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ON THE INTERACTION BETWEEN CLASSICAL AND QUANTUM SYSTEMS*

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Abstract

We consider an unconventional approach to the measurement problem in quantum mechanics--we treat the apparatus as a classical system, belonging to the macro-world. In order to have a measurement the apparatus must interact with the quantum system. As a first step, we embed the classical apparatus into a large quantum mechanical structure, making use of a Superselection Principle. We now couple the apparatus and system such that the apparatus remains classical (Principle of Integrity), and unambiguous information of the values of a quantum observable are transferred to the variables of the apparatus. Further measurement of the classical apparatus can be done, causing no problems of principle. Thus interactions causing pointers to move (which we do not treat) can be added. We examine the restrictions placed by the principle of integrity on the form of the interaction between classical and quantum systems. We illustrate be means of a simple example in which one sees the principle of integrity at work.

Section 1. Introduction

The structure and formulation of quantum mechanical systems is by now well understood. The interaction of a number of simple quantum systems with each other has been examined repeatedly and is in remarkable quantitative agreement with experimental results. However there is one area which even now gives rise to controversy and heated argumentation, namely the problem of measurement and the interpretation of quantum mechanics.[1] The interaction between the micro-cosm and the macro-world is quite simply not fully understood. In this paper we wish to discuss some topics connected with measurement.

A measurement results from the interaction between a special system (the "apparatus") and the original system in question (the "system"). In our considerations the system would be a purely quantum mechanical system of suitably simple structure and obeying the well-known laws of quantum mechanics. In other applications, for example cosmology, the system will be a general relativistic system. The point of view we wish to advance is that a piece of apparatus can, and should, be described by the laws of classical mechanics. The apparatus one uses in a laboratory, for example, belongs to the macroscopic world. The macroscopic world is apparently so well described by classical physics, be it of a statistical or a deterministic nature, that one can say it is indeed classical.

Thus we shall choose to discuss and describe the apparatus by means of classical physics. The apparatus must be not only classical, but also of a simple enough structure that its relevant configurations can be easily parametrized and these parameters related to appropriate properties of the observed system: the process of making this correspondence is the calibration of the apparatus. Moreover, the configurations of the apparatus must be relatively stable on the one hand so that it can be read, while there should be selective minimal inertia to change of configuration so that the system variables can influence, and reflect themselves in the change of, the configuration of the apparatus. It is understood that we have at our disposal the ability for making and reading pointers, of setting the apparatus into preselected standard configurations and shielding the sensitive apparatus from stray and unwanted influences.

The crux of the measuring process, then, consists in the ability to join together the apparatus and system into a single complex system and then to separate them into two. This process of coupling and decoupling is a legal process and is only approximately carried out by such loosely defined physical processes as spatial juxtaposition and separation. Primarily it amounts to the build-up and the destruction of correlations by a deliberate act.

A true piece of apparatus is, in most cases, a very complicated mechanical system. For this reason we shall not, in the present work, attempt to produce a model of classical physics for a particular piece of apparatus. Rather, we are more interested for the present in understanding the principles involved. For this reason the examples we consider may appear, at first glance, to be of a trivial nature.

For our purposes a "measurement" is achieved if unique information of the value of a quantum "system" observable can be transferred into the classical apparatus in an unambiguous fashion to be read off later by an observer.[2]

If we are to successfully use such an apparatus to measure properties of a quantum system, it is clear that we must first understand how to engineer interactions between classical systems and quantal systems. It is the purpose of this paper to pursue further an approach to this problem presented previously by Sudarshan.[3] In this approach we embed, in a highly non-trivial way, the classical system into a quantum system with twice the number of degrees of freedom. However this quantum (-enlarged) system is richer than is allowed by measurements on the primitive classical system. The classical system <u>per se</u> is a "section" of the quantum (-enlarged)

system with only a subset of dynamical variables being perceived. There is a richness that is inherent to the system which is ignored in the classical perception. This richness corresponds to the existence of dynamical variables which cannot be observed, which remain "hidden." In this sense we may picture classical mechanics as "quantum mechanics with hidden variables."

One may ask at this point: Why are certain operators not observable? What criterion demands that these and no other shall remain unobservable? The simplest and most effective answer is to say that any choice of hidden variables corresponds to a superselection principle; in particular, when the superselection principle is maximal then we obtain classical mechanics. (The Superselection Principle[4] is discussed in the next section.) We find that classical physics describes very well a large domain of our experience and hence we give special importance to it in addition to the historical claim that it has.

The question may now be raised: Could it be that the unobservable dynamical variables were so only in practice and not in principle? Could it be that we may subsequently be able to observe the hidden variables? The answer to this is that this is always conceivable: when it happens the model changes to accommodate the new advance. As examples we may point to cases of macroscopic manifestations of quantum effects like superfluidity and the Josephson effect. As to whether all classical systems correspond only to a provisional state of our knowledge we have no answer: it is a question to be decided empirically. But we can say that at the present time it is convenient to act as if this is not so. And from among the persistently classical systems we shall choose our pieces of apparatus.

We must emphasize that we talk of superselecting operators, not superselection rules; that is to say, the superselecting operators are not required to be constants of motion. So the eigenvalues of the superselecting operators need not be preserved, and the system can "move" from one superselected sector to another in the course of time. In a conventional quantum theory where the Hamiltonian is an observable this distinction between superselecting operators and superselection rules is unnecessary. This comes about as follows: the Hamiltonian being an observable it must commute with all the superselecting operators. But this implies that all the superselecting operators are constants of the motion! It follows that in any theory where the superselecting operators are not constants of the motion the Hamiltonian cannot be an observable. In the exotic quantum theories that we will be considering the time evolution operator is not observable even though the evolution it generates is a perfectly natural one.

Classical mechanics is characterized, in contrast to quantum mechanics by the absence of superposed states--two legitimate pure states of a classical system cannot be

superposed to yield another pure state of the system. Of course this is because all of the dynamical variables in a classical system commute. And so a pure state is characterized by precise values for all of the dynamical variables. That is not to say that superposed states do not occur. Rather, it tells us that the superposition states corresponding to different values of the relative phases are physically indistinguishable. Thus, such a state is, of necessity, an incoherent state.

This point is of some importance to us, as our quantum version of a classical theory allows us to write down superposition states. For example, we can write down superpositions of different eigenvectors of the superselecting operators. In this case, the relative phases in the superposition state are unmeasurable, due to the properties of superselection, and so the states are incoherent.

The question of superselection is examined in detail in Section 2. In Section 3 we describe the procedure whereby a classical mechanical system can be embedded in a noncommutative structure. To illustrate this procedure we discuss, in Section 4, some simple examples from elementary classical mechanics, for example a freely moving particle, and the problem of a classical particle undergoing simple harmonic motion. The Heisenberg picture is most useful for describing the dynamics, as opposed to the statics, of such models, as the operator equations of motion most closely resemble the equations of motion of the original classical theory. The interesting part is the structure and evolution of the unobservable dynamical variables. However, as most people are more familiar with a Schrödinger picture time development, we discuss this in Section 5.

The most interesting facet of this development is that we can now couple together the classical and the quantum systems, in such a way that they have an effect on one another. Such a coupling could be totally arbitrary in nature--so that one could envisage that due to the coupling the classical system could lose all the nice properties it had to begin with--it may be no longer a classical system. One might say in such cases that some of the quantum characteristics are interchanged due to the interaction. However it is in those interactions between the two which allow the classical system to remain classical even after the interaction has taken place that we are interested.

Clearly such a "Principle of Integrity" of the classical apparatus system will place constraints on the coupling functions between the classical and quantum systems. The apparatus system is characterized by superselecting operators we denote by $\{\omega^\mu\}$, and the unobservable operators $\{\pi^\mu\}$ canonically conjugate to them. Let us denote the quantum system variables by $\{\xi\}$. Then it turns out, immediately, that the coupling between the systems can be at most linear in the unobservable operators π^μ, in the Hamiltonian operator. We can write the

couplings terms, then, as

$$\phi^\mu(\omega,\xi)\pi^\mu + h(\omega,\xi) \quad . \qquad (1.1)$$

By examining the equations of motion of the combined system, and requiring that the integrity of the apparatus be rigorously maintained, it is possible to derive a set of Integrity Criteria. These criteria must be satisfied by the coupling (1.1) so that the apparatus remain classical. From the form of the criteria it is clear that if the primary coupling functions depend on at most a commuting subset of the quantum variables $\{\xi\}$, then they reduce to conditions imposed on the allowed form of the undisturbed quantum system Hamiltonian $X(\xi)$ and the secondary coupling function $h(\omega,\xi)$. However, the criteria are <u>not</u> obviously violated if the primary coupling functions depend on a non-commuting subset of the quantum variables. We examine this case in detail and find that the criteria can be written in such a form that they can easily be checked case by case. While this is not an altogether satisfactory result, nevertheless it allows the possibility that the measurement criteria alone are not sufficient to guarantee that a measurement can actually be made. One must then consider the question of how uniquely the information stored in the classical variables is related to an observable of the quantum system.

To illustrate the use of the criteria, and to see that in some cases they are a powerful tool, in Section 7 we turn our attention to a simple example.[5] The Quantum System we have in mind is an inert quantum spin system. This quantum system is carried along as internal degrees of freedom by a particle whose translatory motion obeys the laws of classical mechanics. As such the two systems are noninteracting. interaction between the apparatus, which here is just the classical translatory motion of the particle, and the quantum spin system, whcih displays itself in the form of a magnetic moment, is brought about by causing the particle to move into an inhomogeneous external magnetic field. We consider an arbitrary magnetic field

$$\underset{\sim}{B}(\underset{\sim}{q}) = (B_1(\underset{\sim}{q}), B_2(\underset{\sim}{q}), B_3(\underset{\sim}{q})) \tag{1.2}$$

and apply the Integrity Criteria to the coupling functions. The coupling functions which we choose for (1.1) in this case are those necessary and sufficient to reproduce the correct equations of motion for a particle with a magnetic moment passing through an inhomogeneous magnetic field. In this case the Integrity Criteria are quite stringent. We derive the result that, if it is a regular function, the magnetic field must take the form

$$B(q)\underset{\sim}{n} \tag{1.3}$$

where $\underset{\sim}{n}$ is a constant unit vector and B(q) is an arbitrary regular function of q_1, q_2 and q_3.

