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NACHOS -- A FINITE ELEMENT COMPUTER PROGRAM FOR INCOMPRESSIBLE FLOW PROBLEMS

PART I - THEORETICAL BACKGROUND

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PART I - THEORETICAL BACKGROUND

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ABSTRACT

The theoretical background for the finite element computer program, NACHOS, is presented in detail. The NACHOS code is designed for the two-dimensional analysis of viscous incompressible fluid flows, including the effects of heat transfer. A general description of the fluid/thermal boundary value problems treated by the program is described. The finite element method and the associated numerical methods used in the NACHOS code are also presented. Instructions for use of the program are documented in SAND77-1334.

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INTRODUCTION

This document describes the theoretical and numerical background for the finite element computer code called NACHOS. The NACHOS program is designed for the analysis of two-dimensional, viscous, incompressible fluid flow problems including the effects of heat transfer. The code has been designed with the dual purpose of being an easily modified research tool for finite element methods in fluid dynamics and as a user-oriented analysis package for typical engineering problems.

The present document presents a description of the theoretical fluid mechanics background for the code as well as a discussion of the most important numerical procedures utilized by NACHOS. This volume is intended to serve as a background document for the NACHOS user's manual,[1] SAND77-1334, "NACHOS -- A Finite Element Computer Program for Incompressible Flow Problems, Part II - User's Manual." Potential users of NACHOS are encouraged to become familiar with the present report before attempting to use the program.

During the development of the present report, it has been assumed that the reader has a background in the areas of fluid mechanics and heat transfer. A basic knowledge of numerical methods is also essential for the proper understanding and use of the computer code. Though it has not been assumed that the reader has a background in finite element methods, a general acquaintance with the method is highly advantageous. Since by necessity many of the topics covered here are discussed only in the context of the present code application, the reader interested in general finite element methods is referred to the standard texts by Zienkiewicz,[2] Huebner,[3] Gallagher[4] or Oden.[5]

In the following section, the general class of flow problems analyzed by NACHOS is discussed and the equations for the initial, boundary value problem are presented. Section 3 presents a brief description of the finite element method (FEM) and the Galerkin formulation for the present class of problems. Sections 4 and 5 discuss computational details of individual element formulation and solution procedures for the matrix equations, respectively. The last section describes some of the special procedures used in NACHOS for auxillary calculations, e.g., stress calculations, stream function calculations, etc.

FORMULATION OF THE CONTINUUM PROBLEM

A necessary prerequisite to the development of a general purpose computer program such as NACHOS is the careful definition of the class of problems to which the code will be applied. In terms of general categorization, the NACHOS program is designed for the analysis of non-isothermal fluid flow problems. Included in this category are isothermal flows, forced convection, free convection and mixed convection heat transfer problems. Solid body heat conduction effects are also treated in this development.

To be more specific, the following restrictions and assumptions have been used to define the problem areas of interest.

1) The geometry of the fluid/solid region is limited to two dimensions, either planar or axially symmetric.
2) The fluid is assumed to be Newtonian and incompressible within the Boussinesq approximation.[6]
3) All materials are assumed homogeneous and isotropic; the fluid is assumed to be composed of a single species.
4) The fluid motion is assumed to be laminar.
5) The effects of viscous dissipation are assumed negligible.
6) Fluid flows with free surface boundaries are not considered.

To simplify the derivation of the equations in later sections, only the case of plane two-dimensional flow will be treated in detail. Derivation of the axisymmetric form of the equations follows in a straightforward manner. With the assumptions noted above, the appropriate mathematical description of the fluid motion is given by the Navier-Stokes equations,

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} - \rho g_i + \rho g_i \beta (T - T_{ref}) - \frac{\partial \tau_{ij}}{\partial x_j} = 0 \quad , \tag{1}$$

with,

$$\tau_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad .$$

The condition of fluid incompressibility is enforced through the equation,

$$\frac{\partial u_i}{\partial x_i} = 0 \quad . \tag{2}$$

The transport of thermal energy in the fluid is described by,

$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p u_j \frac{\partial T}{\partial x_j} + \frac{\partial q_j}{\partial x_j} - S = 0 \quad , \tag{3a}$$

with,

$$q_j = -k \frac{\partial T}{\partial x_j} \quad ,$$

while in the solid region,

$$\rho C_p \frac{\partial T}{\partial t} + \frac{\partial q_j}{\partial x_j} - S = 0 \quad , \tag{3b}$$

is the appropriate equation.

In Equations (1) through (3), t is the time, u_i is the velocity component in the x_i coordinate direction, P is the pressure, T the temperature, ρ the density, τ_{ij} the stress tensor, q_i the heat flux vector, S the volumetric heat source, μ the viscosity, C_p the heat capacity, k the thermal conductivity and β the coefficient of volume expansion. Also, T_{ref} is a reference temperature for which buoyancy forces are zero, δ_{ij} is the identity tensor and g_i the gravitational constant (assumed to act in the negative y direction).

The material properties such as μ, k and β are in general functions of two thermodynamic variables such as pressure and temperature. In the present case, the dependence on pressure is assumed to be negligible; the material properties are assumed to vary with temperature only. The volumetric heat source is assumed to vary with temperature and time.

To complete the formulation of the initial/boundary value problem, suitable boundary and initial conditions for the dependent variables are required. For the hydrodynamic part of the problem, either the velocity components or the total surface stress (or traction) must be specified on the boundary of the fluid region. The thermal part of the problem requires a temperature or heat flux condition to be specified on all parts of the boundary. Symbolically, these conditions are expressed by,

$$u_i = f_i(s) \text{ on } \Gamma_u \quad ,$$
$$t_i = \tau_{ij}(s) n_j(s) \text{ on } \Gamma_t \quad , \tag{4}$$

for the fluid and,

$$T = g(s) \text{ on } \Gamma_T \quad ,$$
$$\left[q_a(s) + q_c(s) + q_r(s)\right] = \left(k \frac{\partial T}{\partial x_i}\right) n_i(s) \text{ on } \Gamma_q \quad , \tag{5}$$

for the heat transfer problem. In Equations (4) and (5), s is the coordinate along the boundary, n_j is the outward unit normal to the boundary, $\Gamma_f = \Gamma_u + \Gamma_t$ is the total boundary enclosing the fluid region Ω_f, and $\Gamma_e = \Gamma_T + \Gamma_q$ is the total boundary enclosing the energy transfer region, Ω_e. The flux boundary condition for the energy equation has been expressed as the sum of three parts where q_a is the applied heat flux, q_c is the heat flux due to convection and q_r is the heat flux due to radiation. Typically, the convective and radiative heat fluxes are given by,

$$q_c = h_c(T - T_c) \quad ,$$
$$q_r = h_r(T - T_r) \quad , \tag{6a}$$

where h_c and h_r are convective and radiative heat transfer coefficients and T_c and T_r are equilibrium temperatures for which no convection or radiation occurs. The radiation coefficient is given by,

$$h_r = \varepsilon\sigma(T^2 + T_r^2)(T + T_r) \quad , \tag{6b}$$

in which ε is the emissivity and σ is the Stefan-Boltzmann constant.

The initial conditions for the boundary value problem consist of specifying the value of each dependent variable at the initial time for all points in the appropriate region. That is,

$$\left.\begin{aligned} u_i &= a_i(x_i) \\ P &= b(x_i) \end{aligned}\right\} \text{ at } t = 0 \text{ on } \Omega_f \quad ,$$

$$T = c(x_i) \qquad \text{at } t = 0 \text{ on } \Omega_e \quad , \tag{7}$$

where Ω_f and Ω_e are the fluid and energy transfer regions, respectively.

Equations (1) through (3) with the boundary and initial conditions, Equations (4) through (7), form a complete set for the determination of the velocity, pressure and temperature fields in a fluid and the temperature field in a solid. In general, the velocity field is coupled to the temperature field through the body force term (buoyancy) and the appearance of temperature dependent material properties in Equation (1). The relative importance of this coupling is a function of the magnitude of the material properties and the types of boundary conditions imposed on the problem. A discussion of the specialization of the equations for cases of strong or weak coupling is included in the section on solution methods.

FORMULATION OF THE FINITE ELEMENT EQUATIONS

The boundary value problem outlined in the previous section is generally not amenable to closed form solution except in cases where problem geometry may be regularized and/or physical phenomena neglected. For the solution of realistic problems, one is forced to consider approximate solution methods of which the computerized numerical schemes are the most powerful. The currently popular numerical methods are generally divided into two groups--finite difference methods (FDM) and finite element methods (FEM). The objective of both approaches is to reduce the continuous problem (infinite number of degrees of freedom) described by a partial differential equation to a discrete problem (finite number of degrees of freedom) described by a system of algebraic equations. Though the ultimate results of both procedures are very similar, the procedures are sufficiently different in their philosophies and implementation to be considered distinct.

It is beyond the scope of the present discussion to describe either method in general. Rather, the approach followed here will be to describe in some detail the particular FEM used in developing the NACHOS program.

The finite element procedure begins with the division of the continuum region of interest into a number of simply shaped regions called finite elements as shown in Figure 1. Since the Eulerian description of the fluid motion was used in the field equations (1) through (3), these elements are assumed to be fixed in space. Within each element, the dependent variables (u_i, P and T) are interpolated by continuous functions of compatible order, in terms of values to be determined at a set of nodal points. For purposes of developing the equations for these nodal point unknowns, an individual element may be separated from the assembled system.

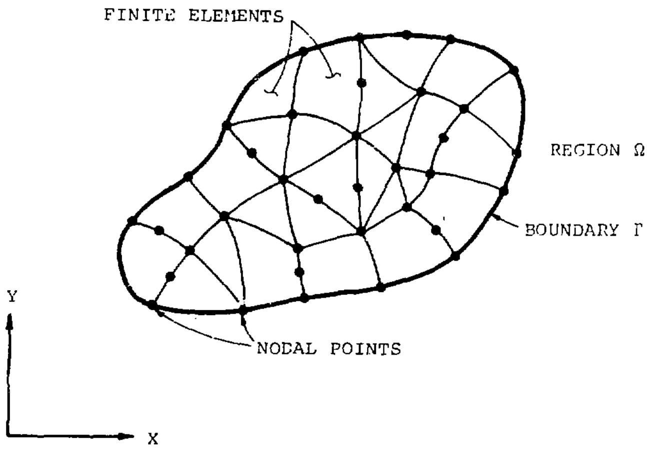

FIGURE 1

Within each element, the velocity, pressure and temperature fields are approximated by,

$$u_i(x_i,\ t) = \underset{\sim}{\Phi}^T(x_i) \cdot \underset{\sim}{u}_i(t) \quad ,$$

$$P(x_i,\ t) = \underset{\sim}{\Psi}^T(x_i) \cdot \underset{\sim}{P}(t) \quad , \qquad (8)$$

$$T(x_i,\ t) = \underset{\sim}{\Theta}^T(x_i) \cdot \underset{\sim}{T}(t) \quad ,$$

where the $\underset{\sim}{u}_i$, $\underset{\sim}{P}$ and $\underset{\sim}{T}$ are vectors of element nodal point unknowns, $\underset{\sim}{\Phi}$, $\underset{\sim}{\Psi}$ and $\underset{\sim}{\Theta}$ are vectors of interpolation functions and superscript T denotes a vector transpose.

