand Co., Inc. Princeton, New Jersey, 1963, 3rd edition, Secs. 30-32.

(22) J. Doob, Ann. Math. 37, 363 (1936).

(23) N. Dunford and J. Schwartz, Linear Operators, Part I Wiley Interscience Publishers, Inc., New York, 1957, Chap. III, Section 5.