Finally, in Section 9, we consider the role played by the secondary coupling function, $h(\omega,\xi)$, which appear in (1.1). To begin with these are needed in the case that ϕ^μ depends on non-commuting quantum variables, to ensure that the Integrity Criteria are satisfied. In the example considered in Section 7, $h(\omega,\xi)$ has a dynamical role to play, as the energy of the particle in a magnetic field. Another point to note is that their presence is required to preserve the form-invariance of the total Hamiltonian of the combined Apparatus and System, under "gauge" transformations $U(\omega,\eta)$, with $\{\eta\}$ some subset of the quantum System dynamical variables. The interesting question which we pose is--can one make a "gauge" transformation $U(\omega,\eta)$ which will transform away this secondary coupling function.

Section 2. Superselection and non-observables

An important ingredient in our model is the usage of superselection. However, we do not follow the conventional usage. Let us first briefly review the conventional scheme.[4]

Within the context of a Hilbert space of states (to which all physical states should belong), there is defined the concept of a selection rule. This has to do with the dynamics of an isolated quantum system. Given two distinct subspaces of the total Hilbert space, a selection rule operates between them if the vectors of one remain orthogonal to the vectors of the other in time--no spontaneous transitions can occur if the system is left to evolve on its own.

The concept of a Superselection rule is a generalization of this idea--a superselection rule operates if a selection rule operates and further, no observable can connect states belonging to the different subspaces.

In many cases the selection rule can be understood to arise as a result of a symmetry of the dynamics of the system. The dynamical symmetry, if it arises from a continuous group of transformations, gives rise to invariant charge operators. It is between the eigenspaces of the charge operators that the selection rule operates. Similarly, the superselection rule can be understood to arise from a symmetry, not restricted just to the dynamics however. There is an associated operator --the superselecting operator--and it is between its eigenspaces that the superselection rule operates.

The existence of a superselection rule has certain consequences. The selection rule means that a state which belongs to a particular eigenspace h_i of the operators initially, will always belong to h_i. The superselection aspect has the following consequences: in a superposition over distinct subspaces of the superselecting operators, the relative phases are not measurable. No observable can connect the subspaces, and so the relative phases cannot be measured by any observable. Such a state we call "an effective mixture state." The subspaces h_i of the Hilbert space are then said to be superselected. Furthermore, those operators which fail to commute with the superselecting operators are not observable.

We can state the essential properties of a superselecting operator A, as follows

(i) A commutes with the time evolution operator.

(ii) A commutes with all the observables.

(iii) A is an observable.

We notice that in conventional quantum systems the time evolution operator is an observable. In such theories (i) is a special case of (ii). In our application of the superselection concept we have been led to look at non-conventional theories. We actually gain in generality by dropping condition (i) in its explicit form.

Following the conventional usage, the existence of superselecting operators gives a sufficient, but not necessary, reason for dynamical operators to be observable or non-observable.

In other words the set of all observables is included within the commutant of the set of superselecting operators. We shall use a different approach. We alter (ii) to the status of an 'if and only if' condition, and we furthermore drop (iii). For the application we will discuss in this paper these two changes are not relevant--they merely illustrate our philosophy. With these changes we can view the Superselection principle as a Principle of Impotence--for, an operator will be unobservable if and only if it fails to commute with some superselecting operator.

As a result of the first change [dropping condition (i)] the consequences of superselection are altered. Clearly since we no longer insist on the selection rule aspect, the Hilbert space of states is no longer broken up into separate subspaces. We first notice that the superselecting operator need not be a constant of the motion. Then in the evolution of time states can spontaneously move [continuously or discontinuously] from one superselected subspace to another. Also, since the superselecting operator is not a constant of the motion, the Hamiltonian operator must be unobservable. Evidently, it will only be in a highly non-conventional theory that such a concept will have application.

An interesting result of this alteration is that the relative phases in a superposition over distinct superselected subspaces, while not observable may well be measureable. This apparently paradoxical situation can occur only because the

time evolution operator is not an observable. A given such superposition may evolve at a later time into a state belonging to just one of the superselected subspaces. More generally, it can happen that two effective mixture states on which the measurement of all observables gives identical results may develop in the course of time into two distinct states in which the measurements of observables give different results. In other words, different superpositions would lead to different states which could be distinguished. However, this difference can only be detected after the state has been altered. The information contained is potentially observable, though not observable at the instant. This is reminiscent of classical optics where only photometric measurements are considered.[6] Two fields of illumination with the same intensity distributions but different spatial coherence cannot be distinguished by photometric measurements. But on propagation the two develop into differing fields of illumination on a later surface. Coherence pertains, in this context, to unobservable, but potentially observable, illumination.

Section 3. Classical Systems as Quantum Systems with *Superselection*

At first glance it may seem that the structures of quantum and classical mechanics are very different. Indeed, in quantum mechanics not all of the dynamical variables commute, whereas in classical mechanics they form a commuting set. Another difference, related to the first, is that one can superpose coherently two pure states to form another pure state in quantum mechanics, but this cannot be done in classical mechanics. Here a pure state is characterized by exact values for all the dynamical variables, and so any combination of such states must fail to satisfy the criterion for purity. In the usual formulation of classical mechanics it is not possible to talk about superposed states.

However, it is precisely these differences which show us how to embed a classical system in a quantum system. As we have seen in Section 2, there are situations in the formulation of quantum mechanics where coherent linear combinations of pure states are not of their own right pure states. This occurs when a theory possesses superselection rules. The superselecting operators decompose the Hilbert space of states such that no observable operator can map states in one eigenspace into a different eigenspace. Furthermore, in a superposition the relative phases between states belonging to different eigenspaces are *not measurable*. As explained previously, it is possible to view the superselection as a statement about unobservable operators.

Let us denote the algebra of dynamical variables of an arbitrary system by A. This algebra is in general non-commutative. We denote the superselecting operators by $\hat{Z}$. Let O be the commutant of the set $\hat{Z}$--the set of elements of A which commute with each element of $\hat{Z}$. Let U be the set of the remaining elements of A. Then, for each $u \in U$ there exists at least one $z \in \hat{Z}$ such that $[u,z] \neq 0$. U is the set of non-observable operators while O is the algebra of observable operators.

In general, O also is a non-commutative algebra. However, $O \supset Z = O \cap \hat{Z}$, which is an abelian algebra. Thus one can enlarge $\hat{Z}$ by adjoining elements of O not already belonging. Consider an element o_1 which we adjoin to $\hat{Z}$. Then all those elements of O which fail to commute with o_1 must now become part of U. Thus we have new sets O', U', $\hat{Z}'$ such that

$$Z \subset Z' \subset O' \subset O , \quad U' \supset U \quad \text{and} \quad \hat{Z}' \supset \hat{Z} \quad (3.1)$$

The important point to note is that we can enlarge $\hat{Z}$ to the point where $O = Z$, so that the algebra of observables is commutative.

We now have a suggested quantum framework in which the two most different properties of a classical system can be realized. The observable sector of a particular non-commutative algebra will be a truly classical theory. In the remainder of this section we will illustrate how this is to be achieved.

Consider first a simple classical mechanics system. In the Hamiltonian formulation, a classical system with n degrees of freedom is characterized by n co-ordinates $(q_1,\ldots,q_n)$ and n conjugate momenta $(p_1,\ldots,p_n)$. The pure states of the system are given by specifying a point in phase space, i.e., we specify a value for each of the n pairs of canonical co-ordinates $(q_1,p_1)\ldots(q_n,p_n)$. This defines the statics. To discuss the dynamics of the system one needs to write down the Hamiltonian function, $H(q,p)$, of the system. Then, Hamilton's form of the equations of motion is

$$\dot{q}_i = \frac{\partial}{\partial p_i} H(q,p) \ ; \qquad \dot{p}_i = -\frac{\partial}{\partial q_i} H(q,p) \ . \tag{3.2}$$

By using the Poisson bracket

$$\{X,Y\} \equiv \sum_i \left[\frac{\partial X}{\partial q_i}\frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial q_i}\frac{\partial X}{\partial p_i}\right] \tag{3.3}$$

these equations can be rewritten as

$$\dot{q}_i = \{q_i,H\} \ , \qquad \dot{p}_i = \{p_i,H\} \ , \tag{3.4}$$

where the apparent asymmetry, due to the minus sign in (3.2), has vanished.

We are now in a position to write down the quantum system in which we find this classical system embedded. We shall

follow a constructive method to illustrate the choice. Let

$$\omega \equiv \{\omega^1,\ldots,\omega^{2n}\} = \{q_1,\ldots,q_n,p_1,\ldots,p_n\} \ . \tag{3.5}$$

These were the dynamical variables of the classical system. We now view them as a set of commuting operators, acting on a Hilbert space of vectors. We introduce operators

$$\pi \equiv \{\pi^1,\ldots,\pi^{2n}\} \tag{3.6}$$

which are conjugate to the ω-operators, with respect to commutation. That is to say,

$$[\omega^\mu,\pi^\nu] \equiv \omega^\mu\pi^\nu - \pi^\nu\omega^\mu = +i\hbar\delta^{\mu\nu} \tag{3.7}$$

and we will henceforth use natural units with $\hbar \equiv 1$. This sense of conjugacy is not to be confused with the conjugacy concept in the Hamiltonian formulation of classical mechanics. There, conjugacy was with respect to the Poisson bracket, so that

$$\{q_i,p_j\} = \delta_{ij} \ .$$

Our conjugacy, on the other hand is defined with respect to commutation. A representation of the π-operators which is an aid to visualizing this method, is

$$\pi^\mu = -i\frac{\partial}{\partial\omega^\mu} \quad . \tag{3.8}$$

So we now have an algebra of dynamical variables which consists of 2n pairs of canonically conjugate operators. In the spirit of our earlier discussion we introduce a set of Superselecting operators. We define the set Z to be $\{\omega^1, \ldots, \omega^{2n}\}$. This is equivalent to the prescription that $\{\pi\}$ be unobservable operators. Thus A is the algebra generated by $\{\omega\} \cup \{\pi\}$, O is generated by $\{\omega\}$, and $U = A \backslash O$, i.e., the conjugate "momenta" π^μ are unobservable operators. The operators ω^μ are the observables. Since the algebra of observables includes all functions of the observables, it is clear that an operator belonging to A is observable if and only if it is independent of the unobservable operators π^μ.

