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A CONFORMAL INVARIANT MODEL OF LOCALIZED SPINNING TEST PARTICLES

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ABSTRACT : A purely classical model of massless test particle with spin s is introduced as the dynamical system defined by the 10 dimensional $O(4,2)$ co-adjoint orbit with Casimir numbers $(s^2,0,0)$. The Mathisson Papapetrou et al. equations of motion in a gravitational field are recovered, and moreover the particle appears to travel on null geodesics. Several implications are discussed.

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I/ INTRODUCTION

In the conventional special relativistic treatment(x) of "zero rest mass" spinning test particles [1,2] , geometrical conformal invariance appears in a somewhat accidental way (xx) (just as for the O(4,2) unexpected symmetry of the hydrogen atom problem [3,4]). It is well known that such particles dwell in a null hyperplane (existence of a wave-front even at the classical level). Whence the related problem of localization. Furthermore one runs into severe difficulties when trying a general relativistic approach [2,5] for the massless particle would so eventually gain one supplementary degree of freedom (localization by means of gravitational scattering).

The aim of this article is an attempt to reconcile, to some extent, localization and geometrical conformal invariance(xxx). We mean to say that, if one insists on localization along a curve of space-time for any test particle (with or without spin) and if one furthermore invokes conformal invariance(xxxx) as a guiding principle, the model intro-

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(x) The phase space (or the space of motions [1]) endowed with a 6-dimensional symplectic manifold structure can be viewed either as a co-adjoint orbit of the (restricted) Poincaré group with vanishing Casimir numbers ($P_\alpha P^\alpha = W_\alpha W^\alpha = 0$ with $W^\alpha = \chi s P^\alpha$ where s stands for the spin and χ $(\chi^2 = +1)$ for the helicity) or as the normalized twistors space $(Z^\alpha \bar{Z}_\alpha = 2\chi s)$ modulo $U(1)$ [2].

(xx) The O(4,2) action on the space of motions is not equivariant to the O(4,2) conformal action on spacetime.

(xxx) Since it might appear as a fundamental symmetry of physics rather than an accidental one [6].

(xxxx) In the sense that conformal rescalings $g_{\alpha\beta} \to \Omega^2 g_{\alpha\beta}$ must be physically insignificant. Conformal invariance is here considered on the same footing as the general covariance principle on which our investigations heavily rely.

duced in the sequel appears in an unambiguous way. Whether our model (which has already been investigated, at least from one point of view by Mashhoon [7]) can account for masslessness of spinning test particles seems to be still under dispute, provided that no clearcut definition of that concept has been proposed so far.
We review in section II a well known procedure [8] which yields the Mathisson, Papapetrou et al. equations of motion of a test particle with spin in a gravitational field [9]. In order to get a deterministic set of equations of motion, one must impose supplementary conditions on the skew symmetric spin tensor $S^{\alpha\beta}$, the linear momentum P^{α} and perhaps the velocity V^{α}. Let us recall briefly the different possibilities.

As shown by Dixon [10], the condition $S^{\alpha\beta}P_{\alpha}=0$ seems to be more appropriate than the condition $S^{\alpha\beta}V_{\alpha}=0$ in the case of massive particles. A "natural" limiting procedure leads to the following conditions for massless particles, $S^{\alpha\beta}P_{\alpha}=0$ and $P_{\alpha}P^{\alpha}=0$ [5,8,11]. As a matter of fact these are the basic constraints entering the previously quoted conventional model.

On the other hand, according to our program, once assumed that some localized action functional for a spinning test particle (see (5) below) be *conformally invariant*, *extra conditions*, *namely* $S^{\alpha\beta}V_{\alpha}=0$ and $P_{\alpha}V^{\alpha}=0$, readily follow under the sole requirement that $S^{\alpha\beta}$ be singular. It is then straightforward to derive that the particle travels on null geodesics. We claim that Mashhoon's arguments in favour of the latter supplementary conditions are essentially pervaded by conformal invariance assumptions. It must be emphasized that the O(4,2) invariance can be ruled out in the flat spacetime case. Section III is thus devoted to the construction of the symplectic structure of the space of motions which is shown -in the free particle case- to be symplectomorphic to a certain 10 dimensional coadjoint orbit of the "conformal group" O(4,2). This, of course, turns out to be no longer an accidental invariance.

Note that we get rid, at the same time, of the instability near the zero curvature of those models introduced by Künzle [5] and

Saturnini [5].