Substitution of these approximations into the field equations (1) through (3) and boundary conditions, Equations (4) and (5), yields a set of equations of the form, Momentum: $\underset{\sim}{f}_1(\underset{\sim}{\Phi}, \underset{\sim}{\Psi}, \underset{\sim}{\Theta}, \underset{\sim}{u}_i, \underset{\sim}{P}, \underset{\sim}{T}) = \underset{\sim}{R}_1$;

Incompressibility: $f_2(\underset{\sim}{\Phi}, \underset{\sim}{u}_i) = R_2$; (9)

Energy: $f_3(\underset{\sim}{\Theta}, \underset{\sim}{\Phi}, \underset{\sim}{T}, \underset{\sim}{u}_i) = R_3$;

where R_i are the residuals (errors) resulting from the use of the approximations in Equation (8).

The Galerkin form of the Method of Weighted Residuals[7] seeks to reduce these errors to zero, in a weighted sense, by making the residuals orthogonal to the interpolation functions (Equation (8)) over each element. These orthogonality conditions are expressed by,

$$\langle \underset{\sim}{f}_1, \underset{\sim}{\Phi} \rangle = \langle \underset{\sim}{R}_1, \underset{\sim}{\Phi} \rangle = 0 \quad ;$$

$$\langle f_2, \underset{\sim}{\Psi} \rangle = \langle R_2, \underset{\sim}{\Psi} \rangle = 0 \quad ; \tag{10}$$

$$\langle f_3, \underset{\sim}{\Theta} \rangle = \langle R_3, \underset{\sim}{\Theta} \rangle = 0 \quad ;$$

where $\langle , \rangle$ denotes the inner product, defined by,

$$\langle a, b \rangle = \int_V a \cdot b \, dV \quad ,$$

with V being the volume of the element.

The detailed manipulations involving the integrals defined in Equation (10) are presented in Appendix A. The results of those computations can be expressed by the matrix equations,

Momentum: $\underset{\approx}{M}\dot{\underset{\sim}{V}} + \underset{\approx}{C}(\underset{\sim}{u})\underset{\sim}{V} + \underset{\approx}{K}(\underset{\sim}{T})\underset{\sim}{V} = \underset{\sim}{F}(\underset{\sim}{T})$, (11)

and,

Energy: $\underset{\approx}{N}\dot{\underset{\sim}{T}} + \underset{\approx}{D}(\underset{\sim}{u})\underset{\sim}{T} + \underset{\approx}{L}(\underset{\sim}{T})\underset{\sim}{T} = \underset{\sim}{G}(\underset{\sim}{T})$, (12)

with

$$\underset{\sim}{u}^T = (\underset{\sim}{u}_1^T, \underset{\sim}{u}_2^T) \quad ,$$

and,

$$\underset{\sim}{V}^T = (\underset{\sim}{u}_1^T, \underset{\sim}{u}_2^T, \underset{\sim}{P}^T) \quad .$$

The matrix equations in Equations (11) and (12) represent the discrete analogues of the conservation equations for an individual fluid finite element. Note that the $\underset{\approx}{C}$ and $\underset{\approx}{D}$ matrices represent the advection (convection) of momentum and energy, respectively; the $\underset{\approx}{K}$ and $\underset{\approx}{L}$ matrices represent the diffusion of momentum and energy (the $\underset{\approx}{K}$ matrix also contains the incompressibility constraint). The $\underset{\approx}{M}$ and $\underset{\approx}{N}$ matrices represent the mass and capacitance terms in the field equations. The $\underset{\sim}{F}$ and $\underset{\sim}{G}$ vectors provide the forcing functions for the system in terms of volume forces (body force, volumetric heating) and surface forces (stress, heat flux).

For the case where a solid (non-flowing) material is to be represented, then only the capacitance, diffusion and force terms of Equation (12) need to be considered, i.e.,

$$\underset{\approx}{N}\dot{\underset{\sim}{T}} + \underset{\approx}{L}(\underset{\sim}{T})\underset{\sim}{T} = \underset{\sim}{G}(\underset{\sim}{T}) \quad , \tag{13}$$

which is the discrete analog for the transient heat conduction equation.

The above derivation has been concerned with a single finite element and the limited portion of the continuum it represents. The discrete representation of the entire continuum region of interest is obtained through an assemblage of elements such that interelement continuity of the approximate velocity, pressure and temperature is enforced. This continuity requirement is met through the appropriate summation of equations for nodes common to adjacent elements (the so-called "direct stiffness" approach[2]). The result of such an assembly process is a system of matrix equations of the form given by Equations (11) through (13).

The matrix equations given in Equations (11) through (13) symbolically describe the FEM as applied to a general convective heat transfer problem. In the next section, some details of the construction of the matrix equations for particular finite elements are given. The actual solution procedures for Equations (11) through (13) are described in the subsequent chapter.

ELEMENT CONSTRUCTION

Of central importance to the development of a finite element code is the choice of particular elements to be included in the element library. For the NACHOS code, four basic elements were selected: a subparametric and isoparametric quadrilateral and a subparametric and isoparametric triangle. The basic concepts of finite element construction and the isoparametric element formulation are thoroughly described in References 2-5.

Quadrilateral Elements

The basic quadrilateral element used in the NACHOS code is an eight node, twenty degree of freedom element shown in Figure 2. The velocity components (u_i) and temperature (T) are approximated using quadratic interpolation functions; the pressure (P) and material properties (μ, k and β) are approximated by linear functions. The interpolation functions for this element are given by the vectors,

$$\underset{\sim}{\Phi} = \underset{\sim}{\Theta} = \frac{1}{2} \begin{Bmatrix} \frac{1}{2}(1 - s)(1 - t)(-s - t - 1) \\ \frac{1}{2}(1 + s)(1 - t)(s - t - 1) \\ \frac{1}{2}(1 + s)(1 + t)(s + t - 1) \\ \frac{1}{2}(1 - s)(1 + t)(-s + t - 1) \\ (1 - s^2)(1 - t) \\ (1 + s)(1 - t^2) \\ (1 - s^2)(1 + t) \\ (1 - s)(1 - t^2) \end{Bmatrix} \qquad (14)$$

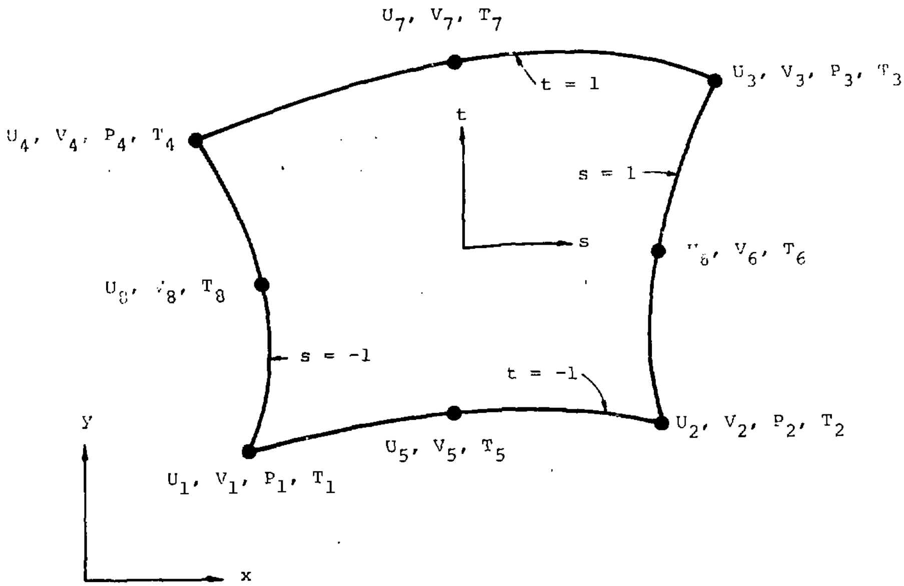

FIGURE 2

$$\underset{\sim}{\Psi} = \underset{\sim}{\eta} = \tfrac{1}{4}\begin{Bmatrix}(1 - s)(1 - t)\\(1 + s)(1 - t)\\(1 + s)(1 + t)\\(1 - s)(1 + t)\end{Bmatrix} \tag{15}$$

where the ordering of the functions corresponds to the ordering of unknowns shown in Figure 2. The shape functions given in Equations (14) and (15) are expressed in terms of the normalized or natural coordinates for the element, s and t, which vary from -1 to +1 as shown in the figure. The relationship between the physical coordinates x, y (or r, z) and the natural coordinates s, t is obtained from the parametric concept discussed by Ergatoudis, et al.[8] That is, the coordinate transformation is given by,

$$x = \underset{\sim}{N}^T\underset{\sim}{x} \ ; \ y = \underset{\sim}{N}^T\underset{\sim}{y} \ ; \ \underset{\sim}{N}^T = \underset{\sim}{N}^T(s, t) \quad , \tag{16}$$

where $\tilde{N}$ is a vector of interpolation functions over the element and $\tilde{x}$, $\tilde{y}$ are vectors of coordinates describing the geometry of the element (generally, nodal point coordinates).

The transformation given in Equation (16) is quite general and allows for the generation of curved-sided elements. In the present case, if $\tilde{N} = \tilde{\Phi}$, a quadratic interpolation of the element boundary is possible and the element is said to be isoparametric (i.e., the functions defining the dependent variables are of the same order as the functions defining the element geometry). If $\tilde{N} = \tilde{\Psi}$, a linear interpolation of the element boundary is possible and the element is subparametric.

