AN ALGORITHM FOR LOCATING THE EXTREMUM OF A MULTI-DIMENSIONAL CONSTRAINED FUNCTION AND ITS APPLICATION TO THE PPPL HYBRID STUDY

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ABSTRACT

A description is presented of a general algorithm for locating the extremum of a multi-dimensional constrained function. The algorithm employs a series of techniques dominated by random shrinkage, steepest descent, and adaptive creeping. A discussion follows of the algorithm's application to a "real world" problem, namely the optimization of the price of electricity, P_{eh}, from a hybrid fusion-fission reactor. Upon the basis of comparisons with other optimization schemes of a survey nature, the algorithm is concluded to yield a good approximation to the location of a function's optimum.

I. INTRODUCTION

In the course of the PPPL Hybrid Study [1], a need arose for the development of an algorithm that optimizes a multi-variable function, hereafter referred to as the objective function.* The domain of the objective function is defined by the lower/upper bound constraints on the variables. The objective function may possess saddle points and several local minima and maxima of which only the global maximum is of interest. The necessity to impose various system constraints upon the function introduces additional complexities into the problem. The constraints are of two types: barrier and penalty. The former type has the net effect of reducing the hyperarea associated with the domain. Also, the bounding hypersurface takes on an irregular configuration characterized by peninsulae. The penalty type of constraint modifies the value of the unconstrained objective function, but only when the constraint is violated. Continuity of the resulting objective function is preserved, but discontinuities in the first derivatives can appear in the objective function at the intersection of the penalized and unpenalized regions. Such junctures could give rise to ridges in the hypersurface defined by the objective function.

Both types of constraints are influential in the choice of the optimality condition, defined as the criterion for determining the solution of the optimization problem. As the maximum may be either a boundary or interior point, the zero-gradient condition

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*The terminology used throughout this report follows that of Jacoby [2].

cannot be solely employed as the optimality condition. Also, the curvature of the constraint functions and the objective function may render other more sophisticated optimality conditions ineffective e.g. the Kuhn-Tucker conditions [3].

In all, the requirements placed upon the search algorithm are formidable. In fact, the problem becomes insoluble in practice when confronted with an objective function possessing buttes (spikes or poles) of small half-width in comparison to the dimensions of the feasible zone, defined as the set of all points or elements of the domain for which the barrier constraints are satisfied. The unconstrained objective function is assumed to be void of such spikes, but the presence of peninsular-type boundaries poses as formidable an obstacle to the optimization process as do spikes. However, the nature of these peninsulae, as observed during the course of the PPPL Hybrid Study, is such that they can be handled successfully. They are seen individually to occupy small areas of the domain as well as to exhibit only small variations in the objective function over the peninsula.

II. DESCRIPTION OF SEARCH ALGORITHM

A. Overview

The algorithm developed explicitly to solve the previously defined problem in full generality is composed of various search techniques whose application is dictated by the state of the optimization. First, the "horizon" is scanned for the tallest hill. Having ascertained its location, an ascent to its summit is then undertaken. It should be emphasized that the scan and ascent are two distinct operations, utilizing different methods. A form of "random shrinkage" is employed during the scan while the ascent is performed predominately via "adaptive creeping" and "steepest-descent" methods (see reference 2 for a brief description of these techniques).

B. Scan Phase

1. Random Shrinkage. In the scan phase of the search algorithm, the topography of the objective function is inferred from the process of sampling in its feasible region. The sampling is performed by evaluating the objective function at a set of randomly chosen locations within a designated region defined to be the sampling region. A strong impetus, derived from the statistics of sampling, exists for concentrating the sample density within the vicinity of the solution, i.e., the maximum. This is accomplished in the algorithm through an iterative feedback process of breaking the sampling process into a series of samplings. The hyperarea of the sampling region is reduced after each sampling in the series. The prescription for such a reduction entails locating the subset of the sampling region that contains all feasible points whose corresponding functional values are above a specified value which is termed a function contour level. This subset, which is an approximation to the function contour, becomes the next sampling region in the series. By using the entire variable domain for the initial sampling and raising the contour level in successive steps, which effectively shrinks the sampling region, the maximum will be enclosed by a series of concentric contours of succeedingly higher value. Through the

appropriate selection of a contour level structure, the final (or highest valued) contour contains a single local maximum which should be the global maximum.

Three supplemental benefits result from the establishment of a predetermined contour level structure. First, the rate of constricture of the sampling area can be controlled so as to avoid the possible exclusion of the global maximum from it. Second, some control is afforded over the degree to which multiple maxima may be resolved. Finally, samples taken during any sampling can be utilized in the construction of all succeeding contours, affording an enhancement in the statistics of the sampling process.

2. Example. This entire process will now be illustrated with a one-dimensional example. Appearing in Fig. 1 are the results of seven random samples of a hypothetical function $f(\underline{x})$ taken within the domain of $\underline{x}$. These samples constitute just one in a series of samplings. From the knowledge derived from the seven samples, a region, which will approximate that region containing all of $f(\underline{x})$ such that $f(\underline{x})>a$, will be constructed for the next generation of sampling. The line connecting samples 1 and 2, i.e. $\underline{x}_1 \leq \underline{x} \leq \underline{x}_2$ serves as a basis from which the new sampling region will be developed. With samples 3 and 4 the region is expanded, as the function $f(\underline{x})$ evaluated at $\underline{x}_3$ and $\underline{x}_4$, respectively, was always greater than a. Sample 5 yields no information concerning contour level a. However, in conjunction with the information obtained with sample 2, another sampling region may simultaneously be constructed for $f(\underline{x})>b$ for use in the third generation of sampling. In addition, the

capacity for distinguishing between multiple local maxima and concentrating on the global maximum via a predefined contour structure (here, $f(\underline{x})=a$ and $f(\underline{x})=b$) is quite evident from Fig. 1. Sample 6 does not expand the domain, but sample 7 does. Furthermore, as a result of sample 7, the boundary of the domain appearing in the top half of Figure 1 is extended to the original lower bound, $\vec{x} = \vec{x}_L$.

Although this hypothetical function exhibits a local maxima to the left of the location of the last sample, the algorithm must be able to contend with the possibility that the function continues to rise until encountering the domain boundary. The resulting boundary maximum could be the global maximum. In order to take into account such a possibility, the algorithm will determine the slope of the objective function at any sample which would be used to expand the future sampling region. If the slope is increasing in the direction of the closest boundary, the sampling region under construction is extended to that boundary. Then, as the iterative scheme moves to a higher contour, in this case $f(\vec{x})=b$, the samples taken on the new sampling region, defined to be the interval $\vec{x}_L \leq \vec{x} \leq \vec{x}_4$ in Fig. 1., will be used to construct another, even smaller sampling region and also resolve the function's behavior at the left boundary. All of the sampling regions are constructed to be continuously connected, even though the feasible regions contained within them may be disconnected. In practice, many contour levels must be employed before singling out the global maximum.

3. Auxiliary Methods. There are additional methods employed in the algorithm which were not illustrated by the previous

example. One is to sample initially in only an outer shell of the domain located at the domain edge so as to contain both of its lower and upper bounds. The sampling process then proceeds by contracting the shell inwardly, toward the central region of the domain, but only in those regions of the shell where samples indicate the maximum is most likely not to be. At some predetermined stage of the sampling process, the region enclosed by the shell is added to the shell's hyperarea to form the sampling region for the remainder of the process. The net effect of this sampling scheme is to concentrate the sample density near the edge of the initial domain. This feature is especially important when a maximum is known to be located near the edge. Two parameters, shell thickness, W, and the number of samplings, NS, for which only the shell is used as the sampling region, afford control of this form of sampling. The shell thickness in conjunction with the number of objective function evaluations, N, per sampling determine the sample density. The number of shell samplings then determines the margin width, $\lesssim NS \cdot W$ measured from the edge, over which the total number of shell samples, $N \cdot NS$, are concentrated. In additon, the number, W, also affects the rate of shrinkage of the sampling region. Determining its value becomes critical in those cases where ambiguity exists as to the possible location of the maximum in relationship to the domain boundaries. Both W and NS are best determined by experimenting with the objective function of interest.

Another technique used during the random shrinkage process is the retention of the location of one or more of the largest values of the objective function obtained during previous generations of sampling. Their locations are observed relative to the boundaries of the sampling region as they are constricted. The boundaries are not permitted to exclude these points during the constriction process unless they are below the corresponding contour level. The incidence of any attempts at constricting the sampling region beyond the location of any of the retained points whose value exceeds the contour level is indicative of a too sparse sample density for the contour structure employed.