In the last section we show that, according to the fairly nice properties of such a model of massless spinning test particle, the fundamental hypothesis of Friedmann's cosmologies need not undergo any modification when spin is admitted for cosmological photons.

II/ A MODEL OF MASSLESS SPINNING PARTICLE IN THE GRAVITATIONAL FIELD

For a more detailed account, see [8]. In the case of a continuous distribution of matter, the identity of conservation of the energy momentum tensor(x)

(1) $$\hat{\partial}_\kappa T^{\kappa\beta} = 0$$

is a consequence of the "general covariance" principle which states that the completely continuous functional

(2) $$\mathcal{T}(\delta g) = \frac{1}{2}\int_M T^{\alpha\beta}\,\delta g_{\alpha\beta}\ \mathrm{vol}$$

must vanish on those perturbations δg of the gravitational potentials which are Lie derivatives with respect to any differentiable vector field with compact support(xx), briefly

(3) $$\mathcal{T}(\mathcal{L}_\Xi g) = 0 \qquad \forall\ \Xi \in \mathfrak{X}_0(M)$$

As for localized test particles there exists a standard procedure which brings, using (3), the following action functional (xxx) [12]

(x) The hat "$\wedge$" denotes the covariant derivative. Vol is the Riemannian volume element of spacetime (M,g).

(xx) or what amounts to the same thing that any local smooth change of coordinates must be physically insignificant.

(xxx) Actually a 1st order distribution with support a 1-dimensional curve Λ of M, the world line of the particle. Γ is the Riemannian connection with curvature tensor R. t denotes some parameter of Λ.

(4) $$\mathcal{T}(\delta g) = \frac{1}{2}\int_{\Lambda}\{T^{\alpha\beta}\,\delta g_{\alpha\beta} + T_{\gamma}^{\alpha\beta}\,\delta\Gamma^{\gamma}_{\alpha\beta}\}\,dt$$

into the final form

(5) $$\mathcal{T}(\delta g) = \frac{1}{2}\int_{\Lambda}\{P^{\alpha}V^{\beta}\,\delta g_{\alpha\beta} + S^{\alpha\beta}V^{\gamma}\,\partial_{\alpha}\delta g_{\beta\gamma}\}\,dt$$

where V is the tangent vector to Λ with respect to t. P and S respectively interpreted as the linear momentum and the skew symmetric spin tensor satisfy in addition the following universal equations(*) (Mathisson, Papapetrou et al. equations of motion)

(6) $$\dot{P} = -\frac{1}{2}R(S).V$$

(7) $$\dot{S} = P\bar{V} - V\bar{P}$$

At this stage one may recall that, in the continuous case (2), the tracelessness of the Maxwell-Poynting tensor (for photons) or the Weyl energy momentum tensor (for neutrinos) can be expressed equivalently by the following conformal invariance(**) of the functional $\mathcal{T}$

(8) $$\mathcal{T}(\lambda g) = 0 \qquad \forall\ \lambda \in C_0(M)$$

It now seems reasonable to maintain equation (8) even in the case of concentrated distributions of matter. It is then easy to deduce from (5)

(*) For the sake of simplicity, intrinsic geometrical expressions are extensively used. The bar "—" denotes the transposition with respect to g. ($\bar{V} := g(V)$). The skew symmetric spin tensor is identified with an antihermitian linear operator. $\bar{V}R(S)W = -\mathrm{Tr}(S.R(V,W))$ for any vectors V and W. The dot "." stands for the covariant derivative with respect to V.

(**) $C_0(M)$ denotes the set of all differentiable functions of M with compact support.

and (8) that

$$S.V = \mu V \tag{9}$$

and

$$\bar{P}.V = \dot{\mu} \tag{10}$$

for some real function μ defined on Λ .

Let us furthermore assume that the spin be singular (the condition that the spin should have rank 2 is widely accepted as a central prerequisite), namely

$$\det(S) = 0 \qquad (S \neq 0) \tag{11}$$

A quick inspection on the spectrum of S shows that whenever[x] $\mathrm{Tr}(S^2) < 0$ at one point of Λ , one gets in view of (7)

$$\mu = 0 \tag{12}$$

$$\mathrm{Tr}(S^2) =: -2s^2 = \text{const.} \tag{13}$$

$$\bar{V}V = 0 \tag{14}$$

$$\dot{V} \text{ parallel to } V \tag{15}$$

The particle <u>travels on null geodesics</u> althrough its (spacelike ! [xx]) linear momentum and its spin are not parallel transported along Λ (6),(7).