The construction of the finite element matrices defined by Equations (A7) through (A15) in Appendix A requires the computation of various derivatives and integrals of the interpolation functions given in Equations (14) and (15). Since the basis functions are given in terms of the normalized coordinates, s and t, and the derivatives and integrals are in terms of the physical x, y coordinates the following relations need to be defined,

$$\begin{Bmatrix} \dfrac{\partial \tilde{\Phi}}{\partial s} \\ \\ \dfrac{\partial \tilde{\Phi}}{\partial t} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \tilde{\Phi}}{\partial x} \\ \\ \dfrac{\partial \tilde{\Phi}}{\partial y} \end{Bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial \tilde{N}^T \tilde{x}}{\partial s} & \dfrac{\partial \tilde{N}^T \tilde{y}}{\partial s} \\ \\ \dfrac{\partial \tilde{N}^T \tilde{x}}{\partial t} & \dfrac{\partial \tilde{N}^T \tilde{y}}{\partial t} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \tilde{\Phi}}{\partial x} \\ \\ \dfrac{\partial \tilde{\Phi}}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \tilde{\Phi}}{\partial x} \\ \\ \dfrac{\partial \tilde{\Phi}}{\partial y} \end{Bmatrix} , \qquad (17)$$

where $[J]$ is the Jacobian matrix. Inverting the Jacobian provides the needed relation for the derivatives of the basis functions,

$$\begin{Bmatrix} \frac{\partial \underset{\sim}{\Phi}}{\partial x} \\ \frac{\partial \underset{\sim}{\Phi}}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial \underset{\sim}{\Phi}}{\partial s} \\ \frac{\partial \underset{\sim}{\Phi}}{\partial t} \end{Bmatrix} \quad . \tag{18}$$

Equation (18) has been written for the quadratic basis function $\underset{\sim}{\Phi}$; similar expressions are available for the remaining functions defined in Equations (14) and (15). To complete the transformation from physical coordinates (x, y) to normalized coordinates, the expression for an elemental area is required. Thus,

$$dx\ dy = \det[J]\ ds\ dt \quad , \tag{19}$$

where det indicates the determinant of a matrix.

Use of the relations given in Equations (18) and (19) allows the element matrices defined in Appendix A to be expressed as integrals of rational functions in the s, t coordinate system. The evaluation of such integrals requires a numerical quadrature procedure. In the NACHOS code, the quadrilateral element is evaluated using a Gauss quadrature formula. Further details of this procedure are given in Appendix B.

Triangular Elements

A companion element to the parametric quadrilateral is the parametric triangle. In the NACHOS code, the triangular element is a six node, fifteen degree of freedom element shown in Figure 3. As in the quadrilateral, the velocity components and temperature are approximated quadratically over the triangle; the pressure and material properties use linear functions. The interpolation functions for this element are given by,

$$\underset{\sim}{\Phi} = \underset{\sim}{\Theta} = \begin{Bmatrix} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_1L_3 \end{Bmatrix} \tag{20}$$

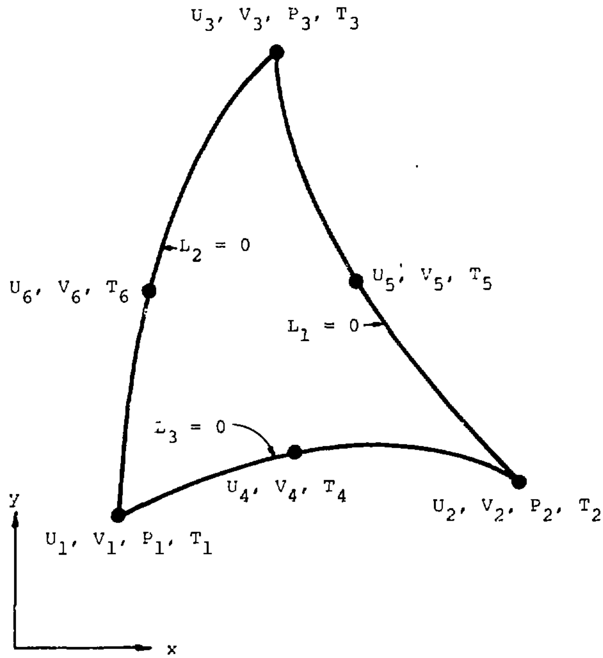

FIGURE 3

$$\underset{\sim}{\Psi} = \underset{\sim}{\eta} = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \qquad (21)$$

where the ordering of the functions corresponds to the ordering of the unknowns shown in Figure 3. The basis functions in Equations (20) and (21) are expressed in terms of the natural or area coordinates for a triangle.[2] Note that the coordinates L_i are not independent but are related by,

$$L_1 + L_2 + L_3 = 1 \quad . \qquad (22)$$

The parametric mapping concept may also be used with the triangular element by defining,

$$x = \underset{\sim}{N}^T \underset{\sim}{x} \; ; \; y = \underset{\sim}{N}^T \underset{\sim}{y} \; ; \; \underset{\sim}{N}^T = \underset{\sim}{N}^T (L_i) \quad , \tag{23}$$

where $\underset{\sim}{N}$ is given by Equation (20) for an isoparametric triangle and by Equation (21) for a subparametric element.

Following the procedure used for the quadrilateral element, the following relations may be defined,

$$\begin{Bmatrix} \dfrac{\partial \underset{\sim}{\Phi}}{\partial L_1} \\ \dfrac{\partial \underset{\sim}{\Phi}}{\partial L_2} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{N}^T \underset{\sim}{x}}{\partial L_1} & \dfrac{\partial \underset{\sim}{N}^T \underset{\sim}{y}}{\partial L_1} \\ \dfrac{\partial \underset{\sim}{N}^T \underset{\sim}{x}}{\partial L_2} & \dfrac{\partial \underset{\sim}{N}^T \underset{\sim}{y}}{\partial L_2} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \underset{\sim}{\Phi}}{\partial x} \\ \dfrac{\partial \underset{\sim}{\Phi}}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \underset{\sim}{\Phi}}{\partial x} \\ \dfrac{\partial \underset{\sim}{\Phi}}{\partial y} \end{Bmatrix} \quad , \tag{24}$$

or,

$$\begin{Bmatrix} \dfrac{\partial \underset{\sim}{\Phi}}{\partial x} \\ \dfrac{\partial \underset{\sim}{\Phi}}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \underset{\sim}{\Phi}}{\partial L_1} \\ \dfrac{\partial \underset{\sim}{\Phi}}{\partial L_2} \end{Bmatrix} \quad , \tag{25}$$

and,

$$dx\, dy = \det[J]\, dL_1\, dL_2 \quad , \tag{26}$$

where L_3 has been expressed in terms of L_1 and L_2 using Equation (22).

The element matrices defined by the integral equations in Appendix A and the above relations require evaluation by numerical quadrature. The NACHOS code uses a seven point quadrature formula for the triangular element. Details of the evaluation procedure are given in Appendix B.

<u>Element Boundary Conditions and Source Terms</u>

The previous sections have described the most important points in the evaluation of the finite element coefficient matrices for a particular choice of elements. However, in constructing the nodal point force vectors, defined by Equations (A11) and (A15) a number of additional assumptions are used that require further comment.

The force vectors for both the momentum and energy equations consist of two parts; a part due to volumetric forces (sources) and a part due to surface forces (fluxes). Considering first the volumetric terms, of which the heat source is typical,

$$\underset{\sim}{G}_S = \int_V \underset{\sim}{\Theta} S \; dV \quad . \tag{27}$$

As given previously, the elemental volume (area for two-dimensional problems) can be expressed in terms of normalized or natural coordinates by Equations (19) or (26). To complete the evaluation of the integral, the variation of the source over the element in terms of the element coordinates is required. In the present version of NACHOS source terms (i.e., S and g_i) are assumed to be uniform over each element allowing Equation (27) to be evaluated in a straightforward manner.

The evaluation of the boundary integrals containing applied surface forces or fluxes also requires an assumption about the variation of the integrand with position in the element. As an example, consider the heat flux term,

$$\underset{\sim}{G}_F = \int_A \underset{\sim}{\Theta} q_j n_j \; dA = \int_A \underset{\sim}{\Theta} q_n \; dA \quad ,$$

which for a plane two-dimensional problem becomes,

$$\underset{\sim}{G}_F = \int_\Gamma \underset{\sim}{\Theta} q_n \; d\Gamma \quad , \tag{28}$$

where Γ is the boundary of the element and q_n the heat flux normal to the boundary. The boundary segment $d\Gamma$ may be expressed as,

$$d\Gamma = \frac{\partial x}{\partial s} ds + \frac{\partial y}{\partial s} ds \quad , \tag{29}$$

where s is the coordinate along the boundary (see Figure 4). As given previously,

$$x = \underset{\sim}{N}^T \underset{\sim}{x} \; ; \; y = \underset{\sim}{N}^T \underset{\sim}{y}$$

which allows Equation (28) to be written as,

$$\underset{\sim}{G}_F = \int_{-1}^{1} \underset{\sim}{\Theta} \left(\frac{\partial \underset{\sim}{N}^T \underset{\sim}{x}}{\partial s} + \frac{\partial \underset{\sim}{N}^T \underset{\sim}{y}}{\partial s} \right) q_n \, ds \quad . \tag{30}$$

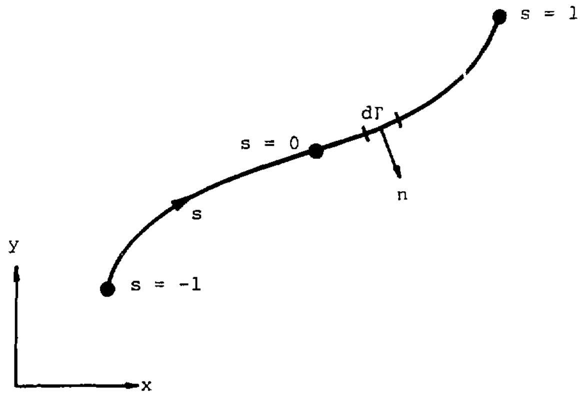

FIGURE 4

In Equation (30) the $\underset{\sim}{N}$ vector may be a set of either linear or quadratic functions depending on the type of element. Note that all functions in Equation (30) are restricted to an element boundary (edge function). That is,

$$\underset{\sim}{\Theta} = \begin{Bmatrix} s(s-1)/2 \\ (1-s^2) \\ s(s+1)/2 \end{Bmatrix} \; ; \; \underset{\sim}{N} = \begin{Bmatrix} s(s-1)/2 \\ (1-s^2) \\ s(s+1)/2 \end{Bmatrix} \text{ or } \begin{Bmatrix} (1-s)/2 \\ (1+s)/2 \end{Bmatrix} \quad . \tag{31}$$

To complete the specification of $\underset{\sim}{G}_F$, the variation of the heat flux, q_n, along the element boundary is required. From Equations (5) and (6), the heat flux is given by,

$$q_n = q_a(s) + h_c(T - T_c) + h_r(T - T_r) \quad . \tag{32}$$

In the present version of NACHOS, the applied heat flux is assumed constant along an element boundary; the convective heat transfer coefficient and sink temperature are also assumed constant. For the radiative flux, the emissivity and sink temperature are assumed constant (see Equation (6b)). Since the temperatures given in Equation (32) are temperatures on the element boundary, they are interpolated using the edge functions defined in Equation (31). With the above considerations, the $\underset{\sim}{G}_F$ vector may be expressed by,

$$\begin{aligned} \underset{\sim}{G}_F = & \int_{-1}^{1} \underset{\sim}{\Theta}\left(\frac{\partial \underset{\sim}{N}^T \underset{\sim}{x}}{\partial s} + \frac{\partial \underset{\sim}{N}^T \underset{\sim}{y}}{\partial s}\right) q_a \, ds \\ & + \int_{-1}^{1} \underset{\sim}{\Theta}\left(\frac{\partial \underset{\sim}{N}^T \underset{\sim}{x}}{\partial s} + \frac{\partial \underset{\sim}{N}^T \underset{\sim}{y}}{\partial s}\right) h_c (\underset{\sim}{\Theta}^T \underset{\sim}{T} - T_c) \, ds \\ & + \int_{-1}^{1} \underset{\sim}{\Theta}\left(\frac{\partial \underset{\sim}{N}^T \underset{\sim}{x}}{\partial s} + \frac{\partial \underset{\sim}{N}^T \underset{\sim}{y}}{\partial s}\right) h_r (\underset{\sim}{\Theta}^T \underset{\sim}{T} - T_r) \, ds \quad . \end{aligned} \tag{33}$$

Note that some of the terms defining $\underset{\sim}{G}_F$ in Equation (33) contain unknown element temperatures. For solution purposes, these terms are moved to the left hand side of the matrix equation given by Equation (A17).