The operators of A act on a Hilbert space of vectors, h. We shall concentrate on the observable operators ω^μ, as they are of primary importance to us. These operators possess a continuous spectrum. Following conventional treatments we will consider the space h to be extended so that it contains also the unnormalizable eigenvectors for ω^μ. The space of states h will thus no longer be a Hilbert space. If we denote such eigenvectors by $|\omega_o\rangle$ where

$$\omega^\mu|\omega_o\rangle = \omega_o^\mu|\omega_o\rangle \tag{3.9}$$

then the ω-representative of this state is

$$\langle\omega|\omega_o\rangle \equiv \prod_\mu \delta(\omega^\mu-\omega_o^\mu) = \delta(\omega^1-\omega_o^1)\delta(\omega^2-\omega_o^2)\ldots\delta(\omega^{2n}-\omega_o^{2n}) \ . \tag{3.10}$$

These are the wave functions for an eigenstate of ω. Now, since the different eigenspaces of the ω^μ are superselected, we see that to the extent direct observation is concerned, h actually decomposes into a product of one-dimensional state spaces corresponding to precise values for all of the observables ω^μ.

The dynamics of the quantum system must also be specified. We note, in this regard, the following result. If we define

$$H_{op} = i\sum_j \left\{ \frac{\partial H(q,p)}{\partial q_j} \frac{\partial}{\partial p_j} - \frac{\partial H(q,p)}{\partial p_j} \frac{\partial}{\partial q_j} \right\} \tag{3.11}$$

then the equations of motion (3.4) appear as

$$\dot{q}_j = -i[q_j, H_{op}] \ ; \qquad \dot{p}_j = -i[p_j, H_{op}] \ . \tag{3.12}$$

These equations are very suggestive of the Heisenberg picture equations of motion. For this reason we will discuss, in this section, the time development of the quantum system in the Heisenberg picture, and treat the Schrödinger picture viewpoint later.

The time development of the operators of the quantum theory is defined in the usual manner, $A(o) \equiv A$

and
$$A(t) = e^{iH_{op}t} A(o) e^{-iH_{op}t} . \qquad (3.13)$$

The equation of motion for A(t) is

$$\dot{A}(t) = -i[A(t),H_{op}] .$$

Here H_{op} is the Hamiltonian operator of the model. We define

$$H_{op} = \frac{\partial H(\omega)}{\partial \omega^{\nu}} \varepsilon^{\mu\nu} \pi^{\mu} \qquad (3.14)$$

with $\varepsilon^{\mu\nu} \equiv \{\omega^{\mu},\omega^{\nu}\}$ and $H(\omega)$ the Hamiltonian function of the classical theory. Then the equations of motion for $\omega^{\mu}(t)$ will be just equations (3.12). We will also have a set of equations of motion for the unobservable operators π^{μ}.

The quantum theory we have constructed is highly non-conventional in character, so some qualifying comments are necessary. First, we should note that the Hamiltonian operator (3.14) is not an observable. This might appear strange at first sight, as we normally associate the Hamiltonian with the energy operator. We must remember that here the energy function of the "physical" sector is $H(\omega)$ and <u>not</u> (3.14). Secondly, although H_{op} is functionally dependent on the π^{μ}, it is only linear in π^{μ}. Thus, as we can see from the equations of motion for $\omega^{\mu}(t)$, these operators will remain in the observable

sector of the algebra of dynamical variables, for all times. Thus, for all t the values of all the observables of the theory can be precisely specified.

As we have noted, the state space upon which our observables act is a product of superselected one-dimensional subspaces. Thus, in any superposition over states the relative phases are not measurable. However, since the Hamiltonian operator is not observable, the selection rule aspect no longer applies, as explained in Section 2. Thus if a system is described at time t, by a state belonging to one superselected subspace, at a later time t_2 the system will be represented by a state belonging to a different subspace. We will illustrate this point in particular examples in the next section.

Section 4. Some Simple Examples

In the last section we introduced a general method for embedding a classical mechanical system in a quantum mechanical system. It is instructive at this stage to examine some simple classical mechanical systems in the light of this embedding procedure. We shall consider very elementary problems, namely that of a classical freely moving particle, that of a particle moving in a conservative field of force, and in particular a simple harmonic oscillator.

Consider first the case of the freely moving classical particle. The conjugate dynamical variables are chosen to be $\underset{\sim}{q} = (q_1, q_2, q_3)$ and $\underset{\sim}{p} = (p_1, p_2, p_3)$, and the dynamics are given by the Hamiltonian energy function

$$H(q,p) = \frac{\underset{\sim}{p}^2}{2m} . \tag{4.1}$$

In the quantum system the conjugate dynamical variables are

$$\omega = (\underset{\sim}{q}, \underset{\sim}{p}) \quad \text{and} \quad \pi = (\underset{\sim}{\pi}^q, \underset{\sim}{\pi}^p) . \tag{4.2}$$

Following the prescription (3.14) for the time development operator, we find

$$H_{op} = +\frac{1}{m} \underset{\sim}{p} \cdot \underset{\sim}{\pi}^q . \tag{4.3}$$

Notice that this operator is linear in the conjugate momenta π. The Superselection Principle specifies that the ω^μ are superselecting operators whereas the π^μ are unobservable operators. The equations of motion for the dynamical variables in the Heisenberg picture are

$$\begin{aligned} \dot{p}_j(t) &= 0 , \qquad \dot{q}_j(t) = \frac{1}{m} p_j(t) \\ \dot{\pi}^q_j(t) &= 0 , \qquad \dot{\pi}^p_j(t) = -\frac{1}{m} \pi^q_j(t) \end{aligned} \tag{4.4}$$

with corresponding solutions, in terms of the operators at an initial time

$$\begin{aligned} p_j(t) &= p_j(o) \qquad q_j(t) = q_j(o) + \frac{1}{m} p_j(o)t \\ \pi^q_j(t) &= \pi^q_j(o) \qquad \pi^p_j(t) = \pi^p_j(o) - \frac{1}{m} \pi^q_j(o)t . \end{aligned} \tag{4.5}$$

The first pair of equations mimic for the observable operators the classical solutions. The equations and solutions for the unobservables cannot be physically checked, but they are necessary to ensure the internal consistency of the quantum theory.

The solutions (4.5) are operator solutions. Thus, we must specify the state of the system. Classically the state chosen is a pure state--p_j and q_j have exact values initially. Correspondingly in this case we can choose the state representing the system to be an eigenstate of p_j and q_j--a complete commuting set of operators. For example, we could choose

$$|\Psi\rangle = |q_o,p_o\rangle \qquad (4.6)$$

where $$p_j|q_o,p_o\rangle = p_{j_o}|q_o,p_o\rangle$$
$$q_j|q_o,p_o\rangle = q_{j_o}|q_o,p_o\rangle \quad .$$

Then at a later time t the values of these operators are found by taking the expectation values, viz.

$$\langle p_j(t)\rangle_\Psi = p_{j_o} \ , \qquad \langle q_j(t)\rangle_\Psi = q_{j_o} + \frac{1}{m} p_{j_o} t \qquad (4.7)$$

So if initially the particle has co-ordinates p_{j_o}, q_{j_o}, then at time t it has co-ordinates p_{j_o}, $q_{j_o} + \frac{1}{m} p_{j_o} t$--just exactly the classical solution.

If we had worked in the Schrödinger picture the state of the system would be seen to develop from the state $|\Psi,o\rangle \equiv |\Psi\rangle$, defined by (4.6), to the state $|\Psi,t\rangle$, on which q_j and p_j would yield the values (4.7). Clearly $|\Psi,o\rangle$ and $|\Psi,t\rangle$ belong to different superselected sectors of the space of state vectors.

For the classical particle moving in a conservative force field the Hamiltonian energy function is

$$H(q,p) = \frac{p^2}{2m} + V(q) \qquad (4.8)$$

where $$\underset{\sim}{F} = -\frac{\partial}{\partial \underset{\sim}{q}} V(q) \ .$$

Then the corresponding Hamiltonian operator is

$$H_{op} = +\frac{1}{m}\underset{\sim}{p}\cdot\underset{\sim}{\pi}^q - \frac{\partial V(q)}{\partial \underset{\sim}{q}}\cdot\underset{\sim}{\pi}^p \quad . \qquad (4.9)$$

The corresponding Heisenberg equations of motion are

$$\dot{p}_j(t) = -\frac{\partial}{\partial q_j} V(q) = F_j \ , \qquad \dot{q}_j(t) = \frac{1}{m} q_j(t)$$
$$\dot{\pi}^q_j(t) = \frac{\partial^2 V(q)}{\partial q_j \partial q_i} \pi^p_i(t) \ , \qquad \dot{\pi}^p_j(t) = -\frac{1}{m} \pi^q_j(t) \ , \qquad (4.10)$$

again giving us the classical equations of motion for the observables.