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[x] If $\mathrm{Tr}(S^2) \geqslant 0$ at one point of Λ , (13), (14), (15) still hold. We discard this unphysical possibility since, in this case, S would have nothing to do with an intrinsic angular momentum.

[xx] We refer to [7] for a discussion on the way this appears to be a venial shortcoming of the theory.

There exis⁺ whatsoever, conserved quantities of the form

(16) $$\bar{P}.\Theta + \frac{1}{2}\,\mathrm{Tr}\left(S.\frac{\hat{\partial}\Theta}{\partial x}\right) = \text{const.}$$

as long as $\Theta \in \mathcal{X}(M)$ satisfies

(17) $$\mathcal{L}_{\Theta}\, g = \lambda\, g \qquad \lambda \in C(M)$$

(This can be checked using (9-12)).

To sum up the material of this section, our model coincides with Mashhoon's [7]. We just claim that the supplementary conditions

(18) $$S.V = 0$$

(19) $$\bar{P}.V = 0$$

are a mere consequence of the conformal invariance of the action functional (5) for a spinning test particle together with the degeneracy of the spin.

III/ SYMPLECTIC STRUCTURE AND CANONICAL O(4,2) INVARIANCE

Let us start with the observation that the constants of motion (16) associated to any conformal Killing vector field Θ of spacetime provide us with 15 conserved quantities in the flat Minkowski spacetime E. More precisely, the general solution $\Theta \in \mathcal{X}(E)$ of

$$\mathcal{L}_{\Theta} g = \lambda g \qquad \lambda \in C(E)$$

reads [2]

$$\Theta = \Lambda.X + \Gamma + A(\bar{X}X) - 2X(\bar{A}X) + \alpha X \qquad (X \in E) \tag{20}$$

where $\Lambda \in L(E)$ and $\Lambda + \bar{\Lambda} = 0$ (Lorentz rotation), $\Gamma \in E$ (translation), $A \in E$ (special conformal transformation), $\alpha \in \mathbb{R}$ (dilatations). We then define the dual conserved quantities (M,P,K,D) by

$$\bar{P}.\Theta + \frac{1}{2}\mathrm{Tr}\left(S\frac{\partial \Theta}{\partial X}\right) =: \frac{1}{2}\mathrm{Tr}(M\Lambda) + \bar{P}.\Gamma + \bar{K}A + D\alpha \tag{21}$$

and thus find

$$M = S + X.\bar{P} - P.\bar{X} \tag{22}$$

$$K = P(\bar{X}X) - 2X(\bar{P}X) - 2S.X \tag{23}$$

$$D = \bar{P}.X \tag{24}$$

Set(x)

$$Z = \begin{bmatrix} \Lambda & -\Gamma\sqrt{2} & A\sqrt{2} \\ -\bar{A}\sqrt{2} & -\alpha & 0 \\ \bar{\Gamma}\sqrt{2} & 0 & \alpha \end{bmatrix} \tag{25}$$

$$\mu = \begin{bmatrix} M & -\frac{K}{\sqrt{2}} & \frac{P}{\sqrt{2}} \\ -\frac{\bar{P}}{\sqrt{2}} & -D & 0 \\ \frac{\bar{K}}{\sqrt{2}} & 0 & D \end{bmatrix} \tag{26}$$

(x) with respect to a basis of $E \times \mathbb{R}^2$ whose Gram matrix is given by

$$\left[\begin{array}{c|c} \mathbb{1}_E & \\ \hline & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{array}\right]$$

It is easy to show that $Z \in \mathcal{G}_{4,2}$ (the Lie algebra of O(4,2)) and that $\mu \in \mathcal{G}^*_{4,2}$. Moreover the quantity (21) is equal to

(27) $$\frac{1}{2}\,\mathrm{Tr}\,(\mu \cdot Z) =: \mu(Z)$$

Let us then consider the 10 dimensional coadjoint orbit of O(4,2) Ω^s_{10} defined by the equation (minimal polynomial)

(28) $$\mu^4 + \mu^2 s^2 = 0 \qquad (s \in \mathbb{R})$$

A translation

(29) $$a_X : \begin{bmatrix} \mathbb{1}_E & X\sqrt{2} & 0 \\ \bar{0} & 1 & 0 \\ -\bar{X}\sqrt{2} & -\bar{X}X & 1 \end{bmatrix} \qquad (X \in E)$$

brings[(+)] μ into

(30) $$\mu_0 : \begin{bmatrix} S & 0 & \frac{P}{\sqrt{2}} \\ -\frac{\bar{P}}{\sqrt{2}} & 0 & 0 \\ \bar{0} & 0 & 0 \end{bmatrix}$$

