BEAM-BEAM INTERACTION IN $e^+ - e^-$ STORAGE RINGS

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1. Introduction

When two beams collide in a storage ring each of them is affected by the presence of the other. This is due to the space-charge forces which occur during the interaction, at each revolution. Such an effect leads to a current limitation of both beams, and then limits the performance of the machine : maximum luminosity.

The first experimental evidence of the beam-beam limit was observed on the double STANFORD-PRINCETON $e^- - e^-$ storage rings. Later on, the same kind of phenomenon was observed on the $e^+ - e^-$ storage rings and for ten years now it has been intensively studied from an experimental point of view as well as from theoretical approaches. However, at the present time, the beam-beam limit is not well understood and more work has to be done, but the similarity of the effect observed on several storage rings permits one to extrapolate the limitation to new machine designs. No ways have been found up to now to cure this effect in order to improve considerably storage ring performance. Space-charge compensation will be used soon but it seems that even this method will be just a partial cure if successful.

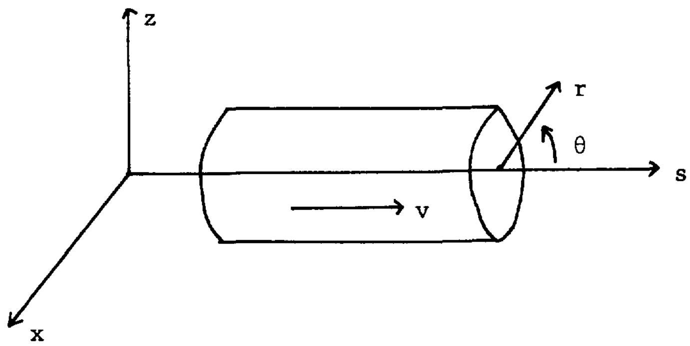
2. The linear beam-beam effect

2.1 The linear tune shift

2.1.1 The space-charge forces

Let us consider a particle from one bunch which crosses the opposite bunch. During the interaction, this particle sees the electromagnetic field which is created by the opposite bunch and its motion, normally defined by the magnetic structure of the ring, will be perturbed. It is usual to say that the particle gets a transverse kick from the other bunch.

Consider first the simple case of a round beam with a Gaussian particle distribution which is often a real situation.



In the laboratory coordinate system, which is the usual frame to look at a particle's motion, the electromagnetic field can be obtained very simply.

From Gauss'law one gets easily the electric field :

$$2\pi\, r\, \varepsilon_o\, E_r = \int_o^r \rho_o\, e^{-\frac{r^2}{2\sigma^2}} .\, 2\pi\, r\, d\, r \qquad (1)$$

$$E_r = -\frac{\rho_o\, \sigma^2}{\varepsilon_o\, r} \left(e^{-\frac{r^2}{2\sigma^2}} - 1 \right) \qquad (2)$$

while Ampere's law applied to the same beam gives :

$$2\pi\, r\, H_\theta = \int_o^r 2\pi\, r\, \rho_o\, e^{-\frac{r^2}{2\sigma^2}}\, v\, d\, r \qquad (3)$$

$$B_\theta = -\frac{\rho_o\, \sigma^2\, v}{\varepsilon_o\, c^2\, r} \left(e^{-\frac{r^2}{2\sigma^2}} - 1 \right) \qquad (4)$$

where :

σ is the r. m. s. of the radial particle distribution.

v is the velocity of the beam.

$$\rho_o = \frac{N\, e}{2\pi\, \sigma^2\, \ell}$$

N is the number of particles in the bunch.

ℓ is the bunch length ($\ell = \sqrt{2\pi}\,\sigma_\ell$)

Notice that this simple approach, where the electric field is purely transverse and the magnetic field purely tangential, neglects the end field of the bunch. But this approximation may be considered in the same way as using equivalent length for the magnetic field of magnets and quadrupoles.

For electrons (or positrons) the velocity is practically equal to the velocity of light and then it becomes obvious that both the electric and magnetic forces, seen by the particle crossing the bunch, are equal in magnitude and direction.

$$\overrightarrow{\Delta F} = e\,(\vec{E} + \vec{v} \wedge \vec{B}) = 2\,e\,\vec{E} \tag{5}$$

or

$$\overrightarrow{\Delta F}_r = -\frac{2\,e\,\rho_o\,\sigma^2}{\varepsilon_o\,r^2}\left(e^{-\frac{r^2}{2\sigma^2}} - 1\right)\vec{r} \tag{6}$$

When expanding this expression for small values of r one gets :

$$\overrightarrow{\Delta F}_r = \frac{e\,\rho_o}{\varepsilon_o}\,\vec{r} \tag{7}$$

which is the linear part of the space-charge force seen by a particle having small betatron amplitude. In cartesian coordinates it becomes :

$$\Delta F_y = \frac{e\,\rho_o}{\varepsilon_o}\,y \tag{8}$$

where y stands for x or z.

2.1.2. The parameter ξ

According now to the finite bunch length which is often very short, we can consider this perturbation as a quadrupole thin lens. The corresponding gradient error is :

$$\Delta K.\ell = \frac{\Delta\,y'}{y} = \frac{\Delta p_y}{p.y} \qquad \text{(with } p = mc\text{)} \tag{9}$$

where the transverse momentum change is just

$$\Delta p_y = \Delta F_y \cdot \Delta t = \Delta F_y \cdot \frac{\ell}{2c} \tag{10}$$

if we don't forget that both the bunch and the particle are moving at the speed c in opposite directions.

Then one gets :

$$\Delta K.\ell = \pm \frac{N r_e}{\gamma \sigma^2} \tag{11}$$

where r_e is the classical electron radius defined as :

$$r_e = \frac{e^2}{4\pi \varepsilon_o m_o c^2} \tag{12}$$

Notice that the effect is focusing ($\Delta K.\ell < 0$) when the particles' charges have opposite sign. In what follows we shall limit the study to $e^+ e^-$ storage rings.

As we know, such a gradient error will produce a change of the betatron wave number :

$$\Delta\nu_y = - \frac{1}{4\pi} \beta_y \cdot \Delta K.\ell \tag{13}$$

where β_y represents the envelope function at the interaction point. However, this formula is only valid when the perturbation is small and when the natural tune of the machine is not too close to an integral resonance. For this reason the $\Delta\nu$ obtained through that formula, even if it is very often called "the linear tune shift", will be represented by another symbol : ξ. Then one has :

$$\xi_y = \frac{N r_e \beta_y}{4\pi \sigma^2 \gamma} \tag{14}$$

In the more general case of an elliptical beam, still having a Gaussian particle distribution, the results are slightly different :

$$\xi_{x,z} = \frac{N r_e}{2\pi \gamma (\sigma_x + \sigma_z)} \left(\frac{\beta}{\sigma}\right)_{x,z} \tag{15}$$

where σ_x and σ_z now represent the r. m. s. in both directions.

This parameter ξ characterizes the space-charge strength. Moreover it is often used to describe the performance of a machine as it is proportional to the beam's transverse density which is directly connected to the useful counting rate for physics experiments.