A computation similar to the above may be carried out for the boundary integrals in the momentum equations. From Equation (A11),

$$\underset{\sim}{F}_i = \int_A \underset{\sim}{\Phi} \tau_{ij} n_j \, dA \quad ,$$

or for a plane two-dimensional problem in component form,

$$F_1 = \int_\Gamma \Phi\tau_{11}n_1 \, d\Gamma + \int_\Gamma \Phi\tau_{12}n_2 \, d\Gamma \quad ,$$

$$F_2 = \int_\Gamma \Phi\tau_{21}n_1 \, d\Gamma + \int_\Gamma \Phi\tau_{22}n_2 \, d\Gamma \quad . \qquad (34)$$

Since boundary conditions for fluid dynamics problems are rarely expressed in terms of applied shear stresses, NACHOS does not allow for the computation of the shear terms defined in Equation (34). In the case of applied normal stresses, the assumption has been made that the applied stress is constant along the element boundary. Thus,

$$F_1 = \int_\Gamma \Phi\tau_{11}n_1 \, d\Gamma = \int_\Gamma \Phi t_1 \, d\Gamma = \int_\Gamma \Phi \, d\Gamma \, t_1 \quad ,$$

$$F_2 = \int_\Gamma \Phi\tau_{22}n_2 \, d\Gamma = \int_\Gamma \Phi t_2 \, d\Gamma = \int_\Gamma \Phi \, d\Gamma \, t_2 \quad .$$

With the definition for $d\Gamma$ given in Equation (29), the above integrals may be directly evaluated for a given element. It should be emphasized that the above boundary condition is in terms of the total normal stress or traction which is different than the pressure. From the constitutive relation, Equation (1),

$$\tau_{ii} = -P + 2\mu\left(\frac{\partial u_i}{\partial x_i}\right) \qquad i \text{ not summed.}$$

In many practical cases, the viscous part of the stress is negligibly small (e.g., small viscosity) and the normal stress is essentially equal to the pressure. When the viscous part is not negligible, the application of a stress boundary condition does not distinguish between contributions from the pressure and viscous parts but simply reflects their net effect. For purposes of input to the NACHOS code[1] this boundary condition is termed a "pressure" boundary condition to reflect its usual role in finite element models. Further comment may be found in Reference 9.

In addition to the "natural" boundary conditions for the equations, "essential" boundary conditions specifying particular values for the dependent variables must be considered. Application of a specified velocity component, pressure or temperature results in the field equation for that particular degree of freedom being replaced by a constraint equation enforcing the proper boundary value. A similar procedure is employed when a boundary condition specifies a certain relationship between dependent variables as in the case of a slip boundary for the fluid. Details of the slip boundary condition procedure are given in Appendix C.

SOLUTION PROCEDURES

For most finite element solutions to field problems, the majority of the computational effort is expended in solving the assembled matrix equations that describe the discrete problem. This is especially true in the case of highly nonlinear problems or problems with coupled physical phenomena. The general convection problem treated here contains both of these latter characteristics.

Before embarking on a description of the various solution procedures contained in the NACHOS code, it is appropriate to describe some of the characteristics of the equation system under consideration. Rewritting Equations (11) through (13), for convenience,

$$\text{Momentum:} \quad \underset{\approx}{M}\dot{\underset{\sim}{V}} + \underset{\approx}{C}(\underset{\sim}{u})\cdot\underset{\sim}{V} + \underset{\approx}{K}(\underset{\sim}{T})\cdot\underset{\sim}{V} = \underset{\sim}{F}(\underset{\sim}{T}) \quad , \tag{35}$$

$$\text{Energy (Fluid):} \quad \underset{\approx}{N}\dot{\underset{\sim}{T}} + \underset{\approx}{D}(\underset{\sim}{u})\cdot\underset{\sim}{T} + \underset{\approx}{L}(\underset{\sim}{T})\cdot\underset{\sim}{T} = \underset{\sim}{G}(\underset{\sim}{T}) \quad , \tag{36}$$

$$\text{Energy (Solid):} \quad \underset{\approx}{N}\dot{\underset{\sim}{T}} + \underset{\approx}{L}(\underset{\sim}{T})\cdot\underset{\sim}{T} = \underset{\approx}{G}(\underset{\sim}{T}) \quad . \tag{37}$$

The above equations describe a rather large class of problems and for purposes of discussion, it is useful to identify several subclasses denoted as: (a) isothermal problems, (b) weakly coupled convection problems, and (c) strongly coupled convection problems. Isothermal problems refer to those involving the solution of Equation (35) only. Weakly coupled convection problems are those in which the momentum Equation (35) is not a function of temperature, e.g., forced convection with constant material properties. This type problem allows the momentum equation to be solved independently of the energy equation, followed by a solution of the energy equation with a known velocity field. The strongly coupled convection problem denotes the case in which the momentum equation depends on the temperature field, e.g., free convection or forced convection with temperature dependent material properties. With strong coupling, the momentum and energy equations must be solved together.

The choice of a solution strategy for each of these problem categories also requires consideration of whether the analysis is to be a steady state or time dependent computation. In the case of convection problems, the appropriate ordering for solution of the energy and momentum equations must also be evaluated. In the following sections, the solution algorithms currently used in NACHOS for problems in each of the three subclasses are described in detail.

<u>Steady State Algorithms</u>

The choice of an iterative solution method for a general purpose code is governed by several considerations. First, the chosen algorithm should be applicable (convergent) for a wide range of problems with a minimal sensitivity to variations in flow conditions and geometry. The rate of convergence of the method should also be reasonably high for economy. Finally, the algorithm should be user-oriented in the sense that input data (e.g., over-relaxation factors) for the procedure is minimal. The algorithms currently available in NACHOS represent the best understood methods that meet the above criteria.

<u>Isothermal Problems:</u>

For problems involving a steady, isothermal flow, the appropriate matrix equation is,

$$\underset{\approx}{C}(\underset{\sim}{u})\underset{\sim}{V} + \underset{\approx}{K}\underset{\sim}{V} = \underset{\sim}{F} \quad . \tag{38}$$

The NACHOS program employs two fixed point iteration schemes for the solution of Equation (38). A particularly simple scheme with a large radius of convergence is the successive substitution or Picard iteration algorithm given by,

$$\underset{\approx}{C}(\underset{\sim}{u}^{n})\underset{\sim}{V}^{n+1} + \underset{\approx}{K}\underset{\sim}{V}^{n+1} = \underset{\sim}{F} \quad , \tag{39}$$

where the superscript n indicates the iteration level. The second iterative algorithm available in NACHOS is a generalized Newton-Raphson procedure given by,

$$\underset{\sim}{f}(\underset{\sim}{V}^{n}) = - \left.\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{V}}\right|_{\underset{\sim}{V}_n} (\underset{\sim}{V}^{n+1} - \underset{\sim}{V}^{n}) \quad , \tag{40}$$

where,

$$\underset{\sim}{f}(\underset{\sim}{V}) = \underset{\approx}{C}(\underset{\sim}{u})\underset{\sim}{V} + \underset{\approx}{K}\underset{\sim}{V} - \underset{\sim}{F} \quad .$$

A comparison of the Picard and Newton-Raphson procedures[10] has indicated a superior rate of convergence for the Newton method. However, the Newton method has a smaller radius of convergence (i.e., is more sensitive to the initial solution vector, $\underset{\sim}{V}^{o}$) and, therefore, inclusion of both algorithms in a particular code is warranted. Note that in NACHOS, the starting vector for both algorithms is the solution to the creeping or Stokes flow problem,

$$\underset{\approx}{K}\underset{\sim}{V}^{o} = \underset{\sim}{F} \quad . \tag{41}$$

Weakly Coupled Convection Problems:

When the energy equations (36) and (37) are decoupled from the momentum equation, the NACHOS program treats the problem by first solving for the velocity and pressure fields using the algorithms described above for an isothermal flow. With a known velocity field, the energy equations are solved directly for the temperature field. If nonlinear boundary conditions on temperature (e.g., radiation) are present, the energy equations are solved iteratively using a Picard algorithm.

Strongly Coupled Convection Problems:

For cases where the momentum equation depends on the temperature field, Equations (35) through (37) can no longer be treated independently. A particularly simple algorithm for this type of problem is an alternating solution scheme given by,

$$\underset{\approx}{D}(\underset{\sim}{u}^n)\underset{\sim}{T}^{n+1} + \underset{\approx}{L}(\underset{\sim}{T}^n)\underset{\sim}{T}^{n+1} = \underset{\sim}{G}(\underset{\sim}{T}^n)$$

$$\underset{\approx}{C}(\underset{\sim}{u}^n)\underset{\sim}{V}^{n+1} + \underset{\approx}{K}(\underset{\sim}{T}^{n+1})\underset{\sim}{V}^{n+1} = \underset{\sim}{F}(\underset{\sim}{T}^{n+1})$$

$$\underset{\approx}{D}(\underset{\sim}{u}^{n+1})\underset{\sim}{T}^{n+2} + \underset{\approx}{L}(\underset{\sim}{T}^{n+1})\underset{\sim}{T}^{n+2} = \underset{\sim}{G}(\underset{\sim}{T}^{n+1}) \quad (42)$$

$$\vdots$$

etc.