4. Summary. In summary of this phase of the algorithm, a flow chart of the random shrinkage technique complete with auxiliary techniques is presented in Fig. 2. All of the processes comprising the scan phase are mathematically equivalent to a Monte Carlo solution of the following integral equation:

$$\lim_{C \to f_m(\underline{r})} \int_{B_1}^{B_u} [C - f(\underline{x})]\, d\underline{x} = 0$$

where C defines the contour level, $f(\underline{x})$ is the objective function, the function to be maximized, $f_m(\underline{r})$ is the value of the maximum whose location in the domain is $\underline{r}$, and $\underline{B}_u$ and $\underline{B}_l$ represent upper and lower bounds on the location of $\underline{r}$. Furthermore, the sample taken in the entire series of samplings (i.e., in the vicinity of the maximum) with the largest value will serve as a good starting point for the ascent phase of the algorithm.

C. Ascent Phase

1. Steepest Descent. In the ascent phase of the search, the local topography of the objective function is determined through

a series of exploratory samples. Each sample represents an evaluation of the objective function at a distance from a designated reference point of prescribed length, termed the step size. The samples are located one step size in either the positive or negative direction along the basis vectors spanning the domain of the objective function. If a pair of samples along a particular basis vector is feasible, then the numerical tabulation of $\frac{\partial f(\underline{x})}{\partial x_i}$ and $\frac{\partial^2 f(\underline{x})}{\partial x_i^2}$, where i denotes which basis vector is involved, is possible. Hence, a quadratic polynomial of one variable x_i may be fit through a pair of feasible samples and the reference point. The location of the quadratic's maximum may then be inferred if one exists. The maxima in each variable are combined to yield a location approximating the location of the objective function's maximum. A search vector is then constructed which points from the reference point to this approximation to the location of the maximum of the objective function.

In the event that a maximum in a particular variable cannot be inferred from the pair of samples, then the appropriate set of contingency plans is invoked. As the contingency plans involve many conditional statements and, hence, can be confusing, a detailed flow chart of the derivative and search vector calculations is presented in Fig. 3. A delineation of the contingency plans follows: When the second derivative is non-negative, the magnitude of the corresponding search vector component is taken to be twice the step size. The component's direction is the one in which the first derivative is positive. Should the first derivative be zero, the component's direction is the one in which the variable is decreasing in value*. When the second derivative is negative and the first derivative is zero, the search vector component is set to zero.

It is also possible that neither of the derivatives is calculatable as one or both of the pair of samples is infeasible. When both samples of a pair are infeasible, the search vector component is set to zero. When only one sample of a pair is infeasible, the value of both the reference point and the only feasible point of the pair is used to resolve which course of action is to be pursued. If the reference point has the larger value of the two points, then the search vector component is set equal to zero. If the reference point is the smaller of the two, then the component is twice the step size in the direction of the only feasible point of the pair of samples.

Due to the properties exhibited by the objective function, e.g. discontinuities in slope, multiplication of the search vector by a scale factor has been found useful. Experience with the objective function of the PPPL Hybrid Study suggests that the largest component of the search vector not exceed more than four multiples of the step size. However, at no time may the search vector extend beyond the domain.

Having completed the calculation of the search vector, it is checked for the existence of at lease one non-zero component. If all of its components are zero, the algorithm takes the first of many branches appearing in the coding of the ascent phase (see Fig. 4). This particular branch permits the immediate application

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* This choice of direction is based upon a systems judgment that "smaller (less) is better".

of the side stepping technique without any reduction in the step sizes present in the algorithm.

Upon discovery of at least one non-zero component in the search vector, evenly spaced samples are then calculated along this vector. The distance between the samples defines another, distinct step size. The systematic process of calculating these samples is analogous to tracing a path along the search vector, starting from the purported maximum and proceeding toward the reference point, thereby defining a direction of travel along the search vector. After taking the first two samples, the direction of travel is reversed, but only if the samples indicate a downward slope in the direction of travel and the second sample is of larger value than the reference point. The "walk" along this path is terminated when either a descent is started, provided the value of that point is larger than that of the reference point, or the walk returns to the reference point.

If the walk is unsuccessful i.e., no point is found whose value is larger than that of the reference point, another walk is attempted along the component of the search vector associated with the largest first derivative of the objective function. If either of these walks is successful, the current reference point is replaced by the newly attained point of higher value, a new search vector is calculated; and the search vector walk process is repeated. If both walks were unsuccessful, the two distinct step sizes employed in the derivative calculations and the vector walk are reduced. Then, the vector walk process is repeated with a newly calculated search vector. Limits are imposed upon the extent to which these step sizes may be reduced in order to avoid numerical noise in the samples and to avoid the trap of approaching the maximum yet never attaining it.

The process of iterating through the search vector walk process is terminated when both of the following two criteria are met. The first criterion is that the search vector walks along both the full vector and just one of its components fail. The second criterion is that both step sizes have attained their lower limits.

2. Adaptive Creeping - Side Stepping. Two means exist for arriving upon the threshold of the next technique to be evoked (see Fig. 4). One is the termination of the search vector walk process. The other is a coding branch invoked by the discovery of a search vector of zero length. The technique to be initiated is a side stepping approach which is similar to a technique termed adaptive creeping. Its execution is contingent upon the existence of at least one feasible point being calculated during the derivative calculations. If all of the exploratory samples were infeasible, the coding of the algorithm branches to that section dealing with the verification that a maximum has been discovered.

Proceeding with the side stepping approach, if permitted, the current position of the reference point is temporarily replaced by the largest sample taken during the derivative calculations. The old reference point is retained, however, for future reference. The iterative process of calculating a search vector and "walking" along it is then initiated from the new reference point. As a result of this iterative process, the new reference point evolves into yet another reference point. Upon abandonment of this iterative scheme, the resulting temporary reference point is compared to the old reference point which was retained in storage. If the temporary reference point is of higher value than that of the old reference point, the old reference point is abandoned and the temporary reference point is adopted as the "permanent" reference point. Then, this successful sidestep is followed by another attempt at side stepping. This process of iterating through the side stepping approach is permitted to continue until an unsuccessful side step is encountered. A side step is termed unsuccessful if the resulting temporary reference point is of lower value than the former reference point.

The occurrence of an unsuccessful side step results in a reduction in the two step sizes employed in the derivative calculations and in the search vector walk process. If the new step sizes do not violate the lower limit constraints placed upon them, then the search vector calculation and walk processes are performed. In the event that either the step sizes' lower limit constraints are violated or the number of unsuccessful side steps exceeds a predetermined limit, the side step approach is terminated.

As an iterative scheme, the side step approach is an effective means of following or creeping along a boundary that exists between the feasible and infeasible zones of parameter space when the maximum of the unconstrained surface lies within the infeasible zone. Its effectiveness is attributable to its capacity for avoiding a premature termination of the search along a boundary, which is a basic criticism of direct search methods employing step size reduction (see pg. 194-195 of Reference 4). However, when the maximum is contained within the feasible zone, this process becomes superfluous and is quickly abandoned in the algorithm as is evident from Fig. 4.

3. Ray Tracing. The next technique to be attempted in the ascent phase of the search algorithm is a form of ray tracing. It is applicable only when the maximum is on a boundary, i.e. within an exploratory step size of the boundary (see Fig. 4). In the construction of these rays, one point in parameter space is chosen as the ray's origin. A second point is chosen to define the ray's orientation. In order to qualify as one of the points used in the ray construction, it must be the lone, feasible point from a pair of samples taken along a basis vector during the exploratory search (or derivative calculation) segment of the search vector calculation. If two such points exist, the one of lower value defines the ray's origin; the other point defines the ray's orientation. The ray is then traced uphill from its origin, in the direction of the second point. The trace is terminated when either a boundary is encountered or the ray reaches a crest from which any further tracing would proceed downhill. If more than two points qualify as points that can be used in the construction of rays, then all such possible rays that can be constructed from the ensemble of qualified points are constructed and traced. The end points of the ray tracing process are then compared to the reference point. If any one end point is of higher value than the reference point, that end point replaces the current reference point as the permanent reference point. With a new reference point, all of the previously described techniques of the ascent phase are applied, beginning with the search vector calculation.

Eventually, either an application of the ray tracing process will not yield a point whose value is larger than that of the reference point, or the method will not be applicable. Then, the current reference point is accepted as the location of the global maximum. Before terminating the ascent phase of the algorithm, the reference point needs to be confirmed as a local maximum.