(28) yields then the following compatibility relations

(31) $$\det\,(S) = 0 \qquad (11)$$

(32) $$\mathrm{Tr}\,(S^2) = -2s^2 \qquad (13)$$

(33) $$\bar{P} * (S)^2 P = 0 \quad ^{(++)}$$

Note that (32) and (31) are in fact interpreted as the two first Casimir numbers of this orbit, the last one being here zero as can easily be seen

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(+) $\mu_0 = a_X \cdot \mu \cdot a_X^{-1} \quad ; \quad a_X \in O(4,2)$.

(++) $*(S)^2 = S^2 - \frac{1}{2}\mathrm{Tr}(S^2)\mathbb{1}_E$ ($*(S)$ denotes the usual adjoint of S). In fact the only polarization $W := *(\mu)P = *(S)P$ turns out to be null.

on the characteristic polynomial

$$\mu^6 + \mu^4 s^2 = 0 \tag{34}$$

As a final remark, the <u>generic element</u> of Ω^s_{10} is in matrix form (30)

$$\left[\begin{array}{cc|c|c|c} 0 & -\chi s & & & \\ \chi s & 0 & & & \\ \hline & & & & \begin{matrix}1\\ \eta\end{matrix} \\ \hline & & 1 \; -\eta & & \\ \hline & & & & \end{array}\right] \tag{35}$$

where $\chi^2 = \eta^2 = +1$. Ω^s_{10} is thus the union of 4 connected components. χ and η are physically interpreted as the helicity and the sign of the energy as usual for massless particles.
We will however confine ourselves in the consideration of the connected component of the identity in O(4,2) and put $\chi = \eta = +1$. The coadjoint infinitesimal action

$$\delta\mu := [Z, \mu] \qquad Z \in \mathfrak{G}_{4,2} \; ; \; \mu \in \mathfrak{G}^*_{4,2} \tag{36}$$

yields

$$\delta M = [\Lambda, M] + (\Gamma.\bar{P} - P.\bar{\Gamma}) - (K.\bar{A} - A.\bar{K}) \tag{37}$$

$$\delta P = \Lambda.P + 2AD - \kappa P - 2M.A \tag{38}$$

$$\delta K = \Lambda.K - 2\Gamma D + \kappa K - 2M.\Gamma \tag{39}$$

$$\delta D = \bar{P}.\Gamma - \bar{K}.A \tag{40}$$

Define then(*)

$$V_{11} := \{ y = (S, P, X) \; ; \; S \in L(E) \; ; \; S + \bar{S} = 0 \; ; \; \det(S) = 0 \; ; \; \mathrm{Tr}(S^2) = -2s^2 \; ; \; P \in E \; ; \; \bar{P}.A(S)^2 P = 0 \; ; \; X \in E \} \tag{41}$$

(*) according to (31-33).

which is called the evolution space [1] or the space of initial data since each initial condition y defines a "moment" $\mu \in \mathfrak{g}^*_{4,2}$ by the application $y \longmapsto \mu$ (22-24).
The infinitesimal action of $O(4,2)$ on V_{44} can be computed bearing in mind that (20) must hold, and by (37-40)

(42) $$\delta S = \left[\Lambda + 2(A\bar{X} - X\bar{A}),\ S \right]$$

(43) $$\delta P = \left[\Lambda + 2(A\bar{X} - X\bar{A})\right]P - (\kappa - 2\bar{A}X)P - 2\,S.A$$

(44) $$\delta X = \Theta \quad (20)$$

It is well known that every non trivial coadjoint orbit Ω of a Lie group can be endowed with a symplectic structure (Ω, σ) (e.g. [1]).
In the case of a semi-simple Lie group

(45) $$\sigma(\delta\mu)(\delta'\mu) = \mu([Z, Z']) \qquad \mu \in \Omega \subset \mathfrak{g}^* ;$$

$$\delta\mu = [Z, \mu] \ ; \quad \delta'\mu = [Z', \mu] \ ; \quad Z, Z' \in \mathfrak{g} .$$

A tedious calculation shows that in the case of $\Omega^s_{10} \subset \mathfrak{g}^*_{4,2}$