2.1.3. The linear tune shift

We assimilated the beam-beam interaction to a thin quadrupole lens located at the crossing point of the two circulating beams. In the preceding section we have assumed the corresponding perturbation to be small enough that the optical properties of the machine remained unchanged apart from the tune. In fact, this is not always true and the additional thin lens has to be introduced in the lattice structure to get a better approximation. Then the total transfer matrix of the machine, assuming there is just one crossing point, can be written :

$$\begin{vmatrix} 1 & 0 \\ \Delta K . \ell & 1 \end{vmatrix} \times \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix}$$

where the unperturbed tune is given by

$$\cos 2\pi \nu_o = \frac{1}{2} (T_{11} + T_{22}) \qquad (16)$$

while the total tune becomes :

$$\cos 2\pi (\nu_o + \Delta\nu) = \cos 2\pi \nu_o + \frac{1}{2} T_{12} . \Delta K . \ell \qquad (17)$$

If we remember that in TWISS'matrix notation one has :

$$T_{12} = \beta_o \sin 2\pi \nu_o \qquad (18)$$

then one gets :

$$\cos 2\pi (\nu_o + \Delta\nu)_{x,z} = \cos (2\pi \nu_o)_{x,z} - 2\pi \xi_{x,z} \sin (2\pi \nu_o)_{x,z} \qquad (19)$$

or :

$$\xi_{x,z} = \frac{\sin (2\pi\ \Delta\nu)_{x,z}}{2\pi} \{ 1 + tg\ (\pi\ \Delta\nu)_{x,z}\ ctg\ (2\pi\ \nu_o)_{x,z} \} \tag{20}$$

$$\simeq \Delta\nu_{x,z} \{ 1 + \pi\ \Delta\nu_{x,z}\ ctg\ (2\pi\ \nu_o)_{x,z} \} \tag{21}$$

This formula shows that when ν_o is not too close to an integer, ξ is a good approximation for the linear tune shift, otherwise $\Delta\nu$ has to be computed through the previous equation. When there are several interaction regions the formula is still valid if ν_o and $\Delta\nu$ are expressed as the tune and the tune shift between two crossing points, assuming all the crossing points are located at homologous points around the ring.

In addition to the tune shift, the matrix formula shows that the β function at the interaction point will be perturbed :

$$\beta \sin 2\pi\ (\nu_o + \Delta\nu) = \beta_o \sin 2\pi\ \nu_o \tag{22}$$

This perturbation is propagated all around the ring meaning that the invariant of the motion is also perturbed. As a consequence the beam cross section at the crossing point is modified.

2.2 Measurements of the linear tune shift

It is possible under certain conditions to measure directly the linear tune shift, but such a method includes also a measurement of the non-linear beam-beam effect, and it is not very accurate. We will come back to this point in another section.

2.2.1 Measurement using luminosity monitors

The usual way to measure the linear tune shift is to measure the luminosity of the machine. Then, let us review briefly the meaning of this parameter.

When two bunches interact the rate of events which occurs at a given interaction region is given by :

$$\dot{n} = f_r \frac{N^+N^-}{S} \sigma_T$$

where σ_T is the total cross-section of the particular process under study, N^+ and N^- are the number of particles in each bunch, S the effective beam area, and f_r the revolution frequency of the machine.

σ_T is related to the dynamics of the process and to the solid angle of the apparatus used to look at the events, and for many processes this parameter can be computed (Bhabha, single and double bremsstrahlung).

If we write :

$$\dot{n} = L \cdot \sigma_T \tag{23}$$

then we get a parameter which characterizes the machine performance and which is called the luminosity :

$$L = f_r \frac{N^+N^-}{S} \tag{24}$$

A monitoring device is used to measure the rate of events for a given process for which the total cross-section is known. It gives then a direct measurement of the luminosity which, in addition to the measured stored currents, leads to a measurement of the beam's effective area. In the case of Gaussian beams the area is :

$$S = 4\pi \sigma_x \sigma_z \tag{25}$$

For round beams the measured area can be directly introduced into the ξ formula, using also the unperturbed β function. The corresponding ξ value can be used to compute the linear tune shift $\Delta\nu$ as shown in the previous section.

For elliptical beams, in addition to the measured effective area, it is necessary to know the real aspect ratio σ_z/σ_x which may have been perturbed by the beam-beam interaction. The unperturbed β value will still enter into the ξ formula, but the perturbed β value must be used to get σ_z/σ_x.

Two extreme cases are generally considered :

a) The beams are affected by a coupling resonance. Then the two transverse dimensions are energy coupled in such a way that :

$$\sigma_z \,/\, \sigma_x = (\beta_z \,/\, \beta_x)^{1/2} \tag{26}$$

The perturbed aspect ratio is obtained by computing the perturbed β function at the crossing point.

b) The beams are flat, which generally means that mainly the vertical motion is affected by the space-charge effect. The radial dimension σ_x then can be taken as the unperturbed one and the vertical dimension σ_z is obtained from the effective area measurement.

2.2.2 Measurement using beam-dimension monitors

Beam sizes can be measured directly in the bending magnets by means of synchrotron light monitors. Such a monitor is described in Section 3.2. These measurements have to be transferred to the crossing point in order to obtain the real effective interaction area. Here again the technique is to use the unperturbed β function in the ξ formula and the perturbed β function to transfer the beam-size measurements from the magnet to the interaction point. However, the perturbed β functions are only known when the real interaction area is known, so that the method is not very accurate unless some iterative method is used.

2.2.3 Comparison between the two methods

The first method is commonly used on various machines. However, on the $e^+ e^-$ ORSAY storage ring (ACO) the second method was also used and we can compare the results. The experimental conditions corresponding to table I were :

$$E = 510 \text{ MeV} \quad , \quad I^+ \sim I^- = 32 \text{ mA}$$

Table I-a : Luminosity measurements

	unperturbed	perturbed
ξ_x	.02	.022
ξ_z	.031	.031

Table I-b : Beam-dimension measurements

	unperturbed	perturbed
ξ_x	.03	.020
ξ_z	.055	.030

The unperturbed values mean that parameters entering in the ξ formula are the natural optical ones. The perturbed values take into account the linear perturbation of the optics due to the beam-beam effect as explained in the previous section.

From table I it appears that both methods agree when applying the perturbation to the optics which means that the linear beam-beam effect must enter in the expression of the space-charge strength.

2.3 The beam-beam limit

In the existing $e^+ e^-$ storage rings, when single-beam current limitations have been pushed far away, a current limitation is still observed when two beams collide. This limit which is obviously due to the space charge effect is considered actually as the ultimate limit on the $e^+ e^-$ storage ring's performance (maximum luminosity).

The beam behaviour close to the limit appears essentially as a bad lifetime on the weaker beam* but, often, just below the limit an enlargement of the transverse dimensions can be observed, leading to a luminosity saturation.

Even if the luminosity is a good parameter to describe the storage ring's performance, it is clear that it is not a good parameter to describe the beam-beam limit because a machine's characteristics (optics, radius, energy) differ considerably from one machine to another.

The space charge strength ξ looks much better as it describes directly the strength of the electromagnetic force which is generated by the collective interaction of the two beams.

However, for a long time, it was believed that the tune shift should be limited by the presence of natural resonances whatever they are. In other words the tune cannot be shifted too close to a resonance. Then it appeared that the real tune shift could be a good parameter to describe the beam-beam limit.

If the parameter ξ is a good representation of the space-charge forces, the parameter $\Delta\nu$ shows much more the ability of the lattice structure to be perturbed by a gradient error. For instance, for a given ξ value, the linear tune shift will be greater than ξ if the natural tune is below an integer and smaller than ξ if the natural tune is above an integer.

In table II we give the beam-beam limit in terms of ξ_{max} and $\Delta\nu_{max}$ for different storage rings and for typical operating energies.

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* Unless it is clearly specified we always consider the case where the two beams have practically the same intensity ($I^+ \sim I^-$).