The basic iteration scheme in Equation (42) can often be accelerated by suitable choice of an interpolation procedure for the dependent variables used to evaluate the coefficient matrices. For example, the $\underset{\approx}{K}$ or $\underset{\approx}{L}$ matrices can be evaluated at $\underset{\sim}{T}^{*n+1}$ with,

$$\underset{\sim}{T}^{*n+1} = \alpha\underset{\sim}{T}^{n+1} + (1 - \alpha)\underset{\sim}{T}^{n}(0 \leq \alpha \leq 1) \quad . \quad (43)$$

NACHOS uses the above interpolation procedure for the temperature field with $\alpha = \frac{1}{2}$; no interpolation on the velocity field is used. The initial solution vectors for the iteration scheme are obtained from the linear problems,

$$\underset{\approx}{L}(\underset{\sim}{T}^{int})\underset{\sim}{T}^{o} = \underset{\sim}{G}(\underset{\sim}{T}^{int}) \quad ,$$

$$\underset{\approx}{K}(\underset{\sim}{T}^{int})\underset{\sim}{V}^{o} = \underset{\sim}{F}(\underset{\sim}{T}^{o}) \quad ,$$

where $\underset{\sim}{T}^{int}$ indicates an initial temperature estimate supplied to the program.

Transient Algorithms

The choice of a transient analysis procedure is basically governed by the same criteria used in choosing a steady state algorithm. However, the inclusion of the temporal term in the matrix system, Equations (35) through (37), requires several additional criteria to be considered. Equations (35) through (37) represent a discrete space, continuous time approximation to a field problem. A direct time integration procedure replaces the continuous time derivative with an approximation for the history of the dependent variable over a small portion of the problem time scale. The result is an incremental procedure that advances the solution by discrete steps in time. In constructing such a procedure, questions of numerical stability and accuracy must be considered.

For use in a user-oriented code such as NACHOS, the increased stability of an implicit integration scheme was deemed more desirable than the computational speed of an explicit method. Furthermore, as a result of several previous investigations[11,12] an algorithm using a consistent mass and capacitance formulation was considered most appropriate, thus specifically excluding explicit integration methods from consideration. The following sections describe the implicit integration scheme used in NACHOS.

Isothermal Problems:

For problems involving an isothermal flow, the velocity and pressure fields are advanced in time through use of the following algorithm,

$$\left\{\frac{2}{\Delta t} \underset{\approx}{M} + \underset{\approx}{C}(\underset{\sim}{u}^n) + \underset{\approx}{K}\right\}\underset{\sim}{V}^a = \underset{\sim}{F} + \frac{2}{\Delta t} \underset{\approx}{M}\underset{\sim}{V}^n \quad , \tag{44}$$

where,

$$\underset{\sim}{V}^a = (\underset{\sim}{V}^{n+1} + \underset{\sim}{V}^n)/2 \quad , \tag{45}$$

and Δt is the time step and superscript n indicates the time level. The algorithm in Equations (44) and (45) is related to the basic Crank-Nicholson method though in the present case, Equation (45) is not used to extend the solution to the end of the time interval but serves only as a definition for $\underset{\sim}{V}^a$. The above method has somewhat better stability properties than the standard Crank-Nicholson approach.[13] Note that quasilinearization has been used to allow the nonlinear term, $\underset{\approx}{C}$, to be evaluated at $\underset{\sim}{u}^n$ rather than the formally correct (and second order accurate) $\underset{\sim}{u}^a$ value. Such an approach has proved to be valid for reasonable choices of the time step and most types of evolution problems. A derivation of the algorithm given by Equation (44) is given in Appendix D.

Weakly Coupled Convection Problems:

When the energy equations are to be advanced in time but are still uncoupled from the momentum equation, NACHOS treats the equations in a sequential manner. The velocity and pressure fields are advanced in time using the previously described algorithm; the temperature field is then advanced using,

$$\left\{\frac{2}{\Delta t} \underset{\approx}{N} + \underset{\approx}{D}(\underset{\sim}{u}^{n+1}) + \underset{\approx}{L}\right\}\underset{\sim}{T}^{a} = \underset{\sim}{G} + \frac{2}{\Delta t} \underset{\approx}{N}\underset{\sim}{T}^{n} \quad , \tag{46}$$

with,

$$\underset{\sim}{T}^{a} = (\underset{\sim}{T}^{n+1} + \underset{\sim}{T}^{n})/2 \quad , \tag{47}$$

which is the energy equation counterpart to Equations (44) and (45). If the energy equation is nonlinear due to the appearance of radiation type boundary conditions, a quasilinearization process is used, i.e., the boundary condition is evaluated using the latest available temperature field.

Strongly Coupled Convection Problems:

A sequential solution procedure is again used when the momentum equation depends on temperature. The quasilinearization procedure is invoked to allow the solution to advance in time without iteration during a time step. Since most strongly coupled problems are of the free convection type, the energy equation is typically advanced first using,

$$\left\{\frac{2}{\Delta t} \underset{\approx}{N} + \underset{\approx}{D}(\underset{\sim}{u}^{n}) + \underset{\approx}{L}(\underset{\sim}{T}^{n})\right\}\underset{\sim}{T}^{a} = \underset{\sim}{G}(\underset{\sim}{T}^{n}) + \frac{2}{\Delta t} \underset{\approx}{N}\underset{\sim}{T}^{n} \quad , \tag{48}$$

followed by the momentum equation,

$$\left\{\frac{2}{\Delta t} \underset{\approx}{M} + \underset{\approx}{C}(\underset{\sim}{u}^{n}) + \underset{\approx}{K}(\underset{\sim}{T}^{n+1})\right\}\underset{\sim}{V}^{a} = \underset{\sim}{F}(\underset{\sim}{T}^{n+1}) + \frac{2}{\Delta t} \underset{\approx}{M}\underset{\sim}{V}^{n} \quad . \tag{49}$$

Nonlinear boundary conditions are again treated by quasilinearization.

Further Remarks on Solution Algorithms

The solution algorithms for both steady and transient analyses as presented above represent the methods currently available in the NACHOS code. It is not to be implied that these choices represent the best or only methods for this class of problems. Rather they do represent methods that have proved generally reliable and accurate for a wide variety of problems. As solution algorithms for nonlinear finite element analysis is a topic of current research it is anticipated that further refinements to the present methods and/or the inclusion of other procedures will occur in future versions of NACHOS.

Matrix Solution Procedures

When the solution algorithms of the previous sections are applied at a given iteration or time step, the general result is a matrix equation of the form,

$$\underset{\approx}{A}\underset{\sim}{x} = \underset{\sim}{b} \quad . \tag{50}$$

In the problems considered here the $\underset{\approx}{A}$ matrix is large (e.g., several thousand equations), sparse, banded and generally unsymmetric.

The solution of the equation system given by Equation (50) may be approached by two basic methods--iterative and direct. Iterative methods, such as the Gauss-Seidel procedure, have occasionally been used in a finite element context. However, by far, the most prevalent solution methods for Equation (50) are the direct methods such as Cholesky decomposition and Gauss elimination.

The solution procedure used in NACHOS is a form of Gauss elimination developed by Irons,[14] called the frontal solution method. The basic premise of the frontal method is that the process of assembling the system matrix, $\underset{\approx}{A}$, from the individual element matrices and the reduction of $\underset{\approx}{A}$ by standard Gauss elimination may be efficiently intertwined. In processing each element in sequence, the frontal procedure passes through the following basic steps:

a) assembly of element equations into global matrix $\underset{\approx}{A}$ (in actuality $\underset{\approx}{A}$ is stored as part of a one-dimensional working space),

b) check each equation in the assembled system to determine if all contributions to that equation have been made,

c) condense from the system (by Gauss elimination) the equations for all degrees of freedom that have been completely assembled,

d) return to Step a) for the next element.

By combining the assembly and reduction process computer storage is effectively minimized since only the currently "active" (i.e., incompletely assembled) degrees of freedom are retained in core storage. Following reduction of the matrix $\underset{\approx}{A}$ to an upper triangular form, a back-substitution algorithm completes the solution process for the vector $\underset{\sim}{x}$.

Since the frontal method is structured around the individual element, it is especially adaptable for general purpose codes with an element library. The processing of higher order elements (e.g., quadratic basis functions) is also handled efficiently by a frontal method.

The original frontal solution package published by Irons was developed for symmetric, positive definite systems. For incorporation into the NACHOS code, Irons' program was modified to account for nonsymmetry of the system and the appearance of zero coefficients on the diagonal of the $\underset{\approx}{A}$ matrix (the zeroes occur due to the form of the incompressibility constraint, see Appendix A, Equation (A5)). This latter modification consists of essentially re-ordering the elimination sequence of some of the pressure degrees of freedom.

PRE- AND POST-PROCESSING OPERATIONS

The NACHOS code was designed to be a self-contained analysis package with the necessary options to set up a problem, solve for the required dependent variables and analyze the resultant solution in terms of derived quantities and graphical output. The previous sections have described the formulation and solution phases of the NACHOS program; the present section provides a brief description of the methods used in the pre- and post-processing operations.

Mesh Generation

For most analysis situations, the major demand on the user of a finite element program arises from the preparation of the input data and in particular the construction of a suitable element mesh. The NACHOS program was developed with a self-contained, automatic mesh generation scheme that allows very complex geometries to be modelled accurately with a minimum of user input.

The mesh generator is based on an isoparametric mapping technique proposed by Zienkiewicz and Phillips[15] and developed in its present form by Womack.[16] As this mesh generator has been previously documented[16] and found use in several other codes[17,18] only a brief description of its operation will be given here.

For purposes of grid construction, the region of interest is considered to be made up of quadrilateral shaped parts or subregions, which are determined by the user. Within each part, an isoparametric mapping is used to approximate the region boundary. By specifying a number of x, y (or r, z) coordinates on the boundary of each part, the limits of the region and the type of interpolation function used to define the boundary shape are determined. The current version of NACHOS allows a linear, quadratic or cubic description of the boundary shape for each side of the individual parts.

The mesh points (nodes) within a region are generated automatically once the number of nodes along a boundary is specified. Local nodal point spacing is easily adjusted by use of a gradient parameter in the input. Nodal points within a region are identified by an I, J numbering system. Definition of individual finite elements and element connectivity are based on the I, J identification of the nodal points. Further details on the mesh generator are given in Reference 1.

Stream Function Computation

A quantity that is often useful in the graphic display of computed flow fields is the stream function. For two-dimensional incompressible flows, the stream function is the remaining non-zero component of a vector potential which satisfies the equation of mass conservation identically. By definition,

$$u_1 = \frac{\partial\psi}{\partial x_2} \; ; \; u_2 = \frac{\partial\psi}{\partial x_1} \quad , \tag{51}$$

and since the change in the stream function, $\delta\psi$, is an exact differential,

$$\delta\psi = \int_A^B \bar{q}\cdot\bar{n} \, d\Gamma \quad , \tag{52}$$

with,

$$\bar{q} = u_1\bar{e}_1 + u_2\bar{e}_2$$

$$\bar{n} = n_1\bar{e}_1 + n_2\bar{e}_2$$

where $\bar{n}$ is the normal to the integration path, $d\Gamma$, and $\bar{q}$ is the velocity vector along the path.