For the simple harmonic oscillator the potential energy function is $V(q) = \frac{1}{2}m\omega^2 q^2$, and in this case the solutions to the equations are

$$q_j(t) = q_j(o)\text{Cos}\,\omega t + \frac{1}{m\omega} p_j(o)\,\text{Sin}\,\omega t$$
$$p_j(t) = p_j(o)\text{Cos}\,\omega t - m\omega q_j(o)\,\text{Sin}$$
$$\pi^p_j(t) = \pi^p_j(o)\text{Cos}\,\omega t - \frac{1}{m\omega}\pi^q_j(o)\,\text{Sin}\,\omega t \qquad (4.11)$$
$$\pi^q_j(t) = \pi^q_j(o)\text{Cos}\,\omega t + m\omega\pi^p_j(o)\,\text{Sin}\,\omega t$$

Applying these operator solutions to the state vector, for example that given by (4.6), gives us back the correct classical solutions for the observable sector of the theory.

Thus, we have explicitly demonstrated some, admittedly simple, examples of quantum theories, of which the observable sector is exactly a well-known classical system. The Superselection principle ensures that superposed states are not allowed as pure states of the system.

Section 5. The Schrödinger Picture

In the previous sections we have discussed a method whereby a classical mechanical system, whose dynamics can be written in the Hamiltonian formulation, could be embedded as the observable sector of a quantum mechanical theory. Because the dynamics of a quantum theory in the Heisenberg picture more closely resemble those of the classical Hamiltonian formulation, we chose to work in that picture. One is more used, on the other hand, to think in terms of the Schrodinger picture when discussing quantum mechanics problems. For this reason we will now include a discussion of the Schrödinger picture time development.

In discussing the Schrodinger picture, one can either choose to work in a "wave mechanics" formulation, or in terms of the vector space methods. The wave mechanics treats the q-, p-representation of the state of a system. Then the operators conjugate to q and p can be represented by

$$\pi_j^q = -i\frac{\partial}{\partial q_j}\,, \qquad \pi_j^p = -i\frac{\partial}{\partial p_j}\,. \tag{5.1}$$

The system is described by a wave function which we denote in the following ways

$$\psi(\omega) = \psi(q,p) \equiv \psi(q,p,o) \tag{5.2}$$

With the Hamiltonian operator given by equation (3.14)

$$H_{op} = i\left[\frac{\partial H(q,p)}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial H(q,p)}{\partial p_j}\frac{\partial}{\partial q_j}\right]$$

the Schrödinger time development of the wave function is

$$\psi(q,p,o) \rightarrow \psi(q,p,t) = e^{-iH_{op}t}\psi(q,p,o) \ .$$

The Schrödinger equation is, then,

$$i\frac{d}{dt}\psi(q,p,t) = H\psi(q,p,t) \quad . \qquad (5.3)$$

In quantum mechanics it is usual to expand $\psi(q,p,t)$ in terms of the eigenstates of H_{op}, thus simplifying the problem. In this case, H_{op} is not an observable, so we do not follow this procedure. Nevertheless the solution of the equation follows quite simply. Let us define

$$b_i(q,p) = -\frac{\partial H(q,p)}{\partial p_i} \ , \qquad d_i(q,p) = \frac{\partial H(q,p)}{\partial q_i} \ . \quad (5.4)$$

The the equation takes the following form

$$\left[\frac{\partial}{\partial t} - b_i(q,p)\frac{\partial}{\partial q_i} - d_i(q,p)\frac{\partial}{\partial p_i}\right]\psi(q,p,t) = 0 \qquad (5.5)$$

a form reminiscent of the Renormalization group equations in quantum field theory.[7] To solve this equation we introduce

the functions $\bar{q}(q,p,t)$ and $\bar{p}(q,p,t)$, where

$$\frac{\partial}{\partial t}\bar{q}_j(q,p,t) = b_j(\bar{q},\bar{p}) \qquad \bar{q}_j(q,p,o) = q_j$$
$$\frac{\partial}{\partial t}\bar{p}_j(q,p,t) = d_j(\bar{q},\bar{p}) \qquad \bar{p}_j(q,p,o) = p_j \ . \qquad (5.6)$$

Then the solution to equation (5.5) is

$$\psi(q,p,t) \equiv \psi(\bar{q}(q,p,t),\bar{p}(q,p,t)), \qquad (5.7)$$

i.e., the Schrödinger equation puts no restrictions on the allowed form of the wave function. Given at time $t = 0$, the wave function $\psi(q,p)$, then at time t the wave function of the system is $\psi(\bar{q},\bar{p})$. The equations determining $\bar{q}$ and $\bar{p}$ are, using (5.4) and (5.6) very similar to the classical equations of motion, apart from a minus sign in each of the equations. To understand the reason for this result, and to supply a correct interpretation for the wave function (5.7) we now examine the representation-free problem.

We consider the time development of an eigenstate of position and momentum, i.e., we choose an initial state

$$|\psi,o\rangle = |q_o,p_o\rangle \ . \qquad (5.8)$$

We shall not, in this paper, solve its time development directly. Instead, we shall infer its form from the Heisenberg picture solutions. As a result we shall be encumbered with a phase factor which is undetermined by the Heisenberg picture. In a more thorough treatment of the Schrödinger picture, such a phase factor is well defined.[5] At time t the state vector is $|\Psi,t\rangle = e^{-iH_{op}t}|\Psi,o\rangle$. We know certain information about this state--for example, it is an eigenstate of the operators $e^{-iH_{op}t}qe^{iH_{op}t}$ and $e^{-iH_{op}t}pe^{iH_{op}t}$ with eigenvalues q_o and p_o respectively. Since these operators form a complete commuting set of operators, this knowledge specifies the state up to an overall phase factor. However, these two operators are just $\bar{q}(t)$ and $\bar{p}(t)$. Comparing directly with the Heisenberg picture formulation we see that $\bar{q}(t) = q(-t)$. We denote the eigenvectors of $\bar{q}(t)$ and $\bar{p}(t)$ with eigenvalues q_o and p_o by $\overline{|q_o, p_o; t\rangle}$. Then the final state is

$$|\Psi,t\rangle = e^{i\delta}\overline{|q_o, p_o; t\rangle} \quad . \qquad (5.9)$$

We are now in a position to say what the eigenvalues of q and p are on this state. We note the similarity, in this regard, between the following cases we have met

$$\begin{array}{cccc} \begin{pmatrix} q \\ p \end{pmatrix} \xrightarrow{+t} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} & & \begin{pmatrix} \bar{q}(t) \\ \bar{p}(t) \end{pmatrix} \xrightarrow{+t} \begin{pmatrix} q \\ p \end{pmatrix} & \\ |\Psi,o\rangle & & |\Psi,t\rangle & \qquad (5.10) \\ q_o, p_o & & q_o, p_o & \end{array}$$

Clearly then, the eigenvalues of q and p on $|\Psi,t\rangle$ are the same as those of q(t) and p(t) on $|\Psi,o\rangle$. Thus, when we act on the final state with the operators q and p, we get the expected classical results.

The relationship with the wave mechanics approach arises as follows. Corresponding to a state (5.8), the representative in the ω-representation is

$$\langle q,p|\Psi,o\rangle \equiv \delta(q-q_o)\delta(p-p_o) \quad . \qquad (5.11)$$

On the other hand, the wave function, not an eigenstate of q or p, is the representative in the ω-representation of a particular superposed state. Consider the superposed state

$$|\Psi\rangle = \int dq_o dp_o f(q_o,p_o)|q_o,p_o\rangle \qquad (5.12)$$

where the integral runs over the spectrum of $\underset{\sim}{q}$ and $\underset{\sim}{p}$, and f(q,p) is a complex valued function. Then its representative is

$$\begin{aligned} \langle q,p|\Psi\rangle &\equiv \psi(q,p) \\ &= \int dq_o dp_o f(q_o,p_o)\langle q,p|q_o,p_o\rangle \\ &= f(q,p) \qquad (5.13) \end{aligned}$$

This is the wave function of the wave mechanics. We see from (5.12) that $|f(q_o,p_o)|^2 = |\psi(q_o,p_o)|^2$ is the probability of finding the system in the state $|q_o,p_o\rangle$, i.e., at the point (q_o,p_o) of ω-space, at time t = 0.

The final state is

$$|\Psi,t\rangle = \int dq_o dp_o f(q_o,p_o) e^{i\delta_o} \overline{|q_o,p_o;t\rangle} \quad . \quad (5.14)$$

The ω-representative of the "barred" state is

$$\langle q,p\overline{|q_o,p_o;t\rangle} \equiv \delta(\overline{q}(t)-q_o)\delta(\overline{p}(t)-p_o) . \qquad (5.15)$$

Then, at time t, the wave function of the system is

$$\psi(q,p,t) = \int dq_o dp_o f(q_o,p_o) e^{i\delta_o} \langle q,p\overline{|q_o,p_o;t\rangle}$$

$$= f(\overline{q}(t),\overline{p}(t) e^{i\overline{\delta}} \qquad (5.16)$$

If we choose the phase $\overline{\delta}$ = o (mod 2π), we see that this agrees exactly with the solution we found for the Schrödinger equation. From the expansion (5.14) we see that $|f(q_o,p_o)|^2$ is the probability of finding the system in the state $\overline{|q_o,p_o:t\rangle}$ at time t, and so $|f(\overline{q}(t),\overline{p}(t)|^2$ is the probability of finding the system in the state $|q,p\rangle$ at time t, i.e., at the point *(q,p) of ω space at time t.*

We have now given an interpretation of the solution of

the Schrödinger equation. Clearly for classical applications, the vector space methods are more appropriate. We have not yet considered the question of interacting systems. In the next sections we will consider in detail a particular type of interaction, namely interactions with a truly *quantal* system, with a view to understanding the interaction as a measurement.

Section 6. Application to Measurement

In this section we will discuss one possible application of the model presented in Section 3. This application is in the realm of measurement. Here one has two separate systems one of which is the apparatus, or measurer, and the other is the system to be examined. It is our premise that a piece of apparatus can be described by classical Physics. At any time the most one wishes to use is a commuting set of apparatus operations. We propose, then, to treat an apparatus as a truly classical system. As we are more concerned with the principles involved we will restrict our attention to mechanical systems of the type discussed in section 3. At this stage we will consider a measurement achieved if unambiguous information concerning certain variables of the system being measured can be stored in the classical apparatus variables, to be read later by an observer.