Table II

Storage ring	Energy (GeV)	ξ_{max}	$\Delta\nu_{max}$	number of crossings
ACO (Orsay)	.510	.03	.04	2
ADONE (Frascati)	1	.06	.03	6
SPEAR (Stanford)	1.5	.06	.04	2

Notice that in this table $\Delta\nu$ is given per crossing, which means that the real tune shift must be multiplied by the number of crossings. Moreover it happens that for the three storage rings we have considered, the limit appears on the vertical motion although ACO and ADONE work on a coupling resonance (round beams) while SPEAR works with flat beams.

It is clear from table II that the beam-beam limit does not correspond to a constant ξ_{max} which means that the maximum space-charge strength allowed is different from one ring to another. But in the case of an interpretation based only upon a pure linear theory, the parameter ξ is meaningless. On the other hand, still having in mind the linear theory, it is not a surprise that the total linear tune shift could be different from one ring to another because the allowed shift should depend on both the natural tune and the location of the first destructive resonance. However, if this interpretation was good it should be possible to compensate the effect just by moving the natural tune, which is not true in practice.

2.4 Dependence of the beam-beam limit upon ring parameters

For a long time it was believed that the beam-beam limit was essentially due to the linear perturbation caused by the beam-beam effect and corresponded to a maximum permissible value for the total linear tune shift. In the previous section we gave a first indication about the fragility of this model. However we can imagine that the beam-beam limit is partly due to the linear effect and if so it must follow specific laws.

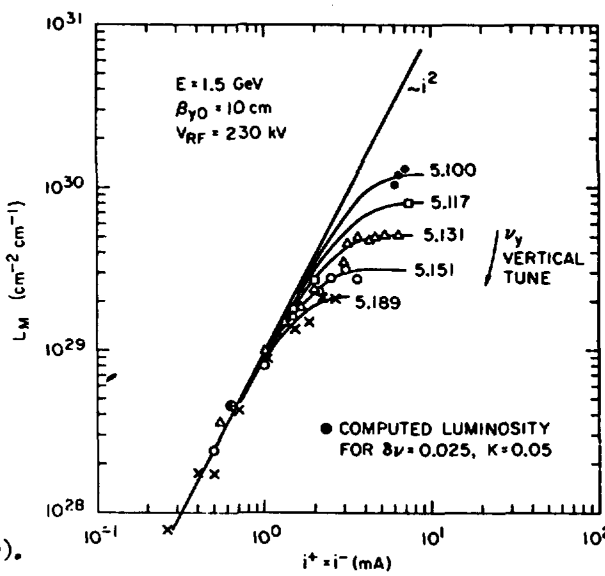
This section is mainly devoted to look for agreements between experiments and linear theory, but we shall not ignore disagreements.

2.4.1 Dependence on the tune

There is concern with the choice of the operating point. The natural tune must be kept as close to an integer as possible and from above. This gives a smaller $\Delta\nu$ for a given space-charge strength, which means that ξ_{max} and then the maximum luminosity should increase when moving towards the integer from above. Experimental evidence of this effect comes from ADONE and SPEAR.

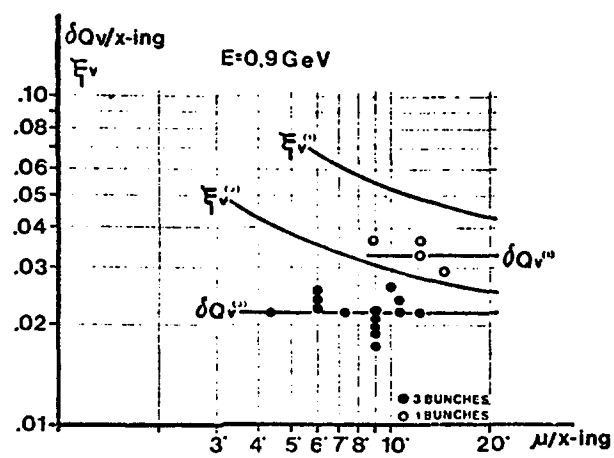
Fig. 1-a shows a gain in luminosity while Fig. 1-b shows an increase of ξ_{max} in the vertical plane ($\mu = 2\pi\ \nu$).

However, Fig. 1-b shows that the total $\Delta\nu$ (called there ΔQ) depends on the number of bunches in each beam. Also the ξ_{max} depends on the number of bunches. The same phenomenon was observed on ACO which can operate either with one bunch per beam or two bunches per beam : with two bunches per beam the ξ_{max} was reduced by a factor $\sim \sqrt{2}$ so the maximum luminosity was independent of the number of bunches.



SPEAR -Luminosity as a function of beam current for different vertical operating points.

Fig. 1-a

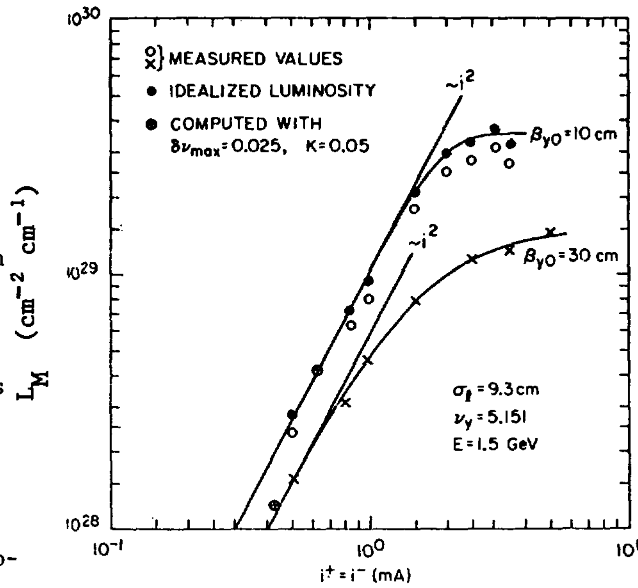


ADONE δQ^V and ξ^V as function of μ, at E = 0, 9 GeV

Fig. 1-b

2.4.2 Dependence on the β's value at the crossing point

Assume now the tune is fixed as well as the number of bunches, so that a constant Δν means also a constant ξ. Then the maximum stored current should be higher when lowering the β at the interaction point. We have a good example of such behaviour in Fig. 2 which comes from SPEAR results. When going from β = 30 cm to β = 10 cm the maximum luminosity increased by a factor 2.5 while at low currents the difference is only $\sim \sqrt{3}$ which corresponds approximately to the expected change in beam cross-section (on these curves y stands for z ; $S \propto \sigma_y \propto \sqrt{\beta_y}$ keeping σ_x practically constant). Another check comes from ACO where the maximum luminosity has been improved by a factor 2 when going from β_z = 400 cm to β_z = 150 cm.



Luminosity for different betatron amplitudes at the interaction point.

Fig. 2

2.4.3 Dependence on beam area

Consider the general case of an elliptical beam. The space-charge strength

$$\xi_z = \frac{2 N r_e \beta_z}{\gamma S (1 + \frac{\sigma_z}{\sigma_x})} \tag{27}$$

shows that the stored current must be higher if the beam's transverse area S and the beam aspect ratio σ_z/σ_x are made larger.

Remembering that the luminosity can be expressed as the product of the current and the density,it is clear that the luminosity will also increased.

Up to now, all the methods used to get an artificial incoherent enlargement of the beams, failed in trying to get more luminosity. The crossing angle, used to increase the effective transverse area of the beam, also failed.

However, on ACO and ADONE it has been possible to increase both the current and the luminosity by increasing the aspect ratio. This is done by moving the tune towards a coupling resonance which makes the beams more or less round. Operation close to a coupling resonance also leads to a natural increase of the transverse area.