4. Confirmation that Result is a Local Maximum. During the confirmation process, a small region encompassing the reference point is examined via three independent sampling methods to verify that the reference point is a local maximum. The first is a repeat of the derivative calculation. The samples necessary to calculate a first derivative centered about the reference point are evaluated and checked against that of the reference point. Checking the value of the first derivatives is not used as the criterion for a maximum as they need not be zero, as in the case of the maximum lying on a boundary. The second method is to sample points at a prescribed distance from the reference point, but of a randomly chosen orientation, and compare values. The third method is to construct an n-dimensional cube and compare the values of the 2n corner points with the reference point. If any one of these comparisons results in the discovery of a point of higher value, a warning message is printed and the entire ascent phase is repeated only once with this new point taken as the initial reference point. Although, theoretically, this should never happen, upon rare occasions it has occurred as the result of walking too close to a boundary. A numerical fuzziness may be associated with the location of a boundary as a result of certain iterative techniques necessary in calculating some of the barrier type constraints found in the

PPPL Hybrid Study. Inadvertently, the algorithm will push its reference point into this region. From such a vantage point, conditions may contrive to give the illusion of being in an infeasible zone. The second sampling method employed in the confirmation process, i.e. randomly sampling about the reference point, affords the algorithm a re-entry point into the feasible region with a high degree of success.

Upon completion of the ascent phase, it is repeated once with the result of the first application of the ascent phase taken as the starting point of the second application. All step sizes are reset to their initial values. The second application of the ascent phase then proceeds as if the first application had not occurred at all. Upon completion of the second iteration of the ascent phase this application of the entire optimization algorithm,defined to be a "game", is concluded. The resulting maximum in the objective function and the "solution vector" at which this maximum occurs is construed as being validated for this particular game.

III. DISCUSSION - APPLICATIONS

A. Series

Upon completion of a game, the question of the validity of the conclusion that its result is the global maximum immediately arises. A relatively simple means of investigating the validity of this conclusion is to "play" additional games and compare the results. Consequently, the procedure of playing several games and reporting the best of the series as the result was adopted in the PPPL Hybrid Study. However, each game in a series is not totally independent of its predecessor. In the process of constructing the sample areas for higher contours, the boundaries at each contour level are not permitted to exclude any portion of a corresponding sampling region from a previous game. Thereby, the rate of constricture of the sampling areas is further reduced.

The merit of retaining the knowledge of the various contours from one game and then utilizing it in all succeeding games can be seen by examining Figure 5. Each bar graph in Figure 5 is a plot of the number of series in which the optimum result was generated in a particular game of a series as recorded over a specified number of series versus the game sequence of a series. A three variable optimization study of 33 series of 6 games each forms the data base of the first bar graph. A five variable optimization study of 33 series of 5 games each forms the data base of the second bar graph. A trend of increasing probability that a particular game yields the optimum result with increasing game number exists in the first two bar graphs. However, this trend is somewhat clouded by the random fluctuations that are characteristic of Monte Carlo calculations. The effect of the fluctuations can be minimized with increased statistics as is attempted in the last two bar graphs. In the third bar graph, the statistics are increased by reporting the number of series for which the optimum result occurred in a particular game of two game intervals. The first two bar graphs are added together to yield the fourth bar graph. When the optimum occurred on the sixth game of the first bar graph, the "second best" game was then plotted in the last bar graph as the optimum of five games.

The observation that different games generate different maxima is not alarming but rather it is to be viewed as a means of measuring the success of the algorithm in its task of finding the maximum. Before proceeding on to the measurement of the algorithm's success, a minor technical point in the application of the algorithm to the PPPL Hybrid Study needs clarification. The ultimate figure of merit employed in the study is the price of electricity, P_{eh}. The optimization problem thus posed is one in which P_{eh} is minimized. As the algorithm is capable of searching only for a maximum, the problem was necessarily translated into maximizing $1/P_{eh}$. Consequently, in the results to be discussed, reference will be made to a minimum rather than a maximum.

Returning to the first bar graph of Figure 5, none of the numbers of series are reported as integers. The reason they are fractions is that in several instances two or more games in a series would share honors for having produced the minimum P_{eh}. Of the thirty three series comprising the basis of this bar graph, five of them (see Table 1) had more than one game which generated the same minimum P_{eh}, that is indistinguishable to five significant figures, but these minima were also the lowest values of P_{eh} found in each of their respective series. Because the minimum step size employed in the derivative calculations was .05% of the vector $\underline{x}$ in all cases reported in the Hybrid Study, the algorithm may stop anywhere within .05% of the location of a maximum and still conclude it has reached the maximum. Consequently, for the five games of the second series reported in Table I, all have arrived at the same conclusion, i.e., the minimum price of electricity and the location in parameter space that produces the minimum price. However, in the third series, the two games yield the same minimum price of electricity but at close, yet distinct locations in parameter space.

B. Verification of Local Minimum with Cuts.

The incidence of such a pair of results whose vector locations are in such proximity to each other calumniates the ascent phase of the algorithm with regard to its optimality condition. More specifically, the premature abandonment of the search for the minimum is suggested. With the aid of "cuts", explorations of the function's behavior in a plane permitting variations in only two variables, the effectiveness of the search algorithm may be ascertained. All such cuts, centered about the result of a 5-dimensional search in T, W_o, n_h/n_e, a, and h, distinct from the results of Table 1, are presented in Tables 2-A through 2-J. Each entry in these tables represents the price of electricity at vector displacements from the solution vector producing a minimum in P_{eh}, termed the "nominal" machine. The vector displacements are integer multiples of 1/2% of the solution vector. The "nominal" machine's price of electricity and its 5-vector location appear as the second game of Table 10.

As is evident from Tables 2 A-J, none of the cuts reveals a lower price of electricity. Consequently, this purported global minimum can at least be construed as a local minimum. Furthermore, a conclusion may be drawn that the ascent phase of the algorithm is capable of successfully completing its task of finding a local optimum.

These cuts as well as others to be discussed below are very useful for determining the characteristics of the objective function and its minima. One such characteristic is the phenomenon of the minimum lying in a corner. This phenomenon is characteristic of most of the results reported in the PPPL Hybrid Study. In producing this particular corner, only one constraint, $n\tau$, is active. An additional property of the objective function is to exhibit a discontinuity in its slope. This property is most visible in Tables 2-E through G. The penetration constraint, which is of the penalty type, is active here. The areas enclosed by a solid line contain those points that violated the penetration criterion and were forced to undergo a reduction in density in order to comply with the penetration criterion, thereby incurring an economic penalty. In those areas encompassed by the dashed line, termed a median area, the points just meet the penetration criterion. The areas not enclosed by either of these lines contain points that fully comply with the penetration criterion. Thus, to either side of the median areas, two distinct means of determining the plasma density, are employed. This circumstance is responsible for the discontinuity in slope, visible in Tables 2-E through G.

Having ascertained the ascent phase to be operating properly, a different explanation for the existence of two distinct, yet approximately equal solutions must be explored. With the aid of Table 3, the terrain about the result of a five dimensional search may be deduced via the four cuts in the n_h/n_e - W_o plane, each at different temperatures, T. The algorithm concluded the minimum price of electricity to be 36.84 mills/kWh. However, these cuts indicate the existence of two corridors within which the ascent phase of the search algorithm will gravitate to distinct local minima, one for each corridor, e.g. 36.99 and 36.84, regardless of the starting point of the ascent phase algorithm within each corridor. Fortunately, the algorithm chose the appropriate minimum in this instance. This particular example illustrates the importance associated with the scan phase of the search algorithm yielding a good starting point for the ascent phase of the algorithm.

C. Comparison with Results of Other Techniques

1. Survey. In order to gauge the effectiveness of the scan phase and hence the reliability associated with the minimum price of electricity, two types of calculations were performed. The first is a survey type calculation. The entire 5-dimensional parameter space was surveyed by tabulating the price of electricity in a grid of equally spaced mesh points (see Table 4 for the grid locations) for an assumed value of the fissile fuel price, F_{FP}. There are 21,504 distinct 5-vectors in each such survey. These data were reviewed and the machine generating the cheapest electricity was determined. These results are presented in Table 5.

The second type of calculation utilized the search algorithm. These results are presented in Table 6. Each entry of Table 6 represents the best result in a five game series. A total of 24,000 samples of the objective function were evaluated in each game during just the scan phase of the search, with an additional approximately 1000 functional evaluations occurring during the ascent phase. The values of P_{eh} in Table 6 are observed to be below the corresponding values in Table 5. In order to make a comparison of Tables 5 and 6 on an individual game basis, the results of the series, upon which Table 6 is based, are displayed in Table 7. In all instances, the game with the most expensive P_{eh} is still cheaper than the survey result shown in Table 5. Obviously, the survey technique is an inferior method. Increasing the number of grid points for the sake of increased accuracy makes the number of functional evaluations excessive. Just halving the distance, L_g, between grid points, L_g ~ 17% to ~ 8% of the 5-vector, raises the number of distinct 5-vectors to 688,128.