To apply these ideas to measurement in quantum theory we must be able to couple together classical and quantal systems. It is here that the model we have developed becomes useful. In this section we are going to examine some questions related to the coupling procedure, in particular how the classical nature of the apparatus is altered by the coupling.

Let us denote the algebra of dynamical variables of a quantum system by $\{\xi\}$, where we write explicitly only the

algebraically independent variables. Then the system is specified when the energy operator, the state vector and the commutation rules of the algebra are known. We write the Hamiltonian of the isolated quantum system as $X(\eta)$, where $\{\eta\}$ is some subset of the quantum variables. The Hamiltonian operator for the uncoupled systems is, then,

$$H_{op} = \frac{\partial H(\omega)}{\partial \omega^{\mu}} \varepsilon^{\nu\mu} \pi^{\nu} + X(\eta) \ . \qquad (6.1)$$

In discussing measurement by including the apparatus within the overall dynamical framework, one wishes to have a Hamiltonian function, or time evolution operator, which takes care of the act of measurement. To generate a coupling between the apparatus and system, we add to (6.1) the interaction term

$$\Phi(\omega,\pi;\xi') \ , \qquad (6.2)$$

where $\{\xi'\}$ is a subset of the quantum variables. The effect of this interaction term will be felt by both apparatus and quantum system, and in the case of the apparatus it may even be sufficient to destroy its classical nature. To be able to treat this in a more quantitative manner we should understand the classical nature of the apparatus system. By examining the model of section 3, we see that its classical nature could be characterized by the following properties.

1. $\omega^\mu(t)$ are observable for all times t.
2. $\omega^\mu(t)$ and $\omega^\nu(t')$ are compatible operators for all times t, t'.

In the classical model it is trivial to show that these properties are true. *Likewise, it follows directly from*

$$[\omega^\mu(t),\omega^\nu(t')] = e^{iH_{op}t}[\omega^\mu(t-t'),\omega^\nu(o)]e^{-iH_{op}t'}$$

in the coupled system that they are equivalent conditions. The first condition tells us that the trajectories of the apparatus observables are observable for all times. The second tells us that we can measure the different trajectories without disturbing the measurable aspects of the system.

We now turn to the general Hamiltonian, the sum of (6.1) and (6.2) to see whether or not in the time development of the system these equivalent properties of the apparatus are retained. *To begin with, we note that these properties are* automatically satisfied in the uncoupled classical model because the Hamiltonian is at most linear in the unobservable operators π^μ. Clearly, then, if Φ is quadratic or higher in π^μ, the apparatus system will no longer be characterized by properties 1 and 2 after the interaction has taken place. Since we are interested in describing measurement, in which case the apparatus must at least remain classical both during and after the interaction between the systems occurs, we can restrict our attention to interaction terms of the form

$$\phi^\mu(\omega,\eta')\pi^\mu + h(\omega,\hat{\varepsilon}) \qquad (6.3)$$

where $\{\eta'\}$ and $\{\hat{\varepsilon}\}$ are two subsets of the quantum variables. However *this alone is not sufficient to guarantee that the apparatus* retains its classical integrity. Both the primary coupling functions ϕ^μ and the secondary coupling function h depend on subsets of the quantum variables. What we need to do is to find what further restrictions, if any, on the functional form of these coupling functions can be deduced by requiring the apparatus to remain classical. Or, if that fails, can we derive criteria which can be used to check different models.

As they stand Properties 1 and 2 are not in a very usable form. For our purposes the following condition is easier to use:

3. $\left[\frac{d^m}{dt^m}\,\omega^\mu(t),\ \frac{d^n}{dt^n}\,\omega^\nu(t)\right] = 0$ for all $m,n \geq 0$ and all times t.

It is clear that this condition can be derived from condition 2. Therefore it is a necessary condition for the apparatus to remain classical. The question as to whether it is also sufficient can be answered as follows. If $\omega^\mu(t)$ is everywhere regular, then it follows from 3 that $\left[\omega^\mu(t'),\frac{d^n}{dt^n}\,\omega^\nu(t)\right] = 0$ for $n \geq 0$ and all t,t'. The n = 0 case gives us condition 2. In these circumstances condition 3 is equivalent to 1 and 2.

However, it may be that $\{\omega^\mu(t)\}$ are not all regular functions, so that they do not have power series expansions valid everywhere. Then it is clear that 3 are necessary, but not sufficient, conditions that the apparatus remain classical.

We can abstract from 3 the following equations:

$$(\alpha)\ \left[\frac{d^m}{dt^m}\,\omega^\mu(t), \frac{d^n}{dt^n}\,\omega^\nu(t)\right] = 0 \qquad \text{for } m,n > 0$$

$$(\beta)\ \left[\omega^\mu(t), \frac{d^n}{dt^n}\,\omega^\nu(t)\right] = 0 \qquad \text{for } n > 0$$

$$(\gamma)\ [\omega^\mu(t), \omega^\nu(t)] = 0$$

Equation (γ) is automatically satisfied as the unitary time development preserves commutation relations. From (α) we see that either all of the time derivatives vanish, or they belong to a commutative algebra. The first alternative cannot even happen in the uncoupled system. Then (β) tells us that $\omega^\mu(t)$ itself must also belong to this same commutative algebra.

We now turn to the coupling between apparatus and system to see what we can learn about them by applying these restrictions. We write the time evolution operator as

$$H_{op} = \frac{\partial H(\omega)}{\partial\omega^\mu}\,\varepsilon^{\nu\mu}\pi^\nu + X(\eta) + \phi^\mu(\omega,\eta')\pi^\mu + h(\omega,\hat{\varepsilon})$$

$$= F^\mu(\omega,\eta')\pi^\mu + G(\omega,\varepsilon)\ . \qquad (6.4)$$

The time derivatives of $\omega^\mu(t)$ take the following form:

$$\dot{\omega}^\mu(t) = -i[\omega^\mu(t), H_{op}] = -F^\mu(\omega(t),\eta'(t)) \qquad (6.5a)$$

$$\ddot{\omega}^\mu(t) = i[F^\mu, H_{op}] = i[F^\mu, F^\nu\pi^\nu + G] \qquad (6.5b)$$

$$\dddot{\omega}^\mu(t) = [[F^\mu, H_{op}], H_{op}] = [[F^\mu, H_{op}], F^\nu\pi^\nu + G] \qquad (6.5c)$$

$$\vdots$$

$$\frac{d^m}{dt^m}\,\omega^\mu(t) = -(-i)^{m-1}[\ldots[[F\ , H_{op}], H_{op}], \ldots, F^\nu\pi^\nu + G] \quad (6.5d)$$

The first restriction which we derive by means of (6.5a) and (α) is that the F^μ belong to a commutative algebra for all μ, i.e., the set $\{F^\mu\}$ or equivalently the set $\{\phi^\mu\}$, must be a commuting set. This follows directly from

$$[F^\mu, F^\nu] = \left[\frac{\partial F(\omega)}{\partial\omega^\rho}\,\varepsilon^{\mu\rho}, \phi^\nu\right] + \left[\phi^\mu, \frac{\partial H(\omega)}{\partial\omega^\lambda}\,\varepsilon^{\nu\lambda}\right] + [\phi^\mu, \phi^\nu] \qquad (6.6)$$

and $[\omega^\mu(o), \eta'(o)] = 0 \Rightarrow [\omega^\mu(t), \eta'(t)] = 0$.

This condition merely tells us that the operators ϕ^μ must commute among themselves. It does not give us any information about the functional form of ϕ^μ. Thus, $\{\eta'\}$ may be a commuting set, in which case ϕ^μ automatically satisfy the requirement, or it is a non-commuting set such that the ϕ^μ commute with each other. In the sequel we will derive other conditions to be satisfied by the coupling functions. In the case that $\{\eta'\}$ is abelian, these will be seen to be simpler.

We introduce some notation at this stage. We denote by

$\{\rho\}$ the maximal algebraically independent subset of the operators $\{\phi^\mu\}$. In combinations we allow the coefficients to depend on the ω variables. Then $[\rho_i,\rho_j] = 0$, and each ϕ^μ can be formed by algebraic combinations of the ρ_i. We consider extensions of this set to a maximal algebraically independent commuting subset of the algebra of dynamical variables. We denote such an extension by $\{\rho\} \cup \{\rho'\}$. We will write for the algebra of operators generated by these two sets $A_\omega(\{\rho\})$ and $A_\omega(\{\rho\} \cup \{\rho'\})$ respectively. These are both commutative algebras. The sets $\{\rho\}$ and $\{\rho'\}$ will in general be finite sets, though the algebras generated by them will be infinite dimensional (in the linear sense). It will be helpful to make use of the linearly independent bases of $A_\omega(\{\rho\})$ and $A_\omega(\{\rho\} \cup \{\rho'\})$. We denote the basis of the first by $\{\zeta\}$, which is just the set $\{\rho_i,\rho_i\rho_j,\rho_i\rho_j\rho_k,\ldots\}$. For $A_\omega(\{\rho\} \cup \{\rho'\})$ we use $\{\zeta'\}$ which is $\{\rho_i',\rho_i'\rho_j',\ldots,\rho_j'\rho_i,\ldots\}$. This notation will simplify slightly our expressions.