2.4.4 Dependence on energy

For a constant ξ_{max} and a given optics, one expects the following energy dependences :

$$N_{max} \propto \gamma^3$$
$$L_{max} \propto \gamma^4 \qquad (28)$$

assuming the beam's transverse area goes like γ^2 as the natural one does.

Experimentally the following γ dependences have been observed

Table III

	current	luminosity	area
ACO	$\gamma^{3.5}$	γ^5	γ^2
ADONE	$\gamma^{4.5}$	γ^7	γ^2
SPEAR	γ^3	γ^4	γ^2

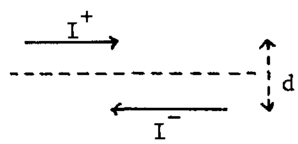
The different behaviour between the three machines makes difficult any interpretation and here again the linear theory is obviously helpless.

Apart from SPEAR, the results can be expressed as a γ dependence for the maximum allowed space-charge strength.

2.5 The effect of small beam separations at the crossing point

If the axis of the two crossing beams are displaced by a quantity d, and if we assume the beam distribution to stay rigid, then the linear beam-beam effect leads to a dipole defect given by :

$$\Delta K \,.\, \ell \,.\, d = \frac{N\, r_e}{\gamma\, \sigma^2} \,.\, d \qquad (29)$$



From the disturbed closed orbit theory it is shown that the displacement of the closed orbit at the location of the dipole error has the following simple form :

$$d - d_o = \frac{N\, r_e}{\gamma\, \sigma^2}\, d\, \frac{\beta}{2\, \mathrm{tg}\, \pi\, \nu}$$

or :

$$d = d_o\, \frac{1}{1 - 2\pi\, \xi\, \mathrm{ctg}\, \pi\, \nu} \qquad (30)$$

where d_o represents the initial displacement which will exist at very low currents.

It is clear that the effect may lead either to a smaller or to a larger displacement of the beams, depending on the choice for the tune. This effect is called "self adjustment of the orbits" and can be useful in the case where the beams circulate in two rings. Unfortunately, this effect has never been observed up to now on existing storage rings.

3. The non-linear beam-beam effect

As seen before it is sometimes hard to interpret part of experimental observations on the basis of the simple linear model. Clearly, the next step is to consider the non-linear part of the space-charge forces which has been neglected up to now. New theoretical considerations need of course new experimental data, but the parameters which have been defined will be still useful.

We shall start presenting a very simple non-linear model for the beam-beam effect which may be useful later on to understand some aspects of the experimental results.

3.1 Tune spread due to the non-linear space charge forces

For a round beam with transverse Gaussian distribution the space-charge force is :

$$\vec{\Delta F}_r = - \frac{2e\,\rho_o\,\sigma^2}{\varepsilon_o\,r^2}\left(e^{-\frac{r^2}{2\sigma^2}} - 1\right)\vec{r} \tag{31}$$

as given in section 2.1.1. Expanding the exponential function one gets :

$$\vec{\Delta F}_r = - \frac{2e\,\rho_o}{\varepsilon_o}\sum_{n=1}^{\infty}(-1)^n\,\frac{1}{n!}\,\frac{1}{2^n}\left(\frac{r}{\sigma}\right)^{2(n-1)}\cdot\vec{r} \tag{32}$$

Let us consider now a test particle, with a pure vertical betatron motion, which crosses the beam. The force seen by the particle is also purely vertical :

$$\Delta F_z = - \frac{2e\,\rho_o}{\varepsilon_o}\sum_{n=1}^{\infty}(-1)^n\,\frac{1}{n!}\,\frac{1}{2^n}\left(\frac{Z^2}{\sigma^2}\right)^{n-1}\cdot Z \tag{33}$$

Assume again the bunch length to be very small and the effect can be assimilated to a non-linear thin lens located at the crossing point. Then, the test particle, after each revolution, will get an angular kick depending on its position. Such a kick will affect the constants of the betatron motion.

The pseudo-harmonic betatron motion can be written :

$$Z(s) = a_z\sqrt{\beta_z(s)}\ \cos\ (\Phi_z(s) + \phi_z) \tag{34}$$

$$Z'(s) = -\frac{a_z}{\sqrt{\beta_z(s)}}\ \sin\ (\Phi_z(s) + \phi_z) + \frac{\beta_z'(s)}{2\beta_z(s)} \cdot Z \tag{35}$$

where a_z and ϕ_z represent the constants of the motion. For instance :

$$a_z = \frac{Z_o}{\sqrt{\beta_z(o)}} \tag{36}$$

where the index "o" represents any azimuthal position which can be of course the crossing point itself, and Z_o is the betatron amplitude at that point.

The particle position being unperturbed, it is easy to show that the angular kick gives the following phase displacement :

$$\Delta\phi_z = -\frac{\beta_z(o)}{Z_o}\ \cos\ (\Phi_z(o) + \phi_z) \cdot \Delta Z'(o) \tag{37}$$

$$\Phi_z(o) = 2\pi\ k\ \nu_z \quad (k = \text{integer}) \tag{38}$$

with (using section 2.1.2) :

$$\Delta Z' = \frac{2N\ r_e}{\gamma\ \sigma^2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} \frac{1}{2^n} \left(\frac{Z^2}{\sigma^2}\right)^{n-1} \cdot Z \tag{39}$$

Putting all these formulae together we get :

$$\Delta\phi_z = -\frac{2N\ r_e\ \beta_z(o)}{\gamma\ \sigma^2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} \frac{1}{2^n} \frac{1}{\sigma^{2(n-1)}} Z_o^{2(n-1)} \cos^{2n}(\Phi_z(o) + \phi_z) \tag{40}$$

The phase displacement takes different values from one turn to another, so let us define the tune shift through the mean value of the phase fluctuations over many turns :

$$\Delta\nu_z = \frac{1}{2\pi} < \Delta\ \phi_z > \tag{41}$$

In doing that, we use trigonometric expansions :

$$\cos^{2n} \alpha = \frac{1}{2^{2n}} \left[\sum_{p=o}^{n-1} 2\, C_p^{2n} \cos 2(p - n)\, \alpha + C_n^{2n} \right] \tag{42}$$

Then the mean effect will only take account of the last term in the bracket. Finally the tune shift can be expressed as :

$$\Delta\nu_z = - \xi_z . \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} \frac{1}{2^{3n-2}} C_n^{2n} \left(\frac{z_o}{\sigma} \right)^{2(n-1)} \tag{43}$$

where C is the binomial coefficient.

It is seen that the tune shift is a function of the betatron amplitude. Fig. 3 shows that the maximum tune shift is obtained for a zero amplitude and corresponds to ξ (or p. ξ if there are p crossings), while the tune shift is zero for large amplitudes. Moreover Fig. 3 takes account of the more general case where the particle has a motion in both transverse planes, but still crossing a round, Gaussian beam.