The calculation of the change in the stream function within a finite element may be carried out using Equation (52) once a suitable integration path is identified. In the NACHOS program, the integration path is taken along the element boundaries. Consider the general element boundary shown in Figure 5 with the following definitions of velocities and coordinates,

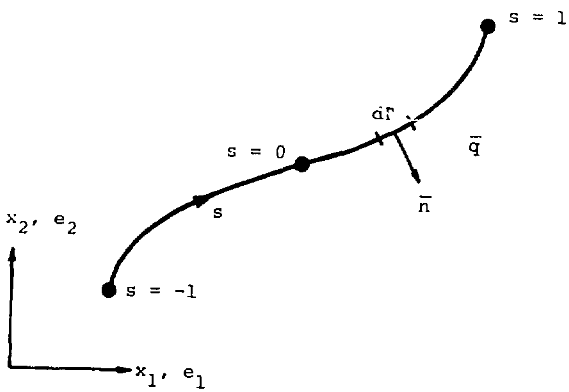

FIGURE 5

$$u_1 = \underset{\sim}{\Phi}^T \underset{\sim}{u}_1 \; ; \; u_2 = \underset{\sim}{\Phi}^T \underset{\sim}{u}_2 \quad ,$$

$$x = \underset{\sim}{N}^T \underset{\sim}{x} \quad ; \quad y = \underset{\sim}{N}^T \underset{\sim}{y} \quad , \tag{53}$$

where $\underset{\sim}{\Phi}$ and $\underset{\sim}{N}$ are interpolation functions along the boundary (edge functions) and $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$, $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are nodal point velocities and coordinates. The normal vector is expressed by,

$$\bar{n} = \frac{\partial y/\partial s}{d\Gamma} \bar{e}_1 - \frac{\partial x/\partial s}{d\Gamma} \bar{e}_2 \quad , \tag{54}$$

with,

$$d\Gamma = \left[(\partial x/\partial s)^2 + (\partial y/\partial s)^2\right]^{\frac{1}{2}} ds \quad .$$

Use of the relations in Equations (53) and (54) in the definition for $\delta\psi$, Equation (52) yields,

$$\delta\psi = \int_{-1}^{1} \left(\underset{\sim}{y}^T \frac{\partial \underset{\sim}{N}}{\partial s} \underset{\sim}{\Phi}^T \underset{\sim}{u}_1 - \underset{\sim}{x}^T \frac{\partial \underset{\sim}{N}}{\partial s} \underset{\sim}{\Phi}^T \underset{\sim}{u}_2\right) ds \quad . \tag{55}$$

In the present application, the quadratic velocity interpolation functions, $\underset{\sim}{\Phi}$, are given by,

$$\underset{\sim}{\Phi} = \begin{Bmatrix} s(s-1)/2 \\ (1-s^2) \\ s(s+1)/2 \end{Bmatrix} \quad .$$

The coordinate interpolation may be either quadratic, $\underset{\sim}{N} = \underset{\sim}{\Phi}$, or linear,

$$\underset{\sim}{N} = \begin{Bmatrix} (1-s)/2 \\ (1+s)/2 \end{Bmatrix} \quad .$$

The shape function definitions for $\underset{\sim}{\Phi}$ and $\underset{\sim}{N}$ allow the change $\delta\psi$ to be computed along any element boundary once the element geometry ($\underset{\sim}{x}$, $\underset{\sim}{y}$) and velocity fields ($\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$) are specified. Computation of the stream function field for an entire finite element mesh is generated by applying Equation (55) along successive element boundaries starting at a node for which a base value of ψ is specified.

The calculation of the Stokes stream function for axisymmetric flow geometries follows a similar procedure with the appropriate definitions for ψ,

$$u_1 = \frac{1}{r}\frac{\partial\psi}{\partial z} \; ; \; u_2 = -\frac{1}{r}\frac{\partial\psi}{\partial r} \; .$$

$$\delta\psi = \int_A^B \bar{q}\cdot\bar{n} \; d\ell \quad ,$$

with,

$$\bar{q} = u_1 r\bar{e}_1 + u_2 r\bar{e}_2$$

$$\bar{n} = n_1\bar{e}_1 + n_2\bar{e}_2 \quad .$$

Stress and Heat Flux Computation

The computation of the stress fields for the fluid flow and the heat flux distributions for the heat transfer problem follow directly from the definition of those quantities.

The fluid stresses for a planar geometry are given by,

$$\tau_{xx} = -P + 2\mu\frac{\partial u_1}{\partial x}$$

$$\tau_{yy} = -P + 2\mu\frac{\partial u_2}{\partial y}$$

$$\tau_{xy} = \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \quad , \tag{56}$$

and the finite element approximations by,

$$u_1 = \underset{\sim}{\Phi}^T \underset{\sim}{u}_1 \; ; \; u_2 = \underset{\sim}{\Phi}^T \underset{\sim}{u}_2 \; ; \; P = \Psi^T \underset{\sim}{P}$$

$$x = \underset{\sim}{N}^T \underset{\sim}{x} \; ; \; y = \underset{\sim}{N}^T \underset{\sim}{y} \quad ,$$

where the $\underset{\sim}{\Phi}$ and $\underset{\sim}{\Psi}$ shape functions are those described in the element construction section. The $\underset{\sim}{N}$ shape functions may be either quadratic or linear depending on the type of element used. Direct substitution of the approximations for u_i and P into Equation (56) shows that derivatives of the shape functions with respect to the physical x, y coordinates are required. As given in a previous section, the needed relations for a quadrilateral are,

$$\begin{Bmatrix} \frac{\partial \underset{\sim}{\Phi}}{\partial x} \\ \frac{\partial \underset{\sim}{\Phi}}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial \underset{\sim}{\Phi}}{\partial s} \\ \frac{\partial \underset{\sim}{\Phi}}{\partial t} \end{Bmatrix} , \tag{57}$$

where,

$$F_{11} = \frac{\partial \underset{\sim}{N}^T \underset{\sim}{y}}{\partial t} \quad ; \quad F_{12} = - \frac{\partial \underset{\sim}{N}^T \underset{\sim}{y}}{\partial s}$$

$$F_{21} = - \frac{\partial \underset{\sim}{N}^T \underset{\sim}{x}}{\partial t} \quad ; \quad F_{22} = \frac{\partial \underset{\sim}{N}^T \underset{\sim}{x}}{\partial s} ,$$

$$|J| = F_{11} \cdot F_{22} - F_{12} \cdot F_{21} .$$

Using these definitions, the stresses are given by,

$$\tau_{xx}(s, t) = -\underset{\sim}{\Psi}^T \underset{\sim}{P} + \frac{2\mu}{|J|} \left\{ F_{11} \frac{\partial \underset{\sim}{\Phi}^T}{\partial s} \underset{\sim}{u}_1 + F_{12} \frac{\partial \underset{\sim}{\Phi}^T}{\partial t} \underset{\sim}{u}_1 \right\} ,$$

$$\tau_{yy}(s, t) = -\underset{\sim}{\Psi}^T \underset{\sim}{P} + \frac{2\mu}{|J|} \left\{ F_{21} \frac{\partial \underset{\sim}{\Phi}^T}{\partial s} \underset{\sim}{u}_2 + F_{22} \frac{\partial \underset{\sim}{\Phi}^T}{\partial t} \underset{\sim}{u}_2 \right\} ,$$

$$\tau_{xy}(s, t) = \frac{\mu}{|J|} \left\{ F_{21} \frac{\partial \underset{\sim}{\Phi}^T}{\partial s} \underset{\sim}{u}_1 + F_{22} \frac{\partial \underset{\sim}{\Phi}^T}{\partial t} \underset{\sim}{u}_1 + F_{11} \frac{\partial \underset{\sim}{\Phi}^T}{\partial s} \underset{\sim}{u}_2 + F_{12} \frac{\partial \underset{\sim}{\Phi}^T}{\partial t} \underset{\sim}{u}_2 \right\} . \tag{58}$$

Note that for a triangular element the s, t coordinates are replaced by the L_i coordinates.

The expressions in Equation (58) allow the stress components to be computed at any point s_o, t_o within an element for a known element geometry and solution field. The NACHOS program evaluates the stress components on the element boundary, midway between adjacent nodes.

Calculation of stress fields for an axisymmetric geometry follow a similar procedure with the definitions of the stress components given by,

$$\tau_{rr} = -P + 2\mu \frac{\partial u_1}{\partial r} \quad ,$$

$$\tau_{zz} = -P + 2\mu \frac{\partial u_2}{\partial z} \quad ,$$

$$\tau_{\theta\theta} = -P + 2\mu \frac{u_1}{r} \quad ,$$

$$\tau_{rz} = \mu\left(\frac{\partial u_2}{\partial r} + \frac{\partial u_1}{\partial z}\right) \quad .$$

The computation of the relevant heat flux quantities follows also from the basic definitions,

$$q_x = -k \frac{\partial T}{\partial x} \quad ,$$

$$q_y = -k \frac{\partial T}{\partial y} \quad , \qquad (59)$$

and,

$$q_n = \bar{q} \cdot \bar{n} \quad ,$$

with,

$$\bar{q} = q_x \bar{e}_x + q_y \bar{e}_y \quad ,$$

$$\bar{n} = n_x \bar{e}_x + n_y \bar{e}_y \quad .$$

Reference to Figure 5 and the previous definitions for the normal vector (see Equation (54)) allows the heat flux normal to a boundary to be given by,

$$q_n = q_x\left(\frac{\partial y/\partial s}{d\Gamma'}\right) + q_y\left(-\frac{\partial x/\partial s}{d\Gamma'}\right) \quad . \qquad (60)$$

As in the case of the stress computations, the following finite element definitions are required,

$$T = \underset{\sim}{\theta}^T \underset{\sim}{T} \quad ,$$

$$x = \underset{\sim}{N}^T \underset{\sim}{x} \quad ; \quad y = \underset{\sim}{N}^T \underset{\sim}{y} \quad ,$$

as well as,

$$\begin{Bmatrix} \dfrac{\partial \tilde{\Theta}}{\partial x} \\ \\ \dfrac{\partial \tilde{\Theta}}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} F_{11} & F_{12} \\ \\ F_{21} & F_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \tilde{\Theta}}{\partial s} \\ \\ \dfrac{\partial \tilde{\Theta}}{\partial t} \end{Bmatrix} ,$$

where J and F_{ij} are defined in an analogous manner to Equation (57).