2. Mixed Mode. An improvement upon the survey results is obtained via a mixed mode, surveying a subset of the 5 parameters while applying the search algorithm to the remainder. The result of just such an operation for a fissile fuel price of 86.5 $/gm appears in Table 8. As P_{eh} is minimized over only three variables, the number of samples taken during the scan phase was reduced to 10,560. The results indicate the presence of possibly two distinct valleys in the P_{eh} versus a-h carpet. One valley stretches from the (a,h) location of (4.0,5.8) to (2.8,8.0), while the other, shorter one, goes from (2.4,3.6) to (2.0,4.7). However, the first valley appears to be a more likely candidate for containing the minimum.

The apparent smoothness of the P_e versus a-h carpet in Table 8 can lead one to the misconception that in traversing a path through a and h on the carpet surface, the variables T, W_o, and n_h/n_e will vary in a continuous fashion. This would be the case only if the objective function $P_{eh}=f\ (T, W_o, n_h/n_e)$ is free from the existence of multiple minima whose relative values are a function of a and h.

As an illustration of the above, the ascent phase of the search algorithm is started from various locations on the carpet. The starting locations are fou. adjoining entries from Table 8, as characterized by their (a,h) locations: (2.8,6.9), (2.8,8.0), (3.2,6.9), and (3.2,8.0). The results of these calculations are displayed in Table 9. They confirm the suspicion regarding the existence of multiple minima. Certainly, a smoothly varying $P_{eh} = f(T, W_o, n_h/n_e)$ surface with only a single minimum would have resulted in the algorithm's repeated convergence upon the same location for the minimum when started from the four different locations.

These results should be compared with those of a standard application of the entire algorithm as employed in the Hybrid Study. The term "standard application" denotes an exploration over the full domain of parameter space:

$$6 \leq T \leq 18 \text{ keV}$$
$$150 \leq W_o \leq 300 \text{ keV}$$
$$0.0 \leq n_h/n_e \leq .3$$
$$1.0 \leq a \leq 5 \text{ m}$$
$$2.5 \leq h \leq 8.0 \text{ m}$$

and six games comprising a series with each game employing 24,000 samples during the scan phase. The results of such a standard application of the search algorithm appear in Table 10. The series in Table 10 served as the basis from which the "nominal" machine was chosen. The "nominal" machine is not lower in price than the fourth entry of Table 9. The existence of a lower minimum is not surprising considering: first, the properties of the objective function, namely, possessing multiple minima and secondly, the interpretation of the result of the algorithm to be an approximation to the global minimum, although it is disappointing.

D. Estimation of Accuracy

The best result of a standard application of the search algorithm is to be accepted as the lowest member of an indefinite set of minima. The remaining task is to gauge the discrepancy between the known minima and the lowest minimum calculated by the algorithm. The basic strategy is to try to beat a standard result by whatever means possible. One such method was the previously described series of calculations starting with the carpet of Table 8, culminating with entry 4 of Table 9. Another is to attempt to enhance the statistics of the sampling process during the scan phase of the search algorithm. Doubling the number of samples taken of the objective function together with increasing the number of games in a series provides one means of enhancing the statistics. The results appear in Table 11. A minor improvement in lowering the price of electricity over the "nominal" machine (Table 10) is reflected by the best game of the series. Furthermore, the location of this minimum lies between that of the "nominal" machine and the best result (Table 9) yet found. This fact cannot be absolutely interpreted as a motion in a continuous manner toward the purported minimum because of the multiple minima argument.

A second application of the algorithm achieved a further increase in sample density by employing the doubled sample size as before, but in a subset of parameter space containing the location of the purported minimum. The subspace employed is defined by the following ranges imposed upon the variables:

$$6 < T < 12 \text{ keV}$$

$$150 < W_o < 225 \text{ keV}$$

$$0.0 < n_h/n_e < .1$$

$$1.0 < a < 4.0 \text{ m}$$

$$4.0 < h < 8.0 \text{ m}$$

The result of this attempt appears in Table 12. Although, the variance between the games of this series is less than that found in previous series, the algorithm failed to improve upon the result of just doubling the sample size.

Apparently, enhancing the sample size does not necessarily improve the results. The validity of this conclusion is demonstrated by the results in Table 13. These results represent a search over the entire original domain with the sample size increased by a factor of six over the standard case. The lowest price of electricity found under these conditions is lower than any found in Tables 10-12. However, the result is still not an improvement over that of Table 9. An examination of the investment in terms of the number of samples of the objective function required to obtain these results reveals 144,000., 864,000., and 2,090,880. functional evaluations in just the scan phases for the "nominal" machine (Table 10), the factor of six case of Table 13, and the mixed mode case of Table 9, respectively. When the requirement of a typical application of the ascent phase, ~ 1000 samples, is added to the previous numbers, the totals become 150,000., 870,000., and 2,292,880., respectively.

From the various attempts at beating the standard result, the discrepancy between the standard result and the known minimum may be gauged to be approximately .6% in P_{eh} and no more than 18.6% in the location of the minimum. Additional results calculated under a new set of economic conditions should also be investigated so as to ascertain whether the conclusions regarding the accuracy of the standard calculation may be applied to all such standard calculations under the various economic conditions encountered in the PPPL Hybrid Study. In this vein, the accuracy of a typical Hybrid Study curve of P_{eh} versus F_{FP} was investigated utilizing the doubled statistic application of the algorithm as a standard of comparison. The information that could be inferred from the dependence of individual results upon F_{FP} was also included in the analysis. The resulting best guess as to the minimum price of electricity is plotted in Fig. 6. For purposes of comparison, a survey curve is included in addition to the standard application of the optimizing algorithm. These results indicate that the accuracy which was deduced from the previous detailed investigation of a single point in fissile fuel price, i.e. 86.5$/gm, is typical of all the various economic conditions encountered in the PPPL Hybrid Study.

IV. CONCLUSIONS

In conclusion, the optimizing algorithm yields a good approximation to the optimum of a multi-dimensional, non-linear function subjected to non-linear constraints. The algorithm's effectiveness seems to suffer with an increase in the number of variables to be examined, as is to be expected. In the 3-dimensional optimizations, the algorithm quite capably performed the task of optimization, while the 5-dimensional case proved to be taxing its abilities. This can in part be attributed to an overly quick rate of constricture of the sampling areas in the 5-dimensional optimizations. Although multiple local minima existed for both the 3 and 5-dimensional optimizations, they were found to be especially cumbersome in the latter instance. It is not known whether the introduction of the additional two variables produced an increased density of local minima. Such a possibility could have made ferreting out the minimum more difficult than usual. As for comparisons with other optimization techniques, resources permitted only the limited comparisons reported here. The properties of the objective function and its constraints, the problem format, and the sheer number of dimensions and constraints conspire to render some of the better known techniques, e.g. Lagrange multipliers, either inapplicable or unattractive. The comparisons undertaken demonstrate the efficiency of the algorithm. The results of Figure 3 emphasize the axiom that "you get what you pay for". The survey requires the least amount of computation and suffices for deducing broad trends. Whereas, the standard application of the optimizer is ideal for investigating finer details but at an increased investment in computations. The "best guess" techniques define the limit of diminishing returns. For a substantial increase in computations, a slightly better result can be obtained. A further increase in computations probably would yield an even better result. However, the increase in computations required to insure a better result would prove to be substantial, if not prohibitive.

ACKNOWLEDGEMENT

The author would like to thank the members of the Fusion Reactor Design Division, especially W. G. Price, Jr. and F. H. Tenney, for their many enlightening discussions and helpful suggestions. This work was supported by U. S. Department of Energy Contract No. EY-76-C-02-3073.