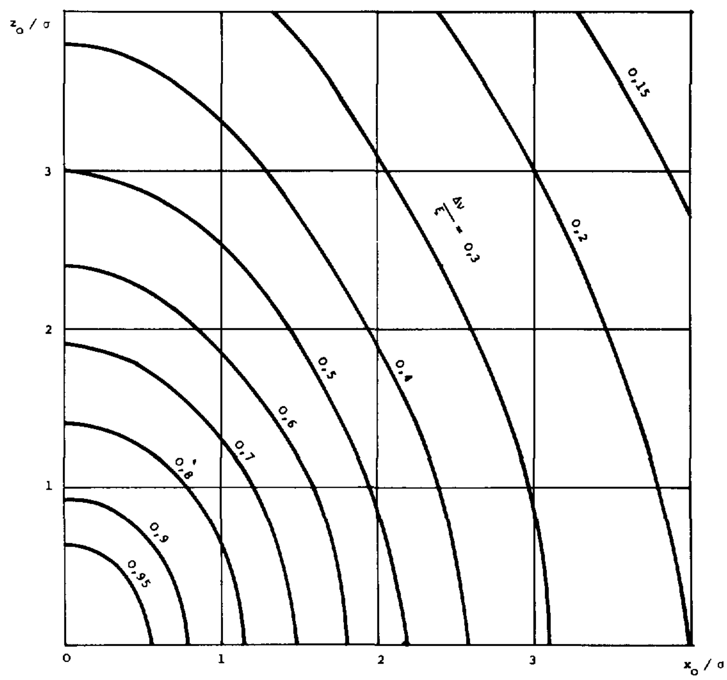


Fig. 3 : Head-on Collision. $\Delta\nu_z = f(x_o, z_o)$

Knowing that the betatron amplitudes are Rayleigh distributed inside the beam, it is possible to get the tune shift distribution using a Monte Carlo method.

Fig. 4 gives the aspect of such a distribution and it is important to notice that the probability of having the maximum or the minimum tune shift iz zero.

As compared to the guide and focusing forces, the non-linear space-charge forces cannot be approximated just by taking the first non-linear terms. For example the octupole term would lead to very large tune shifts for large amplitudes which is an error.

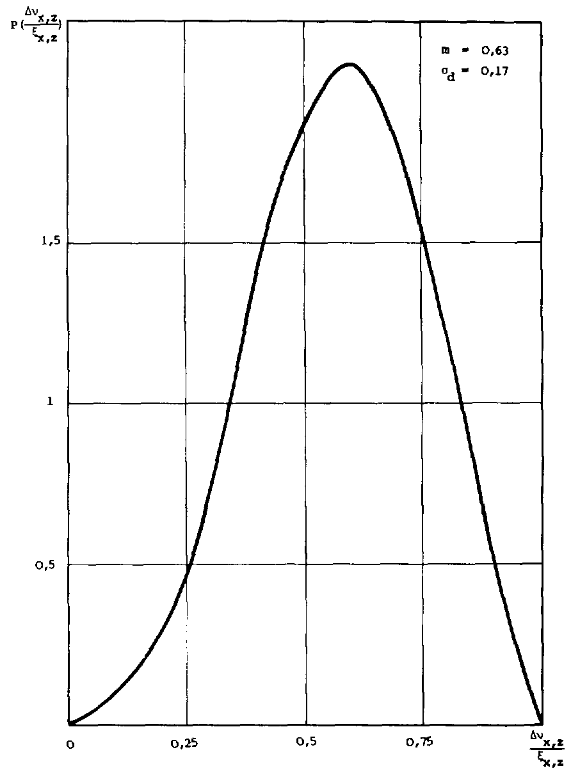


Fig. 4 : Head-On Collision. Tune Shift Distribution Function.

We can understand now the reason why the linear tune shift cannot be compensated by moving the natural tune of the machine. In fact, we have a tune spread and at least the corresponding width must be less or equal to the distance between two dangerous resonances. But this is still a very simple description of the beam-beam effect.

On the other hand the tune spread has some advantages. It is of the same order of magnitude as the tune shift, which means about one or two orders of magnitude above the tune spread one can get using distributed octupoles around the ring. As a matter of fact single-beam transverse instabilities are cured when two beams collide. This effect has been used up to now to operate ACO : during injection, with electrostatic separation of the two beams, the beams are enlarged using noise in order to get more Landau damping to cure the transverse instability (head-tail). The noise is switched off at the same time as the electrostatic separation and the beams are stable with currents much higher than the transverse instability threshold and just below the beam-beam limit.

3.2 Direct measurement of the tune spread

In an electron-positron storage ring the tune is measured using a radio-frequency excitation (R.F.K.O.). When the frequency of the external excitation is equal to the betatron frequency, the particles get transverse energy and the beam dimensions are enlarged. The right frequency is determined by looking at the beam on a T.V. camera. The method is made more sensitive using a synchrotron-light monitor which measures the transverse particle density.

The association of such a monitor with the R.F. excitation is shown in Fig. 5 and works as follows :

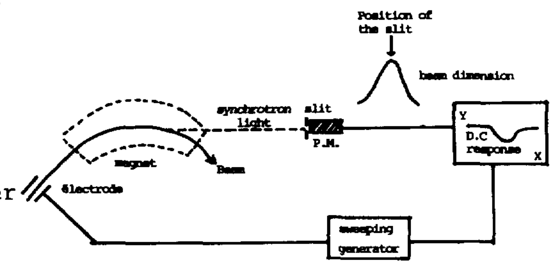


Fig. 5 : Synchrotron light monitor and RF knockout used on ACO.

- An image of the beam is made on a slit, behind which a photomultiplier is installed.
- The slit and the P.M. together can move slowly so that the P.M. gives a signal proportional to the beam density.
- The slit is moved towards the maximum density and the P.M. gives a direct current proportional to that density.
- When the beam is excited the transverse dimension increases and, for a constant number of particles, the density decreases at the observation point.
- Sweeping the frequency of the R.F. excitation, the total number of excited particles varies according to the frequency distribution inside the beam. Then the change in direct current varies more or less proportionally to the number of excited particles.
- Both the sweeping frequency and the P.M. signal feed an x, y plotter which shows the distribution of the betatron frequencies.

The results of such a measurement are shown in Fig. 6 in the case of a strong beam against a weak beam. The narrow peak corresponds to the frequency distribution of the strong beam, unaffected by the space-charge forces, and mainly due to octupole effect or power supplies fluctuations. The wide peak corresponds to the tune spread of the weaker beam.

Fig. 6 : Tune spread as observed on ACO.

This simple method, which was used on ACO, needs however a lot of R.F. excitation and, in the case of two colliding beams, it will disturb the beam-beam effect itself. Experimentally, the method gave nice results only much below the beam-beam limit. Approaching the natural beam-beam limit the beams were destroyed or in other words the beam-beam limit was made lower.

Recently, at SPEAR, a more accurate fast monitor has been successfully used. It is identical in principle to the previous one, but the electronics is more sophisticated and permits detecting the frequency spread without changing the beam sizes by more than 10 %.

3.3 Non-linear resonances induced by the beam-beam interaction

In computing the tune spread we neglected the fact that strong phase relations between successive crossings may not allow a simple averaging over many turns. Let us have a rough look at this problem.

Equations (40) and (42) show that terms like :

$$\cos 2\,(p - n)\,(\Phi_z(o) + \phi_z) \tag{44}$$

enter in the computation of the phase shift. Such terms may contribute to the average over many turns if :

$$2\,(p - n)\,\Phi_z(o) \simeq 2\pi\,p \tag{45}$$

or

$$q\,\nu_z \simeq p \tag{46}$$

and the consequence is a slow variation of $\Delta\phi_z$. A similar effect will happen to the amplitude Z_o as the phase relation can help to increase the transverse energy of the particle. Such a resonant mechanism, in the general case, will occur if the following condition is fulfilled :

$$m \nu_x \pm n \nu_z \simeq p \tag{47}$$

where m + n is called the order of the resonance. It looks obvious in the case of the space-charge forces that high-order resonances are involved, if we remember that it is difficult to approximate the effect by just considering the first non-linear terms in the force.

Studies of the motion close to a given resonance can be performed with usual formalisms but it is painful and many times one prefers to use a computer simulation.