The above relations allow the flux components to be expressed by,

$$q_x = - \frac{k}{|J|} \left(F_{11} \frac{\partial \tilde{\Theta}^T}{\partial s} \tilde{T} + F_{12} \frac{\partial \tilde{\Theta}^T}{\partial t} \tilde{T} \right) ,$$

$$q_y = - \frac{k}{|J|} \left(F_{21} \frac{\partial \tilde{\Theta}^T}{\partial s} \tilde{T} + F_{22} \frac{\partial \tilde{\Theta}^T}{\partial t} \tilde{T} \right) , \quad (61)$$

and the normal flux by a combination of Equations (61) and (60). The expressions in Equation (61) may be evaluated at any point s_o, t_o within an element; the NACHOS program evaluates the heat flux values on the boundary of an element, midway between adjacent nodes.

Plotting

The NACHOS program contains a complete plotting package that allows finite element meshes, contour plots and time history plots to be constructed at the option of the user. Since the generation of the various types of plots follows standard procedures, no attempt will be made to describe those methods here.

CONCLUDING REMARKS

The present document has been designed to provide a moderately detailed description of the methods and numerical procedures employed in the finite element code, NACHOS. The material presented here is intended to serve as necessary background for the user of the NACHOS program. The detailed description of the input for NACHOS is described in Part II of this report.

The version of NACHOS described here represents the first iteration in the process of developing a general-purpose program for the solution of a class of fluid dynamics problems. As the finite element methods applied here are still subjects of current research, it is anticipated that future versions of NACHOS will reflect development activities.

APPENDIX A

FINITE ELEMENT EQUATIONS

The manipulations required to transform the field equations into a discrete system are outlined in this section. Using the definition of the Galerkin procedure, Equation (10), and the finite element approximation, Equation (8), allows the following integral equations to be written:

(Momentum)

$$\left[\int_V \rho \underset{\sim}{\phi} \underset{\sim}{\phi}^T \, dV\right] \frac{\partial \underset{\sim}{u}_i}{\partial t} + \left[\int_V \rho \underset{\sim}{\phi} \underset{\sim}{\phi}^T \underset{\sim}{u}_j \frac{\partial \underset{\sim}{\phi}^T}{\partial x_j} \, dV\right] \underset{\sim}{u}_i - \left[\int_V \frac{\partial \underset{\sim}{\phi}^T}{\partial x_i} \Psi \, dV\right] \underset{\sim}{P}$$

$$+ \left[\int_V \mu \frac{\partial \underset{\sim}{\phi}}{\partial x_j} \frac{\partial \underset{\sim}{\phi}^T}{\partial x_j} \, dV\right] u_i + \left[\int_V \mu \frac{\partial \underset{\sim}{\phi}}{\partial x_j} \frac{\partial \underset{\sim}{\phi}^T}{\partial x_i} \, dV\right] u_j$$

$$= \left[\int_V \rho \underset{\sim}{\phi} g_i \, dV\right] - \left[\int_V \rho \beta g_i \underset{\sim}{\phi} \underset{\sim}{\Theta} \, dV\right] (\underset{\sim}{T} - \underset{\sim}{T}_{ref})$$

$$+ \left[\int_A \underset{\sim}{\phi} \tau_{ij} n_j \, dA\right] \quad . \qquad \text{(A1)}$$

(Incompressibility)

$$\left[\int_V \underset{\sim}{\Psi} \frac{\partial \underset{\sim}{\phi}^T}{\partial x_i} \, dV\right] \underset{\sim}{u}_i = 0 \quad . \qquad \text{(A2)}$$

(Energy)

$$\left[\int_V \rho C_p \underset{\sim}{\Theta} \underset{\sim}{\Theta}^T\right] dV \frac{\partial \underset{\sim}{T}}{\partial t} + \left[\int_V \rho C_p \underset{\sim}{\Theta} \underset{\sim}{\phi}^T \underset{\sim}{u}_j \frac{\partial \underset{\sim}{\Theta}^T}{\partial x_j} \, dV\right] \underset{\sim}{T} + \left[\int_V k \frac{\partial \underset{\sim}{\Theta}}{\partial x_j} \frac{\partial \underset{\sim}{\Theta}^T}{\partial x_j} \, dV\right] \underset{\sim}{T}$$

$$= \left[\int_V \underset{\sim}{\Theta} S \, dV\right] + \left[\int_A \underset{\sim}{\Theta} q_j n_j \, dA\right] \quad . \qquad \text{(A3)}$$

In arriving at the above equations, the Green-Gauss theorem has been used to reduce the second-order diffusion terms in the momentum and energy equations to first-order terms plus an area integral. The appearance of the area integrals containing the applied surface stresses (tractions) and heat fluxes corresponds to the "natural" boundary conditions for the problem.

Several terms in Equations (A1) and (A3) contain the material properties μ, k and β, which are assumed to depend on the temperature. It has been found convenient in the present code to allow these properties to have a spatial variation over each element. Thus, let μ, k and β be approximated within each element by,

$$\mu(x_i) = \underset{\sim}{\eta}^T(x_i)\underset{\sim}{\mu}$$

$$k(x_i) = \underset{\sim}{\eta}^T(x_i)\underset{\sim}{k}$$

$$\beta(x_i) = \underset{\sim}{\eta}^T(x_i)\underset{\sim}{\beta} \quad , \qquad \text{(A4)}$$

where $\underset{\sim}{\eta}$ is a vector of interpolation functions and the $\underset{\sim}{\mu}$, $\underset{\sim}{k}$ and $\underset{\sim}{\beta}$ are vectors of nodal point material property quantities. Since the temperature is also a nodal point quantity, the above form allows the values of μ, k and β to be evaluated in a straightforward manner.

Once the form of the interpolation functions $\underset{\sim}{\Phi}$, $\underset{\sim}{\Theta}$, $\underset{\sim}{\Psi}$ and $\underset{\sim}{\eta}$ is specified the integrals defined in Equations (A1) through (A3) may be evaluated to produce the required coefficient matrices. Combining the momentum equations and the incompressibility constraint into a single matrix equation produces a system of the form:

$$\begin{bmatrix} \mathbf{M} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_2 \\ \dot{\mathbf{P}} \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_1(\mathbf{u}_1) + \mathbf{C}_2(\mathbf{u}_2) & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{C}_1(\mathbf{u}_1) + \mathbf{C}_2(\mathbf{u}_2) & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{P} \end{Bmatrix}$$

$$+ \begin{bmatrix} 2\mathbf{K}_{11} + \mathbf{K}_{22} & \mathbf{K}_{21} & -\mathbf{Q}_1 \\ \mathbf{K}_{12} & 2\mathbf{K}_{22} + \mathbf{K}_{11} & -\mathbf{Q}_2 \\ -\mathbf{Q}_1^T & -\mathbf{Q}_2^T & \mathbf{O} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{O} \end{Bmatrix} . \quad \text{(A5)}$$

The energy equation has the form,

$$\left[\mathbf{N}\right]\left\{\dot{\mathbf{T}}\right\} + \left[\mathbf{D}_1(\mathbf{u}_1) + \mathbf{D}_2(\mathbf{u}_2)\right]\left\{\mathbf{T}\right\} + \left[\mathbf{L}_{11} + \mathbf{L}_{22}\right]\left\{\mathbf{T}\right\} = \left\{\mathbf{G}\right\} . \quad \text{(A6)}$$

The coefficient matrices shown in Equations (A5) and (A6) are defined by,

$$\mathbf{M} = \int_V \rho \boldsymbol{\Phi}\boldsymbol{\Phi}^T \, dV \quad ; \quad \text{(A7)}$$

$$\mathbf{C}_i(\mathbf{u}_i) = \int_V \rho \boldsymbol{\Phi}\boldsymbol{\Phi}^T \mathbf{u}_i \frac{\partial \boldsymbol{\Phi}^T}{\partial x_i} \, dV \quad ; \quad \text{(A8)}$$

$$\mathbf{K}_{ij} = \int_V \boldsymbol{\eta}^T \boldsymbol{\mu} \frac{\partial \boldsymbol{\Phi}}{\partial x_i} \frac{\partial \boldsymbol{\Phi}^T}{\partial x_j} \, dV \quad ; \quad \text{(A9)}$$

$$\mathbf{Q}_i = \int_V \frac{\partial \boldsymbol{\Phi}^T}{\partial x_i} \boldsymbol{\Psi} \, dV \quad ; \quad \text{(A10)}$$

$$\mathbf{F}_i = \int_V \rho \boldsymbol{\Phi} g_i \, dV - \int_V \rho \boldsymbol{\eta}^T \boldsymbol{\beta} \mathbf{g}_i \boldsymbol{\Phi}\boldsymbol{\Theta}^T \, dV \, (\mathbf{T} - T_{ref}) + \int_A \boldsymbol{\Phi} \tau_{ij} n_j \, dA \quad ; \quad \text{(A11)}$$

$$\mathbf{N} = \int_V \rho C_p \boldsymbol{\Theta}\boldsymbol{\Theta}^T \, dV \quad ; \quad \text{(A12)}$$

$$\mathbf{D}_i(\mathbf{u}_i) = \int_V \rho C_p \boldsymbol{\Theta}\boldsymbol{\Phi}^T \mathbf{u}_j \frac{\partial \boldsymbol{\Theta}^T}{\partial x_j} \, dV \quad ; \quad \text{(A13)}$$

$$L_{ii} = \int_V \eta^T k \frac{\partial \Theta}{\partial x_i} \frac{\partial \Theta^T}{\partial x_i} dV \quad ; \qquad \text{(A14)}$$

$$G = \int_V \Theta S \; dV + \int_A \Theta q_j n_j \; dA \quad . \qquad \text{(A15)}$$

Equations (A5) and (A6) may be written symbolically as,

$$M\dot{V} + C(u) \cdot V + KV = F(T) \quad , \qquad \text{(A16)}$$

$$N\dot{T} + D(u) \cdot T + L \cdot T = G \quad , \qquad \text{(A17)}$$

with,

$$u^T = (u_1, \; u_2) \quad ,$$

and,

$$V^T = (u_1, \; u_2, \; P) \quad .$$

APPENDIX B

EVALUATION OF MATRIX COEFFICIENTS

As a result of the manipulations described in Appendix A, the basic field equations were transformed into a set of matrix equations. The coefficients of the various element matrices can be expressed as spatial integrals of various interpolation functions and their derivatives (see Equations (A7)-(A15)). The present section describes the method of evaluation for these coefficients.