From equation (6.5b) we see

$$\ddot\omega^\mu(t) = iF^\nu[F^\mu,\pi^\nu] + i[F^\mu,F^\nu]\pi^\nu + i[F^\nu,G]$$

$$= iF^\nu[\phi^\mu,\pi^\nu] + i[\phi^\mu,G] + iF^\nu\left[\frac{\partial H(\omega)}{\partial\omega^\rho}\,\varepsilon^{\mu\rho},\pi^\nu\right] .$$

The requirement that $\ddot\omega^\mu(t)$ belongs to the commutative algebra tells us that for each m

$$F^\nu[\zeta_m,\pi^\nu] + [\zeta_m,G] \qquad (6.7)$$

must belong to this algebra. Thus if $[\zeta_m,\pi^\nu]$ either vanishes or already belongs to the algebra, we see that $[\zeta_m,G]$ must also belong to the algebra, whereas if $[\zeta_m,\pi_\mu]$ lies outside the algebra the secondary coupling function $h(\omega,\hat\varepsilon)$ must be so chosen that the combination (6.7) belongs to the algebra. If this is not possible then the interaction will destroy the classical nature of the apparatus, and a measurement cannot be achieved. Thus for some choice of $\{\rho'\}$ we have $[F^\mu,H_{op}] \in A_\omega(\{\rho\} \cup \{\rho'\})$ and we can write

$$[F^\mu,H_{op}] = \sum_m a_m(\omega)\zeta_m + \sum_\ell b_\ell(\omega)\zeta_\ell' . \qquad (6.8)$$

From equation (6.5c) we see that $[[F^\mu,H_{op}], F^\nu\pi^\mu + G]$ must also belong to the algebra. We write this commutator as

$$\sum_m \{a_m(\omega)([\zeta_m,F^\nu]\pi^\nu + F^\nu[\zeta_m,\pi^\nu] + [\zeta_m,G]) + F^\nu[a_m(\omega),\pi^\nu]\zeta_m\} +$$

$$\sum_\ell \{b_\ell(\omega)([\zeta_\ell',F^\nu]\pi^\nu + F^\nu[\zeta_\ell',\pi^\nu] + [\zeta_\ell',G]) + F^\nu[b_\ell(\omega),\pi^\nu]\zeta_\ell'\} \qquad (6.9)$$

Now, we know already that $[\zeta_m,F^\nu] = [\zeta_\ell',F^\nu] = 0$ while $F^\nu[\zeta_m,\pi^\nu] + [\zeta_m,G]$, $F^\nu\zeta_m$ and $F^\nu\zeta_\ell' \in A_\omega(\{\rho\} \cup \{\rho'\})$, and so equation (6.9) tells us that we must have

$$F^\nu[\zeta_\ell',\pi^\nu] + [\zeta_\ell',G] \in A_\omega(\{\rho\}\cup\{\rho'\}) \qquad (6.10)$$

for all those ζ_ℓ' occurring in the expansion (6.8). Because we considered a maximal commuting algebra, it is clear that by examining the higher derivatives of $\omega^\mu(t)$ we will be led to condition (6.10) for all those ζ'_ℓ occurring in the expansions of the derivatives of $\omega^\mu(t)$.

Let us now focus on equation (β). The only operator with which $\omega^\mu(t)$ fails to commute is $\pi^\mu(t)$. We have already ensured that the time derivatives of ϕ^μ are independent of $\pi(t)$, and thus equation (β) is automatically satisfied.

The conditions we have derived, (6.7) and (6.10), look like formal conditions, as the linear bases $\{\zeta\}$ and $\{\zeta'\}$ are infinite. However, by making use of their expansion in terms of $\{\rho\}$ and $\{\rho'\}$, we see that we need only enforce (6.7) and (6.10) for these finite sets.

Let us now summarize our results so far. We can state these in the form of both sufficient, and necessary and sufficient conditions for condition 3 to be obeyed. Common to both sets of conditions are

A. $\{\phi^\mu\}$ forms a commuting set

B. $F^\nu[\rho_m,\pi^\mu] + [\rho_m,G] \in A_\omega(\{\rho\} \cup \{\rho'\})$ for all ρ_m

The sufficient conditions are found by adding

C. $F^\nu[\rho'_m,\pi^\nu] + [\rho'_m,G] \in A_\omega(\{\rho\} \cup \{\rho'\})$ for all ρ'_m

Together these tell us that for any $\alpha \in A_\omega(\{\rho\} \cup \{\rho'\})$, $[\alpha,H_{op}]$ also belongs to the algebra.

On the other hand, we have necessary and sufficient conditions by using

D. $F^\nu[\rho'_m,\pi^\nu] + [\rho'_m,G] \in A_\omega(\{\rho\} \cup \{\rho'\})$ for those ρ'_m which occur in the expansions of $[\rho_m,H_{op}]$, $[[\rho_m,H_{op}],H_{op}]$, etc.

In any given model it will be a straightforward task to check these criteria.

The conditions A, B and D will also be necessary and sufficient conditions that the apparatus remain classical, if the interaction is such that $\omega^\mu(t)$ is regular for all times t. These are, then, the Integrity criteria.

However, it may occur[4] that the interaction explicitly precludes the regularity of the apparatus observables. In such a case, A, B and D are necessary, but not sufficient, conditions that the apparatus remain classical. We must go back to condition 2, and examine $[\omega^\mu(t),\frac{d}{dt'}\omega^\nu(t')]$, or equivalently

$$[\omega^\mu(t),\phi^\nu(t')]$$

for t = 0, for example. Then we must have

$$[\omega^\mu(o),\phi^\nu(t')] = 0 \quad \text{for all times } t'.$$

In the non-regular cases, the conditions A, B and D must be supplemented in this fashion to yield the Integrity Criteria.

We must now ask what these conditions tell us about the functional form of ϕ^μ and G. To begin with we see that if the ϕ^μ depend on at most a commuting set of quantum variables, then $[\rho_i,\pi^\nu]$ already belongs to the algebra $A_\omega(\{\rho\} \cup \{\rho'\})$. Furthermore, in this case we can assume ρ_j' also depends on a commuting set of quantum variables. Then conditions B and D reduce to conditions on the allowed $G = X(\eta) + h(\omega,\hat{\varepsilon})$. If ϕ^μ depends in a non-trivial manner on non-commuting quantum variables, then we must check that conditions B and D are satisfied in each case.

We can understand the ϕ^μ depending on a commuting set of quantum variables from the measurement viewpoint. The information stored in the classical variables after the interaction has occurred will be about these operators. On the other hand, if the ϕ^μ depend on non-commuting variables the information stored will be of the $\{\phi^\mu\}$ rather than the set $\{\eta'\}$. Since the $\{\phi^\mu\}$ are not operators of the quantum system, we are led to the conclusion that, whereas an interaction has taken place the effect on the apparatus cannot be identified as a measurement on the original quantum system.

To illustrate the merits of the criteria derived in this section, we will next turn to a simple interaction between a classical system and a quantum system, and see what restrictions on the coupling functions these criteria give us.

Section 7. Illustration of the Use of the Integrity Criteria

In Section 6 we examined some general aspects of the interaction between a quantum mechanical system and a classical apparatus. We concentrated on the effect of the interaction on the behavior of the classical system. In fact we saw that in general the classical system does not retain its integrity (classical nature) as a result of such an interaction. We derived a set of criteria which can be checked for a given interaction to see whether or not the classical system retains its integrity. In this section we propose to look at an example of such an interaction and see what restrictions are placed by the criteria. In particular, we wish to see if the coupling functions can depend in a non-trivial manner on non-commuting quantum variables.

The quantum system we are going to "examine" is an inert quantum Spin System.[5] So the operators of the quantum system are just spin operators S_1, S_2, S_3 and $S^2 = S_1^2 + S_2^2 + S_3^2$, and they satisfy the usual commutation relations

$$[S_i,S_j] = i\varepsilon_{ijk}S_k$$

$$[S^2,S_i] = 0$$

Then since this system is to be inert, the Hamiltonian operator is zero--the system on its own does not change in time. The actual state of the system is specified, in the Heisenberg picture, by choosing a particular initial state, for example an eigenstate of S^2 and S_{i_o}, a complete commuting set of operators of the system. We shall consider that this quantum Spin System is carried along as internal degrees of freedom by a neutral particle whose translatory degrees of freedom are purely classical. Thus the classical apparatus will be a freely moving classical particle, as discussed in section 4. The dynamical variables for the apparatus will then be $\{\omega\} = \{q^i, p^i\}$ and π^μ, the conjugate unobservable operators. The Hamiltonian function of the apparatus is $+\frac{1}{m}\underset{\sim}{p}\cdot\underset{\sim}{\pi}^q$. We couple together the two systems by causing the classical particle to interact with an external inhomogeneous magnetic field. At the purely classical level the potential energy of the particle when under the influence of the magnetic field is $\underset{\sim}{\mu}\cdot\underset{\sim}{B}$ where $\underset{\sim}{\mu} = \gamma\underset{\sim}{S}$ is the magnetic moment of the particle. Then the force exerted on the particle is

$$\underset{\sim}{F} = -\underset{\sim}{\nabla}(\underset{\sim}{\mu},\underset{\sim}{B}) = -\gamma S_i \underset{\sim}{\nabla} B_i \ . \tag{7.1}$$

Now the coupling of the two systems, in order to reproduce the correct classical equations of motion for the apparatus, will be characterized by coupling functions

$$\phi_q^i = 0 \ , \quad \phi_p^i = -F_i \ , \quad h(\omega,\xi) = -\gamma\underset{\sim}{B}\cdot\underset{\sim}{S} \tag{7.2}$$

which give us a Hamiltonian operator

$$H_{op} = +\frac{1}{m}\underset{\sim}{p}\cdot\underset{\sim}{\pi}^q - \gamma\underset{\sim}{B}(q)\cdot\underset{\sim}{S} - \gamma S_i(\underset{\sim}{\nabla}B_i)\cdot\underset{\sim}{\pi}^p \ . \tag{7.3}$$

We allow for the most general coupling between apparatus and system by letting $\underset{\sim}{B}(q)$ be an arbitrary inhomogeneous magnetic field. However we will, in this paper restrict our attention to functions which are regular. We shall comment later on other choices. The coupling functions in (7.2) will now depend on non-commuting quantum variables. We now apply the Integrity Criteria to the Hamiltonian (7.3).