(46) $$\sigma(\delta\mu)(\delta'\mu) = -\frac{1}{s^2}\mathrm{Tr}\left(\delta S . S . \delta' S\right) + \overline{\delta X}.\delta' P - \overline{\delta' X}.\delta P$$

along with (42-44). We can hence define a presymplectic structure on V_{44} which may be still denoted by σ without any ambiguity

(47) $$\sigma(\delta\mu)(\delta'\mu) =: \sigma(\delta y)(\delta' y)$$

The equationsof motion $\delta y \in \ker(\sigma)$ are given by

(48) $$\delta S = P\,\overline{\delta X} - \delta X\,\bar{P}$$

(49) $$\delta P = 0$$

(50) $$\delta X = \tau\ *(S)^2 . P \qquad (\tau \in \mathbb{R})$$

Define $\mathcal{U}_{10} :\overset{\sim}{=} V_{11}/\ker(\sigma)$. σ is an integral invariant of the distribution $y \mapsto \ker(\sigma)$ and thus

(51) $$\sigma(\delta y)(\delta' y) =: \sigma(\delta x)(\delta' x)$$

where x denotes the class of y . $\mathcal{U}_{10}$ is thus a symplectic manifold, the space of motions [1] which uniquely defines a model of "zero rest mass" spinning particle since it can be easily shown to be symplectomorphic to Ω^{s}_{10} (x).

Gravitational interaction can be taken into account. The simplest way of introducing the minimal gravitational coupling is to replace the derivatives δ in (46) by covariant derivatives $\hat{\delta}$. The requirement that σ be still closed leads in fact to the following expression

(52) $$\sigma(\delta y)(\delta' y) = -\frac{1}{s^2}\mathrm{Tr}(\hat{\delta}S.S.\hat{\delta}'S) + \overline{\delta x}\,\hat{\delta}'P - \overline{\delta' x}\,\hat{\delta}P + \frac{1}{2}\overline{\delta x}.R(S).\delta' x$$

The equations of motion read in this case

(53) $$\hat{\delta}S = P.\overline{\delta x} - \delta x.\overline{P}$$

(54) $$\hat{\delta}P = -\frac{1}{2}R(S).\delta x$$

(55) $$\delta x = \tau * (S)^2.P \qquad (\tau \in \mathbb{R})$$

in full agreement with the previous (6,7,9,10,12,14,15) results as soon as one puts $V := \frac{1}{\tau}\delta x$.

(x) It is sufficient to show that the isotropy subgroup of any particular $\mu \in \Omega^{s}_{10}$ is connected ($SO(2) \times \mathbb{R}^4$ in that case). (th.(11.38) in [1]).

IV/ CONFORMAL PHOTONS AND THE COSMOLOGICAL LORE

In the current formulation of homogeneous cosmology, the compatibility of the 3K microwave background and the model of the universe can be expressed by the existence on spacetime of an infinitesimal conformal generator[*] Θ [13]. It can be shown that the fact that matter flows along the orbits of Θ is a consequence of two basic cosmological hypotheses.

i) light rays are null geodesics

ii) the cosmological radiation is an isotropic black body radiation.

Clearly spin is neglected for photons. If spin is taken into account, (14) and (25) apply just as well. The solution to the second hypothesis challenge can be pictured as follows.

Put

(56) $$\Theta = \beta\, U$$

where $\beta = 1/kT$ (k is the Boltzmann constant). U is unit, future pointing and geodesic (matter experiences no net transfer of energy and momentum from the cosmological radiation). (17) yields

(57) $$\mathrm{grad}\,\beta = [\lambda - \overline{u}\cdot\mathrm{grad}\,\beta]\,u$$

and

(58) $$\frac{\hat{\partial} u}{\partial x} - \overline{\frac{\hat{\partial} u}{\partial x}} = 0 \qquad (U \text{ is irrotational})$$

It is then clear that (16) reduces to

(*) Θ is interpreted in the framework of statistical mechanics as the (co-)temperature vectorfield. It is assumed to be timelike and future pointing (the time arrow) [13].

(59) $$\overline{P}\,.\,\Theta = \text{const.}$$

just as if spin were absent.

In other words, if the energy of the photon in the matter frame $\mathcal{U}$ is expressed in terms of frequency ν via Planck's law, we still have

(60) $$\frac{\nu}{T} = \text{const.}$$

The spin of the photon does not modify the cosmological interpretation of the redshift, in complete accordance with Wien's law which guarantees that the cosmological radiation remains a black body radiation in spite of the change of temperature.

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