3.4 Experimental evidence of high-order resonances

The experimental work described in the present and the next section has been performed on ACO at 510 MeV. A systematic investigation of the non-linear effect of the beam-beam interaction has been done, mainly looking at the beam-beam limit as a function of tunes over a wide range of operating conditions.

Notice that on ACO with a single circulating beam, only the integer, half integer, and third-order resonances lead to beam losses. The rest of the ν_x, ν_z diagram is favourable for a single beam.

With two colliding beams, the general features can be described in two steps.

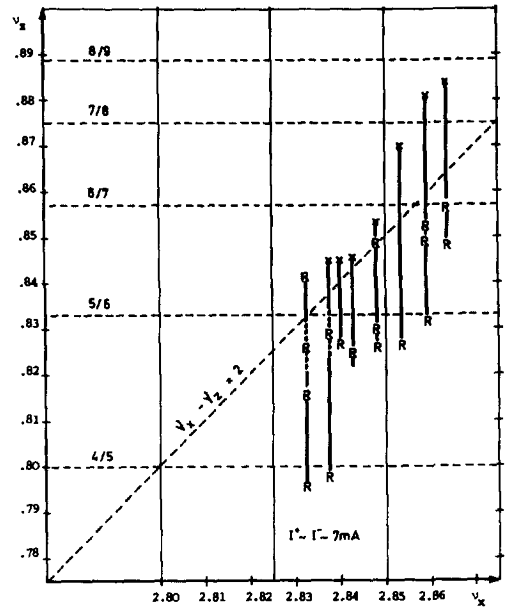
a) At low currents : $I^+ \simeq I^- << I_{max}$

Apart from the three resonances pointed out previously, the beams can be moved in the ν_x, ν_z diagram without any lifetime problem. However, many transverse enlargements of the beam are observed when the tune is approaching a rational value, indicating the excitation of non-linear resonances.

b) <u>At high currents : $I^+ \simeq I^- < I_{max}$</u>

When moving through the $\nu_x\ \nu_z$ diagram, I_{max} depends on the beam shape : flat or round. For instance, at 510 MeV the current limit is 5 mA far away from the coupling resonance $\nu_x - \nu_z = 2$, and 35 mA on the coupling resonance.

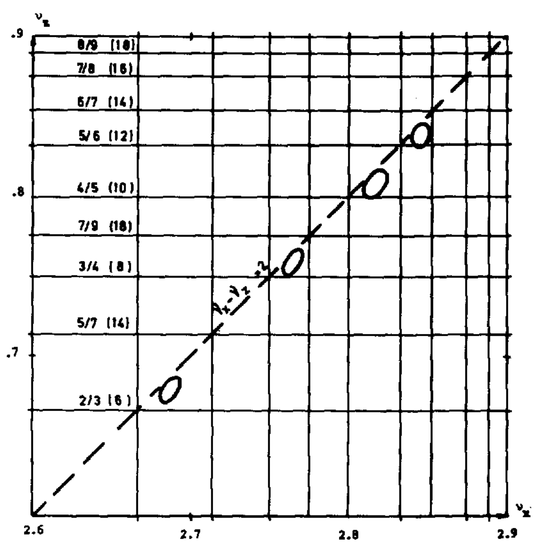
For currents between these two limits Fig. 7 shows that, close to the coupling resonance, a beam enlargement is observed when ν_z is approaching a rational value. Lifetime drops when the beams become flat. The tune values which appear on the figure are the natural ones.



<u>Fig. 7</u> : Range of ν_z versus the radial wave number ν_x.
Enlargement of one beam near a resonance is indicated by R or R.
Low order rational values $\nu_z = p/q$ are indicated by horizontal dotted lines.
(× : limit of the ν_z range when the lifetime drops)

Regions along the resonance line have been explored and Fig. 8 shows small areas, limited by non-linear resonances of high order, where high luminosity can be obtained ($10^{29}\ cm^{-2}\ s^{-1}$). Some areas stayed unexplored because the injection was too difficult there.

At very high currents the boundaries of the available area for the operating point are characterized by a sharp decrease of the beam lifetime.



<u>Fig. 8</u> : ν_x, ν_z Diagram

Studied operating points are indicated by small circles.
Rational values of ν_x and ν_z are indicated by vertical and horizontal lines.
The number between parenthesis gives the order $n = 2q$ of the non-linear resonances associated to the rational number p/q.

The regions far away from the coupling resonance were not studied on ACO for high currents. But recently, this has been done on DCI and Fig. 9 shows that the same behaviour happens everywhere.

It is clear that high-order non-linear resonances play an important part in the beam loss mechanism induced by the beam-beam interaction (ACO is mainly operating with two crossing points).

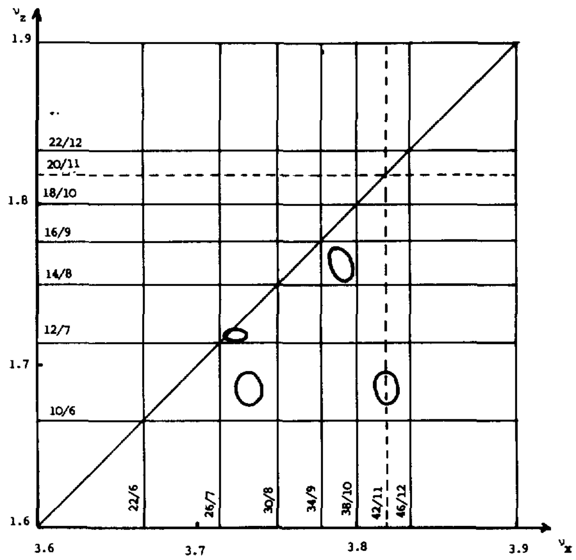


Fig. 9 : Operating zones on D.C.I.

However, it is difficult to separate the effect of simple non-linear resonances from coupled ones.

For a long time, it was believed that ACO was strongly limited because the tunes were sitting just below an integer. (The usual tunes being $\nu_x = 2.845$, $\nu_z = 0.835$). However, practically the same maximum luminosity has been obtained for different areas along the resonance line and it is known that practically no variation of the unperturbed optics occurs in going from one area to another. But we still ignore for this machine the behaviour of two colliding beams just above an integer as it was impossible to inject correctly there.

3.5 Resonance effect as a function of the current

The behaviour of these stable areas in the ν_x, ν_z diagram has been studied as a function of the stored currents. For instance, at the usual operating point of ACO ($\nu_x \simeq 2.845$, $\nu_z \simeq 0.835$), the results are shown on Fig. 10. It is seen that the area decreases when the currents increase. Experimentally, we say that the beam-beam limit is reached when the area is so small that natural power supplies fluctuations make it difficult to stay permanently inside. Good reproducible operating conditions were easily obtained by limiting the width of the area to .003 (in terms of ν). Moreover Fig. 10 shows that the center of the area is displaced when the stored currents increase, and at low currents the area sits practically on the coupling resonance. Notice that, in all cases , the colliding beams are adjusted to have a maximum transverse cross section : beams are practically round.