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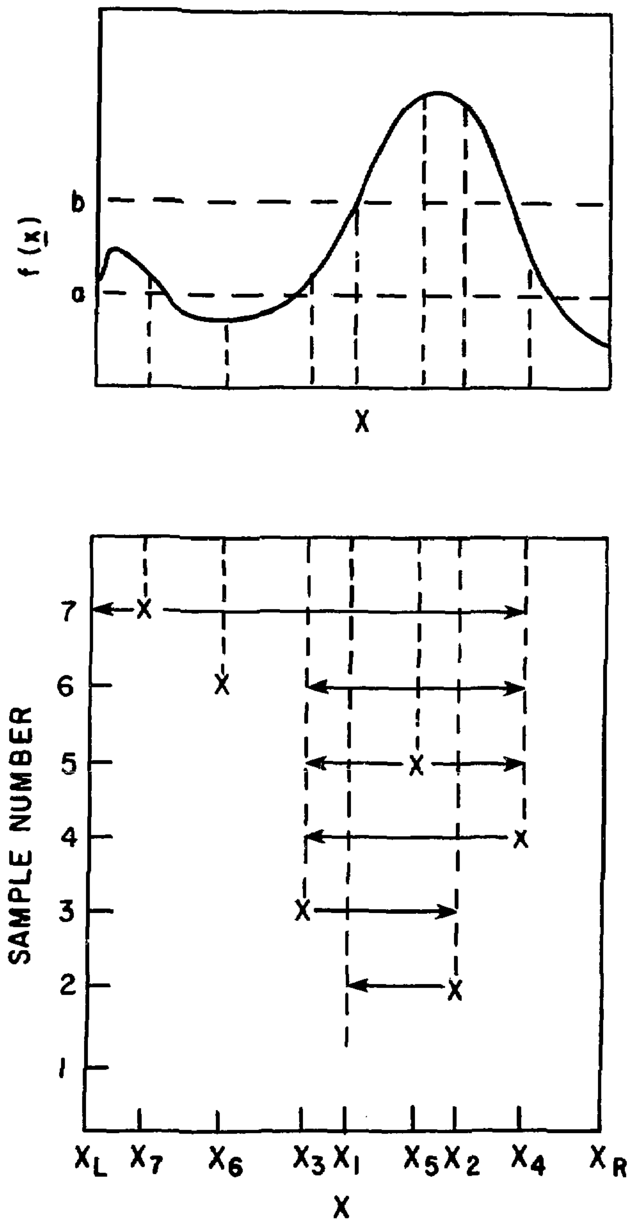
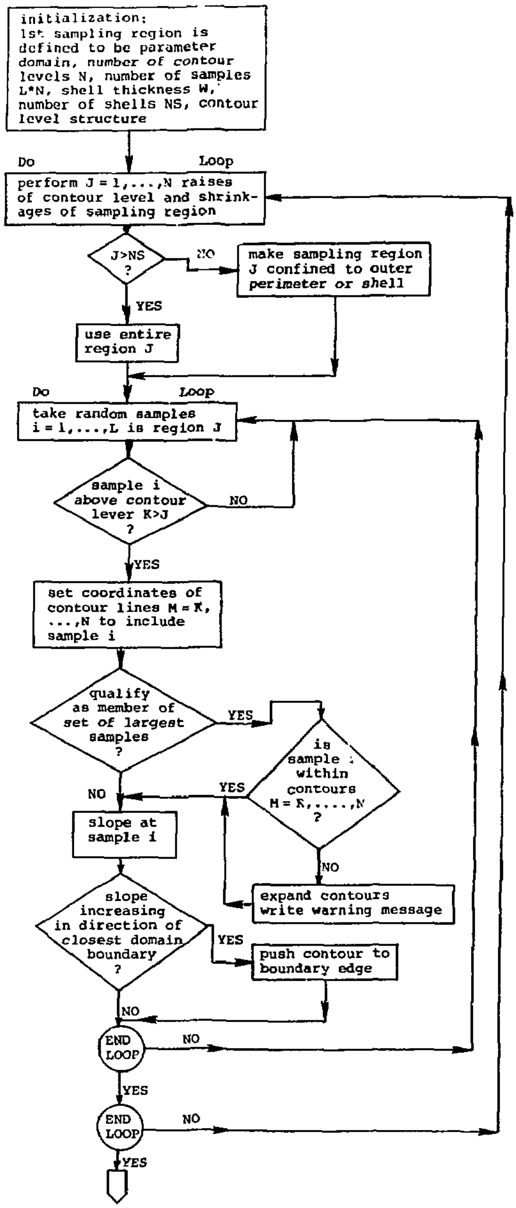


784192

Fig. 1. The figure on the bottom illustrates how a new sampling region containing $f(\underline{x}) > a$ is constructed from seven randomly chosen points within the original domain $\underline{x}_L \le \underline{x} \le \underline{x}_R$. The hypothetical function $f(\underline{x})$ is displayed on the top.



784195

Fig. 2. A flow chart of the scan phase.