The first criterion is that the primary coupling functions should form a commuting set. We can write these functions as

$$\phi_i^p = \underset{\sim}{S}\cdot\underset{\sim}{f}^i \quad \text{with } \underset{\sim}{f}^i = \gamma\frac{\partial}{\partial q_i}\underset{\sim}{B} \ . \tag{7.4}$$

Now $[\phi_i^p,\phi_j^p] = [\underset{\sim}{S}\cdot\underset{\sim}{f}^i,\underset{\sim}{S}\cdot\underset{\sim}{f}^j] = i\underset{\sim}{S}\cdot(\underset{\sim}{f}^i\times\underset{\sim}{f}^j)$, (7.5)

and so they form a commuting set if and only if

$$\underset{\sim}{f}^i \times \underset{\sim}{f}^j = 0 \ . \tag{7.6}$$

In other words, $\underset{\sim}{f}^i$ and $\underset{\sim}{f}^j$ must be parallel.

Now let us go to the next criterion. We must have

$$[[\phi^i,H_{op}],\phi^j] = 0 \quad \text{for all i and j.} \tag{7.7}$$

Using the Hamiltonian operator (7.3) we have

$$[\phi^i, H_{op}] = \frac{i}{m}\, \underset{\sim}{p}\cdot\frac{\partial}{\partial \underset{\sim}{q}}\, \underset{\sim}{f}^i\cdot \underset{\sim}{S} + i\gamma\, \underset{\sim}{B} \times \underset{\sim}{f}^i\cdot\underset{\sim}{S}$$

$$\equiv i\, \underset{\sim}{Df}^i\cdot\underset{\sim}{S} \qquad (7.8)$$

$$\text{where } \underset{\sim}{Df}^i \equiv \frac{1}{m}\, \underset{\sim}{p}\cdot\frac{\partial}{\partial \underset{\sim}{q}}\, \underset{\sim}{f}^i + \gamma \underset{\sim}{B} \times \underset{\sim}{f}^i$$

And so $$[[\phi^i, H_{op}], \phi^j] = -(\underset{\sim}{Df}^i \times \underset{\sim}{f}^j)\cdot\underset{\sim}{S} \,. \qquad (7.9)$$

This equation is satisfied if and only if

$$\underset{\sim}{Df}^i \times \underset{\sim}{f}^j = 0 \,. \qquad (7.10)$$

However, both $\underset{\sim}{B}$ and $\underset{\sim}{f}^i$ are independent of $\underset{\sim}{p}$, so this equation is satisfied if and only if we have both

$$\frac{1}{m}\, \underset{\sim}{p}\cdot\frac{\partial}{\partial \underset{\sim}{q}}\, \underset{\sim}{f}^i \times \underset{\sim}{f}^j = 0$$

$$(7.11)$$

$$\text{and } (\underset{\sim}{B}\times\underset{\sim}{f}^i) \times \underset{\sim}{f}^j = 0 \,.$$

The second of these requires that $\underset{\sim}{B} \times \underset{\sim}{f}^i$ be parallel to $\underset{\sim}{f}^j$. However, $\underset{\sim}{B} \times \underset{\sim}{f}^i$ is $\perp$ to $\underset{\sim}{f}^i$, and hence also to $\underset{\sim}{f}^j$. Then to satisfy this equation we must have

$$\underset{\sim}{B} \times \underset{\sim}{f}^i = 0 \,. \qquad (7.12)$$

Consider the equation $\underset{\sim}{B} \times \underset{\sim}{f}^1 = 0$. This is satisfied if and only if there exists a function $\Psi_1(q)$ such that

$$\underset{\sim}{f}^1(q) \equiv \gamma\frac{\partial}{\partial q_1}\, \underset{\sim}{B}(q) = \Psi_1(q)\underset{\sim}{B}(q)$$

except at points where $\underset{\sim}{B}(q) = 0$. The solution of this equation is $\underset{\sim}{B}(q) = B(q)\underset{\sim}{n}$, in each disjoint region of support, where $B(q) = e^{\int \Psi_1(q)dq_1}$ and n is a constant unit vector. The points at which $\underset{\sim}{B}(q) = 0$ are taken care of by $B(q) = 0$.

The second of equations (7.11) is now automatically satisfied. Since the $\underset{\sim}{f}^i$ are parallel to $\underset{\sim}{B}$, we may simply write

$$\frac{1}{m}\ \underset{\sim}{p}\cdot\frac{\partial}{\partial \underset{\sim}{q}}\, \underset{\sim}{B}(q) = \Phi(q,p)\underset{\sim}{B}(q)$$

$$\text{and} \qquad \frac{1}{m}\ \underset{\sim}{p}\cdot\frac{\partial}{\partial \underset{\sim}{q}}\, \underset{\sim}{f}^i = \Phi_i'(q,p)\underset{\sim}{B}(q)$$

so that the equation is clearly satisfied.

The result (7.15) can be stated as follows: in each disjoint region of support of $\underset{\sim}{B}$, *into which the classical* particle will travel, the magnetic field must take the form

$$B(q)\underset{\sim}{n} \qquad (7.13)$$

where $\underset{\sim}{n}$ is a constant unit vector. In different support regions $\underset{\sim}{n}$ could point in different directions. However, in this section we are considering *only* regular functions, for which the zeroes are isolated, i.e., such functions can have only one region of support. The remaining Integrity criteria are satisfied once this very restrictive form is derived for the external magnetic field. Clearly,

$$[\phi^i,[[\phi^j,H_{op}],H_{op}]]$$

vanishes, as also do the higher criteria. In fact, we see that the primary coupling functions depend now on

$$S_n \equiv \underset{\sim}{S}\cdot\underset{\sim}{n} \qquad (7.14)$$

alone--a commuting set of quantum variables.

We have shown then that the Integrity Criteria are satisfied if and only if the external inhomogeneous magnetic field takes the form (7.13). Since in this case the only quantum variable in the interaction is S_n, we see that the enforcement of the Integrity criteria leads us to an interaction which results in a measurement.

In the derivation of the above result, we have chosen to work with regular functions--in other words $\underset{\sim}{B}(q)$ should have a power series expansion everywhere. However, if we allow choices for $\underset{\sim}{B}(q)$ which are not regular, the result does not immediately follow. In such a case, in place of (7.13) we find that the allowed form of the magnetic field is

$$\sum_m B^m(q)\underset{\sim}{n}_m \qquad (7.15)$$

where $\underset{\sim}{n}_m$ are constant unit vectors and the support regions of $B^m(q)$ and $B^{m'}(q)$ are disjoint for $m \neq m'$. However, as discussed in section 6, in this case there are extra integrity criteria. Those used so far are necessary but not sufficient. In particular we must check

$$[p^i(o),\phi^j(t)] = 0 \quad \text{for all times } t.$$

Let us consider the case $B_1(q)\underset{\sim}{n}_1 + B_2(q)\underset{\sim}{n}_2$. Then the coupling function is, from (7.2)

$$\phi^j = \gamma\left(\frac{\partial B^1}{\partial q_j}\right)\underset{\sim}{S}\cdot\underset{\sim}{n}_1 + \gamma\left(\frac{\partial B^2}{\partial q_j}\right)\underset{\sim}{S}\cdot\underset{\sim}{n}_2 \ .$$

Now, let us suppose the particle passes through the region 1 first. While in this region $B^2(q) \equiv 0$, and so S_{n_1} remains fixed, but $S_{n_2}(t)$ becomes dependent on $\pi(t)$, since the coefficient of π in H_{op} does not commute with $S_{n_2}(t)$. Thus, when the particle arrives in region 2, $P^j(t)$ will depend upon $S_{n_2}(t_2)$ which is as we have seen a function of the unobservables π. Thus the final criterion above cannot be satisfied, unless the magnetic field takes the more restrictive form (7.13). This result will be illustrated in detail in paper II of this series,[5] where we shall be examining particular experiments.

Section 8. The Secondary Coupling Function

In section 6 when we introduced an interaction term in the total Hamiltonian to describe the coupling of the classical apparatus to the quantum system, we made use of both primary and secondary coupling functions, $\phi^{\mu}(\omega,\eta)$ and $h(\omega,\hat{\varepsilon})$ respectively. The role of the primary coupling function is evident--it is responsible for the direct coupling between the quantum system and the apparatus observables $\{\omega\}$. The role played by the secondary coupling is not so transparent. In this section we shall discuss briefly its role.

As we have already seen, if the primary coupling functions ϕ^{μ} depend on a commuting set of quantum variables, the secondary coupling function must be chosen so that, in the language of section 6

$$\text{(i)}\ [\rho, X(\eta) + h(\omega,\hat{\varepsilon})] \in A_{\omega}(\{\rho\} \cup \{\rho'\})$$
$$\text{and (ii)}\ [\rho', X(\eta) + h(\omega,\hat{\varepsilon})] \in A(\{\rho\} \cup \{\rho'\}) \qquad (8.1)$$

for those ρ' which are relevant.

Here, $\{\rho\}$ is the maximal algebraically independent subset of the set of primary coupling functions $\{\phi^{\mu}\}$, where we allow ω-dependent coefficients. It is a commuting set. The set $\{\rho'\}$ contains operators which are algebraically independent of the ρ and which extend $\{\rho\}$ to a maximally commuting set of operators.

Given a particular quantum system Hamiltonian, $X(\eta)$, we are free to choose $h(\omega,\hat{\varepsilon})$ subject to (i) and (ii), if such a function exists. If it does not, the coupling will not allow the apparatus to remain classical.

When ϕ^{μ} depends on non-commuting quantum variables, the restrictions on $h(\omega,\hat{\varepsilon})$ are even more strict, as we must then have

$$\text{(i)}'\ F^{\nu}[\rho,\pi^{\nu}] + [\rho, X(\eta) + h(\omega,\hat{\varepsilon})] \in A_{\omega}(\{\rho\} \cup \{\rho'\})$$

$$\text{and (ii)}'\ F^{\nu}[\rho',\pi^{\nu}] + [\rho', X(\lambda) + h(\omega,\hat{\varepsilon})] \in A_{\omega}(\{\rho\} \cup \{\rho'\}). \qquad (8.2)$$

In section 7 we discussed an application of the model to a simple example. There we chose a secondary coupling function using physical arguments--we chose $h(\omega,\hat{\varepsilon})$ to be the potential energy of a classical particle, with a magnetic moment, moving in an external magnetic field. However, this choice was not essential for logical consistency.