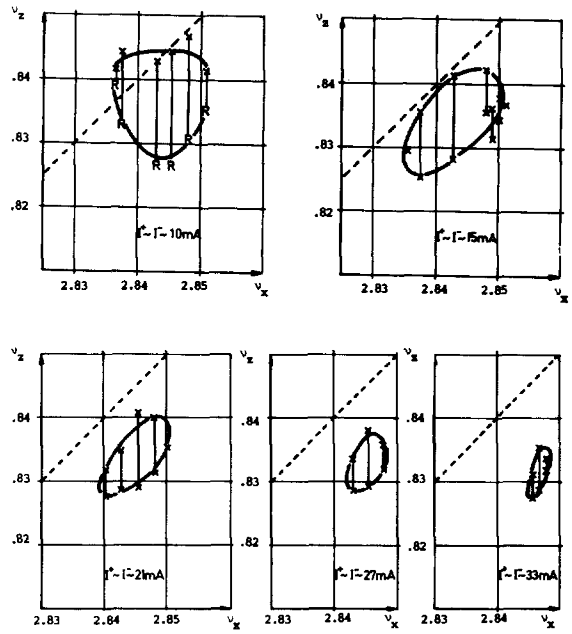


Fig. 10 : Range of ν_z versus the radial wave number ν_x for different intensities.
The coupling resonance $\nu_x - \nu_z = 2$ is indicated by a dotted line.
Enlargement of one beam at an end point of the ν_z range is indicated by R.

This phenomenon can be explained in the following way :

- The operating conditions are such that $\beta_z > \beta_x$ at the crossing point ($\beta_z/\beta_x \sim 2$).

- The beams are round and the tune spread in the vertical plane is expected to be larger than the tune spread in the horizontal plane ($\sim$ a factor 2 again).

- When currents increase, the maximum of the tune spread distributions (approximately ξ) are shifted, and to maintain the beams round (which means most of the particles on the coupling resonance) the natural tunes of the machine must be displaced more vertically than horizontally. As a matter of fact, the area moves down from the coupling resonance (remember the tune shift is positive for $e^- e^+$ collisions). The total displacement in both directions is then limited by non-linear resonances.

Different regions in the ν_x ν_z diagram were investigated in that way. However, to simplify the presentation, the ν_x value in each zone has been fixed, and the range for the maximum ν_z variation as a function of the stored current is shown on Fig. 11. There is no strong difference between these experimental zones.

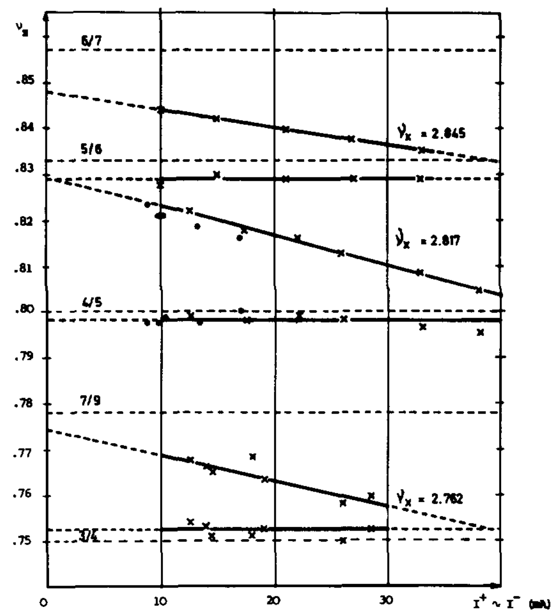


Fig. 11 : Limits of ν_z range versus the intensity of each beam and for 3 values of ν_x.
Low order rational values $\nu_z = p/q$ are indicated by horizontal dotted lines.
x : one bunch per beam
o : two bunches per beam

But an unexpected problem, according to the simple non-linear model, has been observed for the middle zone shown on Fig. 11. With one bunch per beam (2 crossing points) or two bunches per beam (4 crossing points) the boundaries remain unchanged. In fact, speculations would tell us that the dotted lines do not correspond to the same resonance order for the two cases of one or two bunches per beam. From the simple non-linear model, one would expect the following simple non-linear resonances to be excited :

$$m \, \nu_{x,z} = p$$

$$m = \text{odd number}$$

$$p = k \, M \text{ (M number of crossings)}$$

In other words the upper boundary for the middle zone corresponds to the order 12 for the one-bunch case and 24 for the two-bunch case. One may guess that the excitation of high order non-linear resonances decreases as their order increases.

Does that mean the effect is much more complicated than the excitation of simple resonances, but it was difficult to trust the part played by the coupling resonances? We shall put a question mark there and wait for more experimental work.

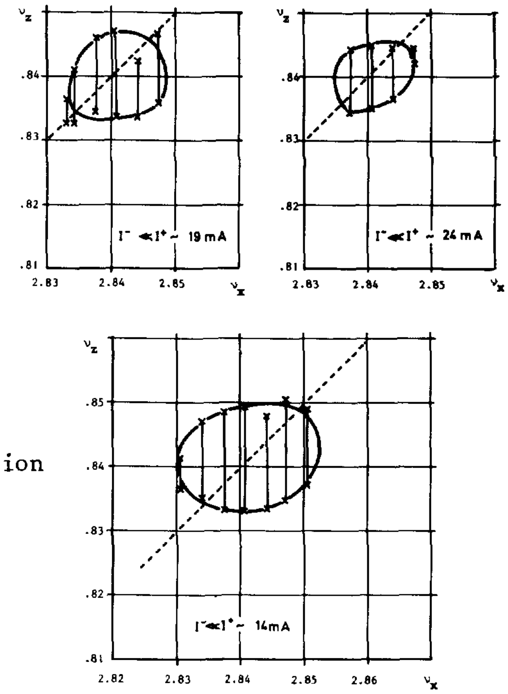


Fig. 12 : Range of ν_z versus the radial wave number ν_x and for different intensities of the strong beam. The coupling resonance $\nu_x - \nu_z = 2$ is indicated by a dotted line.

Before ending this section we must emphasize that there is no different behaviour between the strong beam-strong beam and the strong beam-weak beam interaction, apart from the fact that for the second case (Fig. 12), the center of the stable area stays on the coupling resonance when the strong beam's current increases. But this is not a surprise because the strong beam remains unperturbed if the weak beam's current is chosen small enough.

3.6 A few considerations about the beam-beam effect

The beam-beam effect is a difficult subject on which machine physicists may have differing thoughts. The following section expresses the author's ideas on this topic and, therefore, should be taken with some care.

The experimental results described in the previous section are consistent with an isolated resonance effect, even at the limit where the available space for tune displacements is so small that any fluctuation of the ring parameters will lead to an interaction with one or another resonance.

At low currents, the effect looks quite clear from that point of view. When the tune is moved towards a rational value, beam enlargement is observed which has the same qualitative features as for a single circulating beam when brought on a low-order resonance.

For higher beam currents, beam losses are observed close to the same rational values. Increasing the currents makes it more difficult to come near the rational value, and this can be interpreted as a widening of the resonance stop-band.

The conventional, isolated resonance treatment based on perturbation methods, gives the resonance width but also shows that, inside such a stop-band, the amplitude is weakly modulated and cannot explain the observed losses. The small amplitude modulation is related to the very large tune spread associated with the non-linear beam-beam effect, which means that the damping term (tune spread) is much larger than the driving term of the resonance. This is just opposite to what happens with the non-linearities of the ring focusing fields.

All these considerations show that the non-linearity of the space-charge force plays an important role, but needs the addition of another mechanism to explain the beam losses.

Generally, in an $e^+ e^-$ storage ring, beam losses are generated by random fluctuations through a diffusion process. Howewer, when the walls are far away from the particle distribution, a stationary state is obtained.

Random fluctuations still exist when space-charge forces occur, but the diffusion process may be accelerated due to the trapping effect of the non-linearities. For instance, particles are trapped in small islands in the phase space and tune fluctuations will displace these islands. Then, particles can be brought faster towards large amplitudes which is the condition for loss.