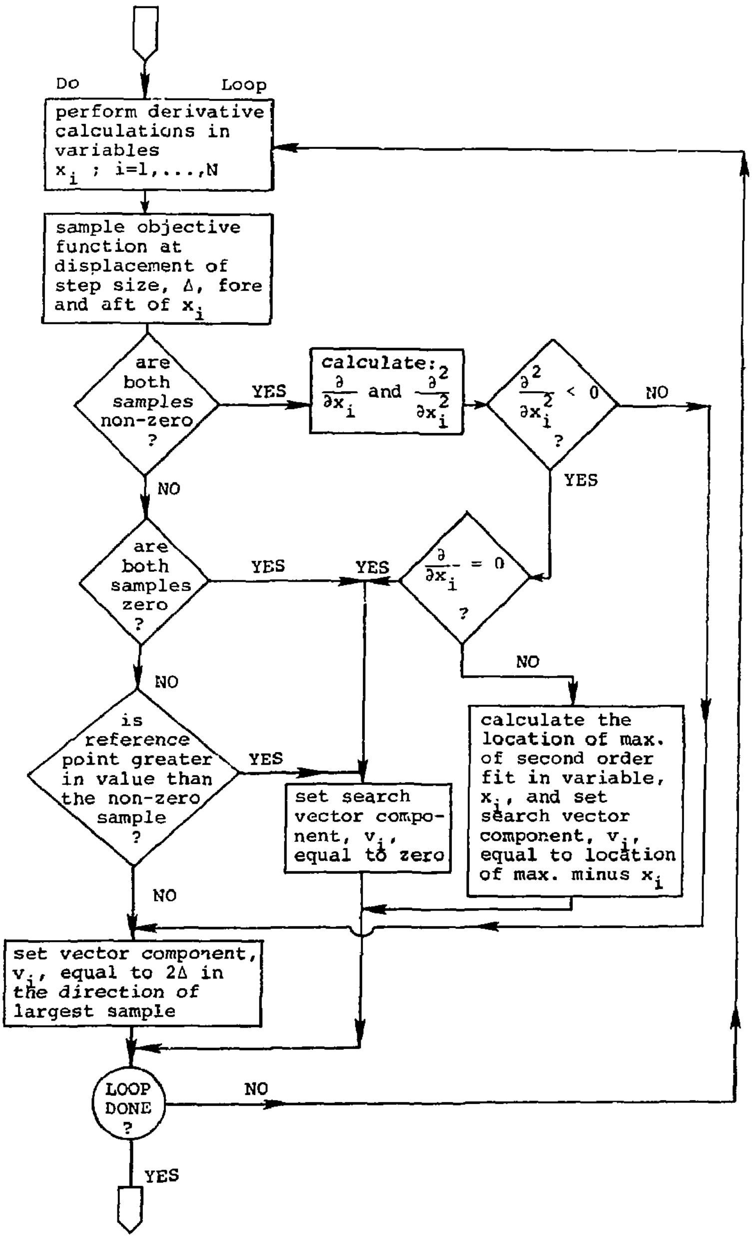


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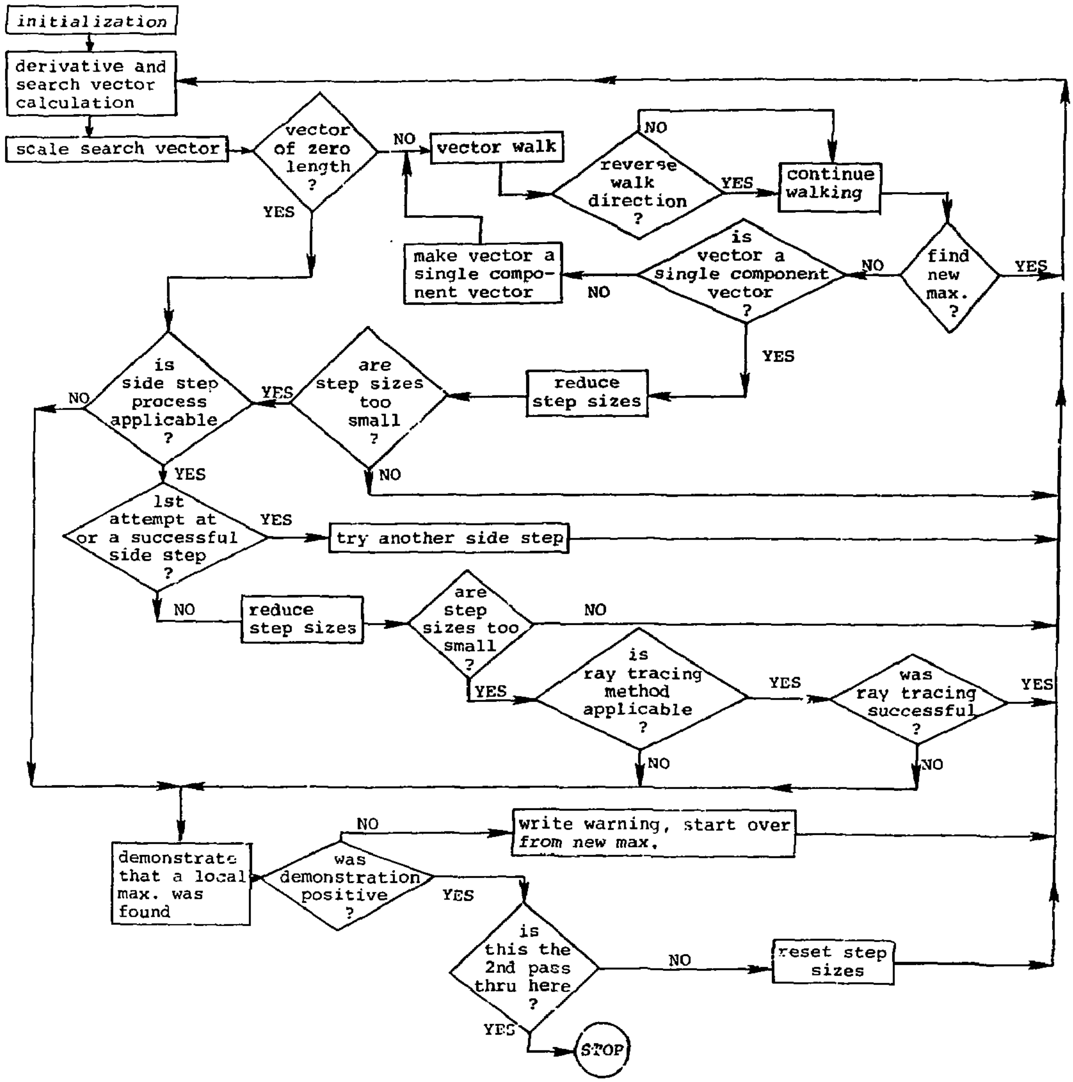
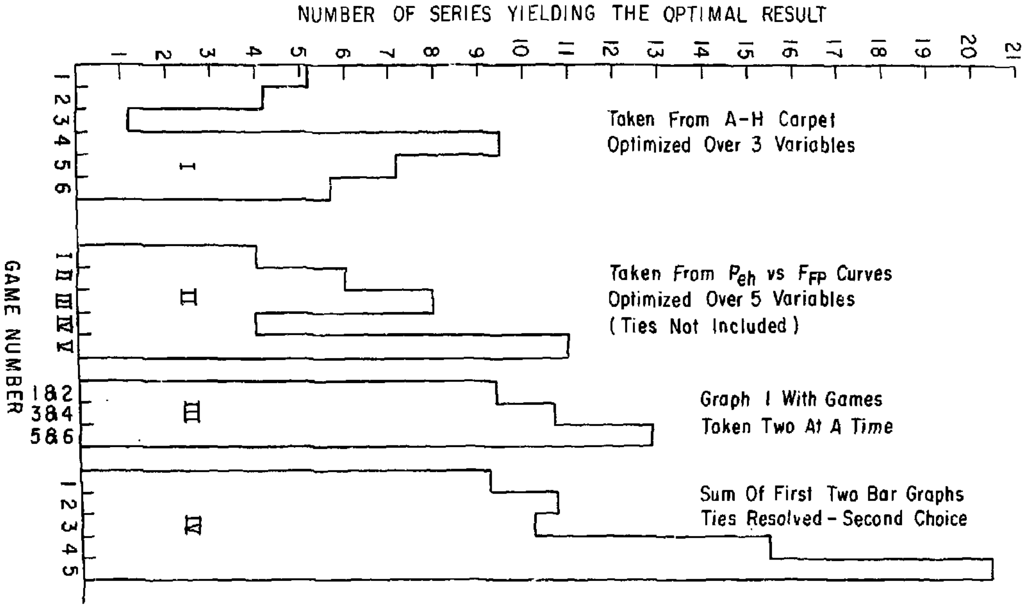
Fig. 3. A flow chart of the derivative and search vector calculations.



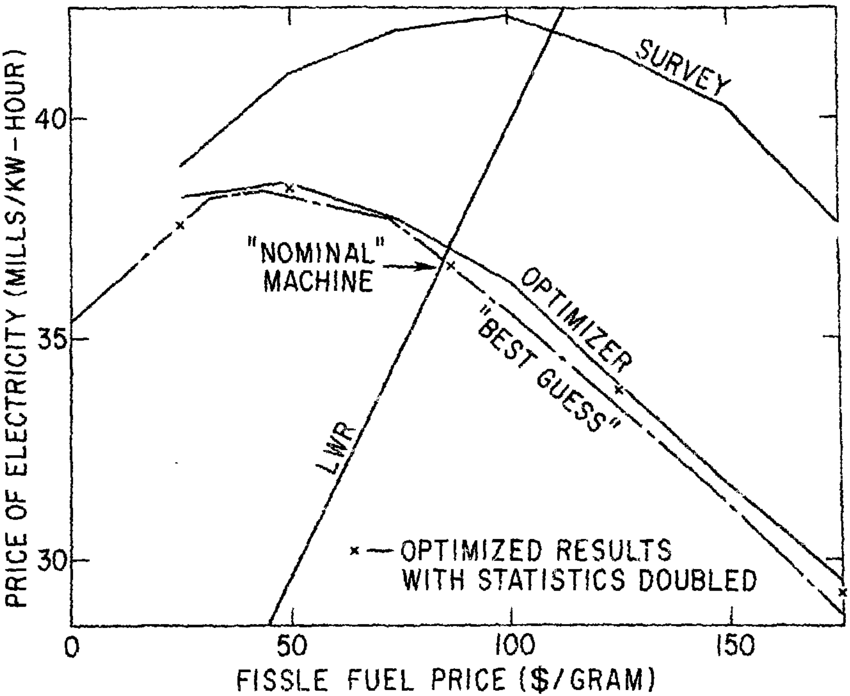
784208

Fig. 4. A flow chart of the ascent phase.



784125

Fig. 5. A demonstration of the benefit of retaining knowledge of contours between games in a series. Bar graph I is based upon a collection of 33 series of 3-D optimization results. Bar graph II is based upon a collection of 33 series of 5-D optimization results. Bar graph III is a replot of Bar graph I data per two game intervals. Bar graph IV is a sum of Bar graphs I and II.



774601

Fig. 6. A comparison of the effects of ...
statistics per game. Also shown ...

TABLE 1.

A list of the results of those games forming the data base of the first bar graph of Fig. 2, that had more than one game in a series generating the optimum.

Series	Game	3-Vector Location of Minimum			Minimum
		n_h/n_e	T	W_o	P_{eh}
I	2	0.048870	7.2551	150.00	44.078
	6	0.048870	7.2551	150.00	44.078
	1	0.065404	7.8996	150.00	43.874
	2	.065405	7.8989	150.00	43.874
II	3	.065396	7.8990	150.00	43.874
	5	.065359	7.8979	150.00	43.874
	6	.065339	7.8973	150.00	43.874
III	4	0.0097097	6.5591	196.02	38.923
	6	.0096996	6.5576	196.06	38.923
IV	4	0.11410	6.3542	150.00	50.292
	6	0.11411	6.3542	150.00	50.292
V	2	0.10804	6.7039	150.00	50.719
	4	0.10808	6.7042	150.00	50.719

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.92	36.92	36.94	36.98	37.01	37.02	37.03
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.81	36.84	36.88	36.89	36.90	36.91	36.92	36.93
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.75	36.78	36.79	36.80	36.81	36.82	36.83	36.84	36.85
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

T

W_0

Table 2-A. An exploratory cut in the W_o-T plane about the "nominal" machine.

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.74	36.76	36.76	36.76	0.	T
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	

n_h / n_e

Table 2-B. An exploratory cut in the n_h/n_e-T plane about the "nominal" machine.