We wish now to discuss another role of the secondary coupling function. This role is connected with ensuring the form invariance of the total Hamiltonian under unitary "gauge" transformations $U(\omega,\varepsilon)$. Let us consider the effect of such a unitary transformation on our theory. To recall, our Hamiltonian is

$$H_{op} = \frac{\partial H(\omega)}{\partial \omega^{\mu}} \varepsilon^{\nu\mu}\pi^{\nu} + X(\eta) + \phi^{\mu}(\omega,\eta')\pi^{\mu} + h(\omega,\hat{\varepsilon}) \ . \qquad (8.3)$$

The dynamical variables transform as follows

$$\omega \to \omega' = U(\omega,\varepsilon)\omega U^{-1}(\omega,\varepsilon) = \omega$$

$$\pi \to \pi' = U(\omega,\varepsilon)\pi U^{-1}(\omega,\varepsilon) = \pi + U[\pi,U^{-1}]$$

$$\xi_\alpha \to \xi'_\alpha = U(\omega,\varepsilon)\xi_\alpha U^{-1}(\omega,\varepsilon) = \xi_\alpha + f(\omega,\varepsilon) \ . \qquad (8.4)$$

Term by term, the total Hamiltonian behaves as follows

$$\frac{\partial H(\omega)}{\partial \omega^\mu} \varepsilon^{\nu\mu}\pi^\nu \to \frac{\partial H(\omega)}{\partial \omega^\mu} \varepsilon^{\nu\mu}[\pi^\nu + U[\pi^\nu,U^{-1}]]$$

$$X(\eta) \to X(\bar{\eta}) = X(\eta) + Y(\omega,\xi)$$

$$\phi^\mu(\omega,\eta')\pi^\mu \to \phi^\mu(\omega,\bar{\eta}')[\pi^\mu + U[\pi^\mu,U^{-1}]]$$

$$= \psi^\mu(\omega,\xi)[\pi^\mu + U[\pi^\mu,U^{-1}]]$$

$$h(\omega,\hat{\varepsilon}) \to h(\omega,\hat{\varepsilon}') = h(\omega,\hat{\varepsilon}) + Z(\omega,\xi) \ . \qquad (8.5)$$

Thus

$$H_{op} \to \frac{\partial H(\omega)}{\partial \omega^\mu} \varepsilon^{\nu\mu}\pi^\mu + X(\eta) + \psi^\mu(\omega,\xi)\pi^\mu$$

$$+ h(\omega,\hat{\varepsilon}) + Z(\omega,\xi) + \frac{\partial H(\omega)}{\partial \omega^\mu} \varepsilon^{\nu\mu}U[\pi^\nu,U^{-1}]$$

$$+ Y(\omega,\xi) + \psi^\mu(\omega,\xi)U[\pi^\mu,U^{-1}] \ . \qquad (8.6)$$

We can group together the interaction terms which are π-independent as follows

$$\hat{h}(\omega,\xi) \equiv h(\ ,\hat{\varepsilon}) + Z(\omega,\xi) + \frac{\partial H(\omega)}{\partial \omega^\mu} \varepsilon^{\nu\mu}U[\pi^\nu,U^{-1}]$$

$$+ Y(\omega,\xi) + \psi^\mu(\omega,\xi)U[\pi^\mu,U^{-1}] \qquad (8.7)$$

and then the transformed Hamiltonian operator takes the same generic form as in (8.3)

$$H'_{op} = \frac{\partial H(\omega)}{\partial \omega^\mu} \varepsilon^{\nu\mu}\rho^\nu + X(\eta) + \psi^\mu(\omega,\xi)\pi^\mu + \hat{h}(\omega,\xi) \ . \qquad (8.8)$$

thus exhibiting *what we mean by form* invariance.

Clearly, if we were to formulate the problem without a secondary coupling function, we could in this manner generate such a coupling by making a unitary transformation of the above type. Similarly, if we were to formulate the problem initially with an h-coupling which depended on only a commuting subset of the quantum variables, we could by means of a unitary transformation generate an h-coupling depending on non-commuting quantum variables. Of course, the primary coupling functions will also change in the process,

$$\phi^\mu(\omega,\eta') \to \psi^\mu(\omega,\xi) \ .$$

There are two interesting questions which one can pose. Can we find a unitary transformation which will cause a secondary coupling function $h(\omega,\hat{\epsilon})$, depending on non-commuting quantum variables, to be replaced (in the sense of the above) by $\hat{h}(\omega,\xi)$ which depends only on a commuting set of quantum variables? This question cannot be answered in general. One can also ask whether or not a $U(\omega,\xi)$ exists which will cause the original coupling $h(\omega,\hat{\epsilon})$, whatever its dependence on quantum variables, to be transformed to zero?--i.e.,

$$\hat{h}(\omega,\xi) = 0 \quad \text{identically} .$$

This question also cannot be answered in general. However, within the context of the application discussed in section 7 we can supply an answer--it can be done!

The Hamiltonian in that case takes the explicit form (after the Integrity Criteria are enforced)

$$H_{op} = \frac{1}{m}\, \underset{\sim}{p}\cdot\underset{\sim}{\pi}^{q} - \gamma B(q) S_3 - \gamma \frac{\partial B(q)}{\partial q_i} S_3 \pi_i^p \,. \tag{8.9}$$

We notice that $X(S) \equiv 0$. Then setting $\hat{h} = 0$ identically in equation (8.7) gives us the following equation for $U(\omega,\xi)$

$$-\frac{i}{m} p_j U\left(\frac{\partial}{\partial q_j} U^{-1}\right) + i\gamma \frac{\partial B(q)}{\partial q_j} U S_3 \left(\frac{\partial}{\partial p_j} U^{-1}\right) = \gamma B(q) U S_3 U^{-1} \,. \tag{8.10}$$

To find a solution to this equation we specialize to a magnetic

field

$$\gamma B(q) = A q_3 + B \,.$$

Also, we assume that S_3 is in an irreducible spin multiplet. *For our argument we also assume the spin of the quantum system to be half-old-integral*, though our result can be modified to hold also for the integer spin case. Thus S_3 can be inverted. We further work in a representation in which S_3 is diagonal.

With these restrictions, equation (8.10) can be solved using the methods for first order partial differential equations.[8] *We find a particular solution of the form*

$$U^{-1} = \exp -i \left\{ \frac{(p_3)^3}{3mAS_3} + \left(\frac{B}{A} + q_3\right) p_3 \right\} \tag{8.11}$$

as we are not interested in finding the most general solution of equation (8.10). Since S_3 is diagonal, $(S_3{}^{-1})_{ii} = \frac{1}{S_{3ii}}$. We can take account of integer spins as follows: for spin s, and for $k \neq s+1$, we define

$$(U^{-1})_{kk} = \exp -i \left\{ \frac{(p_3)^3}{3mA(S_3)_{kk}} - \left(\frac{B}{A} + q_3\right) p_3 \right\} \tag{8.12}$$

while for $k = s+1$, we define

$$(U^{-1})_{s+1,s+1} = e^{i\theta} , \quad \theta \text{ a constant} .$$

In fact θ need just be independent of q_j. In the $s_3 = 0$ case there is no interaction occurring in the model anyway, and then we can choose θ to be zero.

It is interesting to note that since the unitary transformation so constructed is diagonal,

$$S_3 \to S_3' = S_3$$

and the transformed Hamiltonian is

$$H'_{op} = \frac{1}{m}\, \underset{\sim}{p} \cdot \underset{\sim}{\pi}^q - \gamma \frac{\partial}{\partial q_j} B(q) S_3 \pi_j^p$$

--the original Hamiltonian without the Secondary coupling function.

A further interesting comment is that equation (8.10) is the zero-eigenvalue equation for the Hamiltonian (8.9)

$$\text{namely,} \qquad HU^{-1} = 0$$

and thus the matrix elements of U^{-1} are the zero eigenvalue eigenfunctions fo H corresponding to the allowed values of s_3.

Section 9. Concluding Remarks

In the preceding sections we have analyzed the problem of measurement insofar as the transfer of dynamical information from the (quantum) "system" to the (quantum-enlarged) classical "apparatus" is concerned. The concepts of superselecting operators and unobservable dynamical variables have been essential in setting up the formalism for this analysis. The construction of the measurement interaction has been reduced in part to finding coupling functions which satisfy the Integrity Criteria--an algebraic problem. It now remains to analyze the structure of these criteria, and their associated "gauge" transformations, in the context of preselected examples. The simple Stern-Gerlach systems studied in this paper show that even simple systems can give surprising answers.

As we have emphasized, the true measurement problem involves the recording of data and thus the general problem of irreversible processes. It is interesting to note that by virtue of a general theorem of Poincaré, there is no non-negative phase function which can have a monotonic time derivative; hence the second law of thermodynamics must have its dynamical counterpart in the study of dynamical variables other than phase functions. Our quantum-enlarged classical systems automatically provide a rich framework within which a true entropy function may be constructed. That the algebra of dynamical variables of even a classical system manifesting a thermodynamic behavior must be non-commutative is an idea being pursued by Prigogine,[9] particularly in his recent work with Misra. If such ideas prove useful, they will answer in part some of the questions raised in the Introduction.

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References

1. See for example the excellent book by B. d'Espagnat, and the many references contained therein: Conceptual Foundations of Quantum Mechanics, 2nd Edition (W. A. Benjamin, Mass., 1976).

2. This same process of destruction (or modification) of correlations may also underlie the irreversible process involved in the indelible actions produced in measurement. But a discussion of such a topic is beyond the scope of this paper .

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