However, question marks still exist when using such a model. For instance, the effect, as shown in the previous section, does not depend very much on the order of the resonance. The peculiar case of two bunches per beam, compared to one bunch per beam, would be more explicable if the beams were slightly separated at the crossing points (the periodicity of the crossing points is then lower) due, for instance, to a coherent effect. Experimentally, coherent beam-beam effects have been often observed, but with small amplitudes probably due to the strong Landau damping which is related to the beam-beam tune spread.

Another unexplained observation is that the beam-beam effect looks different at low and high energy. On ACO, at low energy, beam enlargement is seen before the lifetime becomes bad, while at high energy the losses occur although the Gaussian distribution is practically unaffected. A partial answer will take account of the damping time in $e^+ e^-$ storage rings, which is much larger at low energy than at high energy.

For a long time, it was believed that the use of beams crossing at an angle would be better because the effective transverse area would be larger. However, this interpretation is based on the linear theory. The non-linear theory will predict, for that case, the excitation of satellite resonances due to the longitudinal motion, and then smaller available area in the ν_x ν_z diagram. It seems that the last point is more in agreement with experimental observations on DORIS.

It is important to note that, even if the beam-beam effect is related to the non-linear nature of the interaction, it can be still expressed as an upper limit on the tune shift. This is due to the fact that the driving terms of the resonances are directly proportionnal to the ξ parameter, as well as the damping term (tune spread).

4. Optimization of storage ring performance

The main question arising from our knowledge of the beam-beam effect is : can we extrapolate the present results to future machines? The main trouble, of course, is that presently operating machines behave quite differently. Moreover, the non-linear nature of the beam-beam effect, as pointed out, does not help very much, apart from the fact that tunes must be chosen inside discrete areas and crossing angles must be avoided.

There remains the use of the parameter ξ for which some conservative upper limit can be taken, say ξ_{max} = 0.025 to 0.05 depending on the optimism of designers. The use of many bunches may give at least a luminosity per interaction region, equal or slightly larger as compared to the one-bunch case. Even without any improvement for the luminosity at a given interaction region, the use of many bunches is interesting to provide luminosity simultaneously at several interaction regions.

In $e^+ e^-$ storage rings there is also concern with a second current limitation due to the RF power. Part of the RF power will feed the cavity and the rest will be used to compensate for synchrotron-radiation energy loss. The stored current will be limited according to the maximum available power for the beams. This limit can be expressed as :

$$I_{max} = \frac{P_{max}}{\delta E} , \qquad (48)$$

δE represents the energy loss per turn, and $P_{max} = P_T - P_J$ where P_T is the total maximum power from the transmitter and P_J the Joule losses in the cavity.

When the power limitation is reached, the stored current becomes less than the space-charge limit and decreases very rapidly when increasing the energy ($L_{max} \propto \gamma^{-10}$). The transition point, where the two limits are equal, will roughly represent the maximum operating energy of a storage ring.

Then, when the transition point has been determined (depending of course on the total R.F. power) the energy range of the machine is known and the currents inside this range will be just space-charge limited.

Now, the optimization of the storage ring is done by using a constant ξ_{max} in the one bunch mode.

Putting together formulae (24) and (15) one gets :

$$L_{max} = \frac{f_r}{4\, r_e^2} \left(1 + \frac{\sigma_z}{\sigma_x}\right)^2 \xi_{max}^2\, \gamma^2\, \frac{S}{\beta_z^2} \tag{49}$$

assuming :

$$\beta_z/\sigma_z \geqslant \beta_x/\sigma_x \tag{50}$$

a condition which is very often fulfilled.

The transverse beam area depends on the envelope function and the coupling effect. In the absence of coupling, the vertical dimension is very small for an isomagnetic ring and the horizontal dimension can be written :

$$\sigma_x = \sqrt{U_{xo} \cdot \beta_x} \cdot \frac{\sigma_E}{E} \tag{51}$$

where σ_E represents the r.m.s. of the particles' energy distribution in the synchrotron phase space, and U_{xo} is the invariant of the horizontal betatron motion related to the choice of the lattice.

The real vertical dimension, in practice, will be determined by the amount of residual coupling, this coupling depending on the proximity of a coupling resonance ($\nu_x - \nu_z = p$). As a consequence, transverse energy is transferred from the horizontal motion to the vertical one and one can write :

$$\sigma_{x,z} = \sqrt{U_{x,z} \cdot \beta_{x,z}} \cdot \frac{\sigma_E}{E} \tag{52}$$

with :

$$U_x + U_z = U_{xo} \tag{53}$$

We can define the coupling constant as :

$$U_z / U_x = k \qquad 0 < k < 1 \tag{54}$$

and we get :

$$L_{max} = \frac{\pi f_r}{r_e^2} \left(\frac{\sigma_E}{E}\right)^2 \xi^2_{max} \gamma^2 . U_{xo} . \frac{\sqrt{k}}{1 + k} . \sqrt{\frac{\beta_x}{\beta_z}} \left(1 + \sqrt{k}\sqrt{\frac{\beta_z}{\beta_x}}\right) \frac{1}{\beta_z} \tag{55}$$

Now, it becomes clear that optimum conditions are :

a) $k = 1$

b) $\beta_x >> \beta_z$ at the interaction point

c) U_{xo} as large as possible.

Notice that σ_E/E only depends on the energy $\left(\frac{\sigma_E}{E} \propto E\right)$.

The first condition is easy to obtain, just by moving the tune on a coupling resonance.

The second condition is made by using long straight sections with matching quadrupoles.

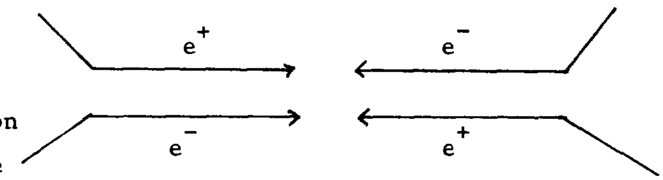
The last condition is made either by adjusting the betatron phase advance in the normal cells (U_{xo} is larger for weak focusing cells), or by changing the transverse damping constants.

However, the beam dimension goes like γ and must be always compatible with the transverse acceptance of the ring (the vertical acceptance is mainly the limiting one). Then, a good optimization would be to vary U_{xo} like γ^{-2} so that the luminosity would go like γ^2. Such a law is a considerable improvement, when the maximum luminosity which occurs at the transition point has been fixed.

The recipes given above do not really cure the space-charge effect. A good cure should include the cancellation of the non-linear terms. Howewer, this is not possible using multipolar magnetic lenses for two reasons :

- The order and the number of multipoles are too high.
- The space-charge force is POISSON-shaped, while the magnetic force is LAPLACE-shaped.

The ORSAY group has proposed a few years ago to compensate the space-charge force by using another beam of opposite charge. This scheme needs two rings having common interaction straight sections where the beams cross as indicated by the sketch below.



A perfect compensation can only be obtained if the companion beams are identical and have the same orbits. But this is impossible for technical reasons and, even then, an instability of the four beams has been predicted for which the cure has not yet been defined.

The expected gain in current is one order of magnitude giving two orders of magnitude in luminosity, as compared to the one-ring case. It should be noted that high currents can be difficult to handle (injection, vacuum, R.F. power, etc...) and this restricts the use of space-charge compensation to intermediate energy storage rings. A double ring system is being set up at ORSAY, and the four-beam operation will be tested soon.

In principle the space-charge compensation scheme could be combined with the small β method, if the lattice can stand it, giving thus an extra gain in luminosity, or at least less current for a given luminosity design.

One of the disadvantages of the space-charge compensation is the reduced Landau damping. Then the interaction can no longer cure single beam transverse instabilities.

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