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.88	36.97	37.02	37.06	37.11	37.16	37.21	37.26	37.31	T
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.04	37.31	37.55	37.61	37.68	37.74	37.81	37.88	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.28	37.63	38.05	38.55	38.68	38.77	38.87	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	38.07	38.59	39.23	39.99	40.48	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0	0.	0.	0.	0.	0.	0.	0.	38.65	39.31	40.11	41.88	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	39.41	40.25	41.26	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	41.46	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	

α

Table 2-C. An exploratory cut in the a-T plane about the "nominal" machine.

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.31
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.04	37.07	37.11	37.14
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.84	36.88	36.92	36.96	37.00	37.05	37.09
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	36.72	36.77	36.82	36.87	36.93	36.98	37.04	37.09	37.15	37.21
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.75	36.84	36.93	37.03	37.13	37.25	37.37	37.46	37.54
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.87	36.97	37.08	37.20	37.33	37.46	37.60
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.04	37.16	37.29	37.43	37.58
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.26	37.41	37.56
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.55
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

T

h

Table 2-D. An exploratory cut in the h-T plane about the "nominal" machine.

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	35.67	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	36.73	36.75	36.75	36.76	36.76	36.77	36.78	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	36.74	36.76	36.79	36.81	36.83	36.83	36.83	36.84	36.84
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.72	36.75	36.77	36.80	36.02	36.85	36.87	36.90	36.91	36.91
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.73	36.76	36.78	36.01	36.83	36.06	36.88	36.91	36.93	36.96
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.74	36.77	36.79	36.82	36.84	36.07	36.89	36.92	36.94	36.97
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.75	36.77	36.80	36.82	36.85	36.08	36.90	36.93	36.95	36.98
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.78	36.81	35.83	36.06	36.00	36.91	36.94	36.96	36.99
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.79	36.82	36.84	36.87	36.89	36.92	36.95	36.97	37.00
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.80	36.83	36.85	36.00	36.90	36.93	36.96	36.98	37.01
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.81	36.84	36.86	36.89	36.91	36.94	36.97	36.99	37.02

W_o

n_h/n_e

Table 2-E. An exploratory cut in the n_h/n_e-W_o plane about the "nominal" machine. The median area bounded by solid and dashed lines separates the penalized, enclosed by solid line, and the unpenalized regions under a penalty type constraint for penetration.

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.80	36.83	36.87	36.91	36.95	36.99	37.03	37.07	37.11	0.	W_0
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.80	36.95	36.99	37.04	37.08	37.12	37.17	37.21	37.26	37.31	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0..	0.	36.81	36.96	37.15	37.20	37.25	37.30	37.35	37.40	37.45	37.50	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.82	36.97	37.17	37.40	37.45	37.51	37.56	37.62	37.68	37.74	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.83	36.98	37.18	37.42	37.70	37.76	37.82	37.88	37.95	38.02	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.84	36.99	37.18	37.43	37.72	38.06	38.13	38.20	38.27	38.35	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.85	37.00	37.19	37.44	37.73	38.09	38.49	38.57	38.66	38.74	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.86	37.01	37.20	37.45	37.74	38.10	38.54	39.01	39.10	39.20	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.87	37.02	37.21	37.45	37.75	38.11	38.55	39.07	39.62	39.73	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.88	37.03	37.22	37.46	37.76	38.12	38.56	39.80	39.70	40.36	
										a											

Table 2-F. An exploratory cut in the a-W_o plane about the "nominal" machine. The median area bounded by solid and dashed lines separates the penalized, enclosed by solid line, and the unpenalized regions under a penalty type constraint for penetration.

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.91	36.95
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.83	36.87	36.92	36.96	37.01
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.75	36.79	36.84	36.89	36.94	36.99	37.04	37.10
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	36.72	36.77	36.82	36.87	36.93	36.98	37.04	37.09	37.15	37.21
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.75	36.83	36.91	36.98	37.04	37.10	37.16	37.22	37.29	37.35
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.76	36.84	36.92	37.01	37.10	37.21	37.31	37.39	37.46	37.53
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.77	36.85	36.93	37.02	37.11	37.21	37.32	37.44	37.56	37.69
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.78	36.85	36.94	37.03	37.12	37.22	37.33	37.45	37.57	37.70
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.79	36.86	36.95	37.04	37.13	37.23	37.34	37.46	37.58	37.71
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.80	36.87	36.96	37.04	37.14	37.24	37.35	37.47	37.59	37.72
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.80	36.88	36.96	37.05	37.15	37.25	37.36	37.48	37.60	37.73
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.81	36.89	36.97	37.06	37.16	37.26	37.37	37.49	37.61	37.74
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.82	36.90	36.98	37.07	37.17	37.27	37.38	37.50	37.62	37.75
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.83	36.91	36.99	37.08	37.18	37.28	37.39	37.51	37.63	37.76

W_0

h

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.82	36.97	37.02	37.07	37.12	37.17	37.22	37.28
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.85	36.94	36.99	37.04	37.09	37.13	37.18	37.24
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.87	36.92	36.96	37.01	37.05	37.10	37.15	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.74	36.85	36.89	36.93	36.98	37.02	37.07	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.77	36.83	36.87	36.91	36.96	37.00	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.77	36.81	36.85	36.89	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.76	36.80	36.84	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	36.75	36.79	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	36.75	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

n_n/n_e

a

Table 2-H. An exploratory cut in the a-n_n/n_e plane about the "nominal" machine.

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	36.80	36.89	36.99	37.09	37.20	37.32
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.74	36.83	36.92	37.02	37.12	37.24	37.36
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.69	36.77	36.86	36.95	37.05	37.16	37.27	37.39
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	36.80	36.88	36.98	37.08	37.19	37.30	37.39
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.74	36.82	36.91	37.01	37.11	37.21	37.28	37.35
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.69	36.77	36.85	36.94	37.04	37.12	37.18	37.25	37.32
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.72	36.79	36.88	36.97	37.03	37.10	37.16	37.22	37.29
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.74	36.82	36.90	36.96	37.02	37.08	37.14	37.20	37.26
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.69	36.77	35.83	36.89	36.94	37.00	37.06	37.12	37.18	37.24
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.72	36.77	36.83	36.88	36.93	36.99	37.05	37.10	37.16	37.22
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	36.72	36.77	36.82	36.87	36.93	36.98	37.04	37.09	37.15	37.21
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.72	36.77	36.82	36.87	36.92	36.98	37.03	37.08	37.14	37.19
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.73	36.77	36.82	36.87	36.92	36.97	37.02	37.08	37.13	37.18
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.73	36.78	36.82	36.87	36.92	36.97	37.02	37.07	37.12	37.18
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.78	36.83	36.87	36.92	36.97	37.02	37.07	37.12	37.17
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.79	36.83	36.88	36.93	36.97	37.02	37.07	37.12	37.17
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.84	36.89	36.93	36.98	37.02	37.07	37.12	37.17
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.85	36.90	36.94	36.98	37.03	37.08	37.12	37.17
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.87	36.91	36.95	36.99	37.04	37.08	37.13	37.17
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.92	36.96	37.00	37.05	37.09	37.14	37.18
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.93	36.97	37.02	37.06	37.10	37.14	37.19

n_h/n_e

h

Table 2-I. An exploratory cut in the h-n_h/n_e plane about the "nominal" machine.

0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.76	36.80	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.76	36.80	36.85	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	36.76	36.82	36.87	36.93	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.72	36.78	36.84	36.91	36.98	37.05	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.68	36.74	36.81	36.88	36.95	37.02	37.07	37.13	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.71	36.78	36.84	36.89	36.95	37.00	37.06	37.11	37.17	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.67	36.72	36.77	36.82	36.87	36.93	36.98	37.04	37.09	37.15	37.21	a
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.76	36.81	36.86	36.91	36.96	37.02	37.07	37.13	37.19	37.25	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.79	36.84	36.89	36.95	37.00	37.05	37.11	37.17	37.23	37.29	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.82	36.88	36.93	36.98	37.04	37.09	37.15	37.21	37.27	37.33	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.91	36.97	37.02	37.07	37.13	37.19	37.25	37.31	37.37	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.95	37.00	37.06	37.11	37.17	37.23	37.29	37.35	37.41	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	36.99	37.04	37.10	37.15	37.21	37.27	37.33	37.39	37.45	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.08	37.13	37.19	37.25	37.31	37.37	37.43	37.50	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.12	37.17	37.23	37.29	37.35	37.41	37.48	37.54	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.16	37.21	37.27	37.33	37.39	37.46	37.52	37.58	
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	37.20	37.25	37.31	37.37	37.44	37.50	37.56	37.63	

h

Table 2-J. An exploratory cut in the h-a plane about the "nominal" machine.