For purposes of discussion consider the evaluation of a typical diffusion term from the momentum equation (Equation (A9)) for a planar quadrilateral element. The matrix coefficients are given by,

$$\underset{\approx}{K}_{xy} = \int_V \underset{\sim}{\eta}^T \underset{\sim}{\mu} \frac{\partial \underset{\sim}{\phi}}{\partial x} \frac{\partial \underset{\sim}{\phi}^T}{\partial y} \, dV \quad , \tag{B1}$$

where $\underset{\sim}{\phi}$ and $\underset{\sim}{\eta}$ are interpolation functions given by Equations (A14) and (A15), $\underset{\sim}{\mu}$ is a vector of nodal point quantities (viscosity values), and V is the volume of the element. For a planar element,

$$dV = dx \, dy \quad ,$$

or from Equation (19),

$$dV = \det[J] \, ds \, dt \quad , \tag{B2}$$

where s and t are the normalized coordinates for the element. From Equations (17) and (18), the derivatives of the interpolation functions can be expressed as,

$$\begin{Bmatrix} \dfrac{\partial \tilde{\Phi}}{\partial x} \\ \\ \dfrac{\partial \tilde{\Phi}}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \tilde{\Phi}}{\partial s} \\ \\ \dfrac{\partial \tilde{\Phi}}{\partial t} \end{Bmatrix} , \tag{B3}$$

where,

$$[J]^{-1} = \frac{1}{\det[J]} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} , \tag{B4}$$

and,

$$F_{11} = \frac{\partial \tilde{N}^T \tilde{y}}{\partial t} \quad ; \quad F_{12} = - \frac{\partial \tilde{N}^T \tilde{x}}{\partial t}$$

$$F_{21} = - \frac{\partial \tilde{N}^T \tilde{y}}{\partial s} \quad ; \quad F_{22} = \frac{\partial \tilde{N}^T \tilde{x}}{\partial s} \tag{B5}$$

$$\det[J] = F_{11} \cdot F_{22} - F_{12} \cdot F_{21} ,$$

where $\tilde{N}$ may be either $\tilde{\Phi}$ (isoparametric element) or $\tilde{\eta}$ (subparametric element). Substituting definitions (B2)-(B5) into (B1) yields,

$$\tilde{K}_{xy} = \int_{-1}^{+1} \int_{-1}^{+1} \tilde{\eta}^T \tilde{\mu} \left(F_{11} \frac{\partial \tilde{\Phi}}{\partial s} + F_{12} \frac{\partial \tilde{\Phi}}{\partial t} \right) \left(F_{21} \frac{\partial \tilde{\Phi}^T}{\partial s} + F_{22} \frac{\partial \tilde{\Phi}^T}{\partial t} \frac{ds\ dt}{\det[J]} \right) . \tag{B6}$$

Similar procedures may be carried out for the other terms in the matrix equations (Equations (A7)-(A15)), as well as other element geometries such as the triangle. When considering an axisymmetric element, the definition of the elemental volume is modified to,

$$dV = r\ d\Theta\ dr\ dz = r\ d\Theta\ \det[J]\ ds\ dt \quad . \tag{B7}$$

The axisymmetric equivalent of Equation (B6) is then,

$$K_{rz} = \int_0^{2\pi} \int_{-1}^{+1} \int_{-1}^{+1} n^T \mu \left(F_{11} \frac{\partial \phi}{\partial s} + F_{12} \frac{\partial \phi}{\partial t} \right)$$

$$\left(F_{21} \frac{\partial \phi^T}{\partial s} + F_{22} \frac{\partial \phi^T}{\partial t} \right) (N^T r) \frac{ds \cdot dt}{\det[J]} \, d\theta \quad . \tag{B8}$$

where the radius, r, has been interpolated according to the parametric concept given in Equation (16). The axisymmetric form of the field equations introduces several additional matrix terms not given in Equations (A7)-(A15), however, all are treated in a manner analogous to Equation (B8).

An examination of the definite integrals given by Equation (B6) or (B8) shows that the integrand is a rational function of the s, t coordinates. The evaluation of these integrals is carried out in the NACHOS program by use of a numerical quadrature procedure. Expressing a generalized definite integral by,

$$I = \int_{-1}^{1} \int_{-1}^{1} f(s, t) \, ds \, dt \quad ,$$

the numerical quadrature algorithm is given by,

$$I = \int_{-1}^{1} \int_{-1}^{1} f(s, t) \, ds \, dt = \sum_{i=1}^{n} \sum_{j=1}^{n} H_i H_j f(s_i, t_j) \quad , \tag{B9}$$

where the weighting coefficients H_i, H_j and the abscissae values s_i, t_j depend on the particular quadrature formula used. The NACHOS program uses a 3 x 3 (i.e., n = 3) Gauss formula[2] for the evaluation of matrix coefficients for quadrilateral elements. Matrix coefficients for triangular elements are evaluated using a seven point quadrature scheme developed by Hammer, et al.[19]

APPENDIX C

SLIP BOUNDARY CONDITIONS

A boundary condition that allows a viscous fluid to flow freely (slip), parallel to a given boundary, is quite useful in some modeling applications. The numerical implementation of the slip condition is different than most "essential" boundary conditions since it specifies a relationship between two dependent variables (velocity components) rather than a specific boundary value.

Consider the boundary geometry shown in Figure 6 where u_t and u_n are the local velocity components tangent and normal to the boundary, u_x and u_y are the local velocity components in the x, y coordinate system and α is the local angle betwen the x coordinate direction and the tangent to the boundary. The slip boundary condition provides that the fluid velocity normal to the boundary is zero, while the tangential velocity is unconstrained. Since the numerical computations are carried out in terms of velocity components in the coordinate directions, a suitable constraint condition in terms of u_x and u_y is required.

The tangential velocity can be expressed by,

$$u_t = u_x \cos \alpha + u_y \sin \alpha \quad . \tag{C1}$$

The normal velocity is set to zero and thus need not be written. Equation (C1) may be transformed back into x, y components by,

$$\bar{u}_x = u_t \cdot \cos \alpha \quad ,$$

$$\bar{u}_y = u_t \cdot \sin \alpha \quad , \tag{C2}$$

or,

$$\begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \end{Bmatrix} = \begin{bmatrix} \cos^2 \alpha & \sin \alpha \cdot \cos \alpha \\ \sin \alpha \cdot \cos \alpha & \sin^2 \alpha \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} , \tag{C3}$$

where the barred components in Equations (C2) and (C3) represent x, y velocity components that have been modified to account for the slip boundary condition.

The constraint matrix in Equation (C3) is used to modify the appropriate coefficients of the equations in the assembled matrix system, e.g., Equation (35).

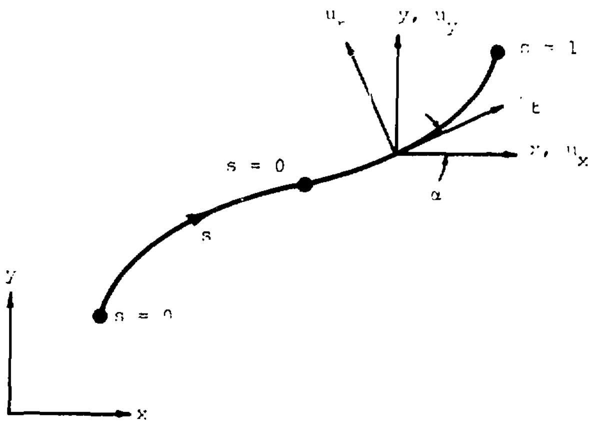

FIGURE 6

APPENDIX D

TIME INTEGRATION METHOD

The time integration method to be considered here is a modification of the basic Crank-Nicholson algorithm. The nonlinear momentum Equation (35) may be evaluated at time level n and time level n + 1,

$$\underset{\approx}{M}\dot{\underset{\sim}{V}}^{n} + \underset{\approx}{C}(\underset{\sim}{u}^{n})\underset{\sim}{V}^{n} + \underset{\approx}{K}\underset{\sim}{V}^{n} = \underset{\sim}{F}^{n} \quad , \tag{D1}$$

$$\underset{\approx}{M}\dot{\underset{\sim}{V}}^{n+1} + \underset{\approx}{C}(\underset{\sim}{u}^{n+1})\underset{\sim}{V}^{n+1} + \underset{\approx}{K}\underset{\sim}{V}^{n+1} = \underset{\sim}{F}^{n+1} \quad . \tag{D2}$$

By an averaging process, Equations (D1) and (D2) may be combined to provide an equation at the mid-point of the time interval, Δt. Thus,

$$\underset{\approx}{M}\dot{\underset{\sim}{V}}^{a} + \underset{\approx}{C}(\underset{\sim}{u}^{a})\underset{\sim}{V}^{a} + \underset{\approx}{K}\underset{\sim}{V}^{a} = \underset{\sim}{F}^{a} \quad , \tag{D3}$$

where,

$$\dot{\underset{\sim}{V}}^{a} = (\dot{\underset{\sim}{V}}^{n+1} + \dot{\underset{\sim}{V}}^{n})/2 \quad ,$$

$$\underset{\sim}{V}^{a} = (\underset{\sim}{V}^{n+1} + \underset{\sim}{V}^{n})/2 \quad ,$$

$$\underset{\sim}{u}^{a} = (\underset{\sim}{u}^{n+1} + \underset{\sim}{u}^{n})/2 \quad ,$$

$$\underset{\sim}{F}^{a} = (\underset{\sim}{F}^{n+1} + \underset{\sim}{F}^{n})/2 \quad .$$

The averaging process has assumed that $\underset{\sim}{V}$ varies linearly over the interval Δt; therefore, $\dot{\underset{\sim}{V}}$ is constant over Δt and can be expressed by,

$$\dot{\underset{\sim}{V}}^{a} = \dot{\underset{\sim}{V}}^{n+1} = \dot{\underset{\sim}{V}}^{n} = (\underset{\sim}{V}^{n+1} - \underset{\sim}{V}^{n})/\Delta t \quad . \tag{D4}$$

Substituting Equation (D4) into (D3) and using the definition of $\underset{\sim}{V}^a$ allows Equation (D3) to be rewritten as,

$$\frac{2}{\Delta t} \underset{\approx}{M}\underset{\sim}{V}^a + \underset{\approx}{C}(\underset{\sim}{u}^a)\underset{\sim}{V}^a + \underset{\approx}{K}\underset{\sim}{V}^a = \underset{\sim}{F}^a + \frac{2}{\Delta t} \underset{\approx}{M}\underset{\sim}{V}^n \quad , \tag{D5}$$

with,

$$\underset{\sim}{V}^a = (\underset{\sim}{V}^{n+1} + \underset{\sim}{V}^n)/2 \quad .$$

Equation (D5) is a nonlinear algebraic equation system. To reduce the cost of the integration procedure, a quasilinearization process can be applied to Equation (D5) to yield,

$$\frac{2}{\Delta t} \underset{\approx}{M}\underset{\sim}{V}^a + \underset{\approx}{C}(\underset{\sim}{u}^n)\underset{\sim}{V}^a + \underset{\approx}{K}\underset{\sim}{V}^a = \underset{\sim}{F}^a + \frac{2}{\Delta t} \underset{\approx}{M}\underset{\sim}{V}^n \quad . \tag{D6}$$

Equation (D6) is the integration algorithm presently available in NACHOS.

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