T keV	W_0 keV	$n_n / n_e \times 10^{-3}$: 1.76	1.80	1.84	1.92	1.96	2.00	2.04	2.08	2.12	2.16	2.20
5.91	146	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	150	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	153	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	156	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	159	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	162	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	165	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	169	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	172	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	175	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
	178	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
6.04		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	36.66	36.69	36.72	0.	0.	0.	0.	0.	0.	0.
		0.	36.68	36.73	36.70	36.92	36.84	36.87	36.90	36.94	36.98	37.03
		0.	36.70	36.75	36.80	36.34	36.89	36.94	36.98	36.99	37.02	37.05
		0.	36.72	36.77	36.81	36.86	36.91	36.96	37.01	37.07	37.12	37.13
		0.	36.74	36.79	36.83	36.88	36.93	36.90	37.03	37.09	37.14	37.19
6.16		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	36.67	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	36.74	36.79	36.83	36.63	36.84
		0.	0.	0.	0.	0.	0.	36.76	36.81	36.86	36.91	36.96
		0.	0.	0.	0.	0.	0.	36.77	36.82	36.88	36.93	35.98
		0.	0.	0.	0.	0.	0.	36.79	36.84	35.89	36.95	37.00
		0.	0.	0.	0.	0.	0.	36.81	36.86	36.91	36.97	37.02
6.28		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

Table 3. Cuts in n_h/n_e-W_o plane at four temperatures for constant a=2.8 m and h=8.0 m. A 5-dimensional search in n_h/n_e, W_o, T, a, and h concluded the minimum price of electricity P_{eh}=36.84 to be located at n_h/n_e=.002, W_o=162 keV, T=6.16 keV, a=2.8 m, and h=8.0 m.

TABLE 4.
Grid Locations Employed in Survey Calculation

T	6	7	8	9	10.5	12	15	18	keV
W_o	150	166	183	200	225	250	300		keV
n_h/n_e	0	.01	.03	.05	.075	.1	12	.3	
a	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	m
h	2.5	3.6	4.7	5.8	6.9	8.0		m	

TABLE 5.
Survey Results of the Minimum Price of Electricity
Calculated at Various Values of F_{FP}

F_{FP}	5-vector location of minimum					Minimum
	T	W	n_h/n_e	a	h	P_{eh}
25	8	150	.075	2.0	4.7	38.9
50	8	150	.075	2.0	4.7	41.0
75	7	183	.010	2.4	6.9	42.0
100	7	200	.010	2.8	5.8	42.3
125	7	200	.010	2.8	5.8	41.5
150	7	200	.010	2.8	6.9	40.2
175	9	166	.030	2.8	5.8	37.5

TABLE 6.
Search Algorithm Results of the Minimum Price of Electricity Calculated at Various Values of F_{FP}.

F_{FP}	5-vector location of minimum					Minimum
	T	W	n_h/n_e	a	h	P_{eh}
25	6.04	150	.0412	2.43	3.58	38.2
50	6.15	176	.0030	2.63	7.41	38.5
75	6.25	175	.0026	2.91	7.03	37.7
100	6.21	176	.0024	2.99	6.89	36.3
125	6.15	164	.0017	2.95	7.71	33.9
150	6.08	162	.0015	3.00	7.80	31.6
175	6.47	157	.0022	2.91	7.89	29.5

TABLE 7.
The P_{eh} as Calculated in all Games of the Series Upon Which Table 6 is Based.

F_{FP}	Game				
	1	2	3	4	5
25	*	38.7	38.3	38.2	38.5
50	40.0	38.8	39.2	39.1	38.5
75	39.7	37.9	37.7	*	38.3
100	36.6	37.3	36.3	36.4	36.7
125	34.2	34.4	34.8	33.9	34.5
150	32.3	32.0	31.9	32.2	31.6
175	29.5	29.5	29.7	29.6	30.0

* Lost surface during ascent phase of search algorithm

*	*	*	*	*	*	*	*	2.5
*	*	6.00 49.30 150 .126	7.26 44.08 150 .048	7.69 45.38 153 .043	8.40 46.75 151 .048	8.50 47.82 159 .040	7.78 48.15 178 .022	3.6
*	*	7.90 43.87 150 .065	8.26 44.41 152 .042	7.59 44.17 181 .020	6.99 42.56 194 .010	6.59 41.29 200 .005	6.18 40.78 207 .003	4.7
*	6.00 52.53 150 129	7.86 43.86 150 .040	6.56 42.83 196 .009	6.92 40.33 184 .007	6.23 38.32 193 .003	6.14 38.92 192 .002	6.04 40.01 195 .002	5.8
*	6.36 50.29 150 .114	6.49 43.67 182 .001	6.96 41.27 174 .008	6.47 37.92 175 .003	6.00 37.29 179 .001	6.01 38.76 183 .002	6.04 40.19 184 .002	6.0
*	6.70 50.72 150 .108	6.36 43.54 172 .008	7.12 40.35 159 .007	6.16 36.84 162 .002	6.15 37.57 167 .001	6.04 39.11 171 .002	6.00 40.32 173 .001	8.0
1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	H / A

Table 8. The price of electricity, P_{eh}, versus a and h. Each entry is the best result of a six game series with the search algorithm working in T, W_o, and n_h/n_e. The location of the minimum in the three variables is displayed about P_{eh} starting with T in the upper-left-hand corner and proceeding counter-clockwise are W_o and n_h/n_e.

TABLE 9.
The Results of an Application of the Ascent Phase of the Search Algorithm to Four Starting Locations Taken from Table 8.

Start	Minimum P_{eh}	5-vector location				
		T	W_o	n_h/n_e	a	h
(2.8,8.0)	36.81	6.1592	162.51	.0019592	2.8000	8.0000
(3.2,6.9)	36.75	6.0022	179.27	.0018814	3.0946	6.8129
(2.8,6.9)	37.15	6.4687	164.73	.0028804	2.8000	7.5900
(3.2,8.0)	36.45	6.0217	162.71	.0015549	2.8800	8.0000

TABLE 10.
The Results of a Standard Application of the Search Algorithm.

Game	Minimum P_{eh}	5-vector location of minimum				
		T	W_o	n_n/n_e	a	h
1	37.15	6.3659	167.86	.0027136	2.8247	7.4387
2	36.67	6.0964	168.47	.0018437	2.9395	7.4919
3	37.89	6.6402	172.24	.0040667	2.8774	6.7239
4	37.012	6.0465	180.45	.0020182	3.1175	6.6802
5	37.33	6.2867	174.11	.0027822	2.8583	7.0863
6	36.88	6.0820	176.32	.0020787	2.9948	6.9953

TABLE 11.
The Results of Taking Twice the Samples per Game as Taken in the Standard Application Together with Increasing the Number of Games in a Series

Game	Minimum	5-vector location of minimum				
	P_{eh}	T	W_o	n_h/n_e	a	h
1	36.61	6.0543	167.74	.0017555	2.9187	7.6019
2	37.37	6.1196	184.82	.0023823	3.1655	6.3404
3	37.86	6.5339	178.04	.0037960	2.9421	6.5210
4	37.00	6.1964	167.45	.0022390	2.8194	7.6230
5	38.81	6.6564	176.64	.0049819	2.7074	6.7638
6	37.33	6.5031	167.24	.0031045	2.8222	7.3638
7	36.85	6.1503	171.55	.0021426	2.9192	7.2858
8	37.47	6.2713	181.11	.0028998	2.9997	6.5462

TABLE 12.
The Results of Doubling the Samples per Game in Combination with Reducing the Size of the Domain.

P_{eh}	T	W_o	n_h/n_e	a	h
36.71	6.1036	169.48	.0019196	2.9295	7.4375
36.74	6.0957	171.51	.0019517	2.9552	7.2952
36.81	6.0124	176.96	.0019252	3.0031	7.0222

TABLE 13.
The Results of Increasing the Sample Size per Game by a Factor of Six Over the Standard Application.

P_{eh}	T	W_o	n_h/n_e	a	h
37.11	6.4076	166.05	.0026724	2.8469	7.4877
36.56	6.0560	163.91	.0016270	2.9063	7.8522
36.94	6.3445	162.14	.0023130	2.8309	7.8220
36.88	6.2678	166.30	.0022620	2.8702	7.5599
36.65	6.0636	168.33	.0018155	2.9076	7.5729
37.11	6.0681	182.38	.0021768	3.1101	6.5697