SINGLE PARTICLE DYNAMICS - HAMILTONIAN FORMULATION

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1. Introduction

In the first two lectures we have seen the linear theory of accelerator dynamics developed in terms of second-order differential equations of motion. One might therefore ask what purpose is served by introducing the Hamiltonian formalism, which at first sight appears to be a roundabout way of obtaining the same equations of motion that one started with.

There are several good reasons. In the first place the Hamiltonian is a "total-energy" type of function and energy is a very familiar concept to physicists and engineers. Since the conjugate variables are closely associated with the kinetic and potential parts of the energy, the Hamiltonian method furnishes a clear insight into the behaviour of a system. Such a presentation can already be useful in fairly simple problems of classical dynamics; for more complicated systems, or for situations where the dynamical variables have a more abstract significance, the Hamiltonian formulation is valuable in suggesting analogies to familiar problems.

It frequently happens that one can readily specify the equations of motion in a simple co-ordinate system which is not the one best adapted to finding the solution. The Hamiltonian method then facilitates the transformation between the two co-ordinate systems by a formalism which ensures the correct equations of motion in the new system, in which they may be unfamiliar.

The canonical transformations of Hamiltonian theory often permit the simplification of a problem by finding invariants of the motion, reducing the number of degrees of freedom, or decoupling two or more degrees of freedom. Nonlinear problems are frequently simplified by transforming away the linear part of the motion.

In a single lecture it is only possible to cover these topics in brief outline, to demonstrate the methods by elementary examples and to indicate the procedures used to handle specific problems in accelerator theory. The formal theory is treated in many textbooks, for example Goldstein [1], and particular applications are covered in some of the references cited below.

2. Outline of Lagrangian and Hamiltonian Formalism

The dynamical behavour of a system may be characterised by a set of generalised co-ordinates q_k and their derivatives $\dot{q}_k$ with respect to some independent variable t, which can conveniently be chosen to be the time. In the formal development of the theory one introduces the Lagrange function or Lagrangian $L(q_k, \dot{q}_k, t)$ which, for a nonrelativistic system in the absence of velocity-dependent potentials is

$$L(q_k, \dot{q}_k, t) = T - U \tag{1}$$

where T and U are respectively the kinetic and potential energies expressed in terms of $(q_k, \dot{q}_k, t)$.

In the relativistic case, and in the presence of electromagnetic fields, the Lagrangian for a single particle is

$$L = - m_0 c[c^2 - \vec{v} \cdot \vec{v}]^{\frac{1}{2}} - e\phi + e\vec{A} \cdot \vec{v} \tag{2}$$

where $\vec{A}$ is the vector potential, ϕ the scalar potential and $\vec{v}$ the particle velocity. Equation (2) has been written in vector form for compactness; it is implicit that the velocity $\vec{v}$ is expressed in terms of the components $\dot{q}_k$ appropriate to the co-ordinate system used, and that ϕ and $\vec{A}$ are functions of the generalised position co-ordinates q_k and of the time t in general. This Lagrangian is <u>not</u> equal to the difference between kinetic and potential energies.

The action integral S is defined by

$$S = \int_{t_1}^{t_2} L \, dt \tag{3}$$

and Hamilton's principle of stationary (often minimum) action states that the variation of the action integral vanishes to first order in the neighbourhood of the dynamically true trajectory (for fixed t_1 and t_2)

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt = 0 \tag{4}$$

Evaluation of (4) by the calculus of variations results in the Lagrangian equations of motion (for a conservative system)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{L}{\partial q_k} = 0 \tag{5}$$

A very clear discussion of (4) is given by Feynman [2].

For a system with k degrees of freedom the Lagrangian is a function of 2k dynamical variables (generalised co-ordinates and velocities) and the time t, k of these variables ($\dot{q}_k$) being the time derivatives of the other k variables (q_k). The Lagrange equations (5) consist of a set of k second-order differential equations describing the motion of the system.

The Hamiltonian formulation of mechanics describes a system in terms of generalised co-ordinates q_k and generalised momenta p_k. The change of basis from the set (q_k, $\dot{q}_k$, t) to the set (q_k, p_k, t) is obtained through a Legendre transformation [1] defined by the function

$$H(q, p, t) = \sum_k p_k \dot{q}_k - L(q, \dot{q}, t) \tag{6}$$

We can take the differential of H on the left-hand side as

$$dH = \frac{\partial H}{\partial t} dt + \sum_k \frac{\partial H}{\partial q_k} dq_k + \sum_k \frac{\partial H}{\partial p_k} dp_k \tag{7}$$

or alternatively, from the right-hand side of (6) as

$$dH = \sum_k p_k d\dot{q}_k + \sum_k \dot{q}_k dp_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt \tag{8}$$

If now in (8) we define canonically conjugate momenta

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{9}$$

the first and fourth summations cancel and the remaining terms can be identified with the corresponding terms in (7)

$$\frac{\partial H}{\partial t} = - \frac{\partial L}{\partial t} \tag{10}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \tag{11}$$

and from (5) and (9)

$$\dot{p}_k = - \frac{\partial H}{\partial q_k} \tag{12}$$

The function $H(q, p, t)$ is the Hamiltonian and equations (11) and (12) are the Hamilton equations of motion. They are of first order, 2k in number for k degrees of freedom, and show a remarkable symmetry of form between the generalised position co-ordinates q_k and their conjugate momenta p_k. This symmetry leads to very flexible transformation properties between sets of dynamical variables.

For non-relativistic motion the Hamiltonian is often, though not necessarily, the sum of potential and kinetic energies

$$H(q, p, t) = T + U \tag{13}$$

A relativistic Hamiltonian for a single particle in an electromagnetic field can be derived from the Lagrangian of equation (2). In cartesian co-ordinates, $k = x, y, z$, the canonical momenta given by (9) are

$$p_k = m_0\gamma\, v_k + eA_k \tag{14}$$

and differ from the component $m_0\gamma\, v_k$ of the mechanical momentum by the contribution eA_k, which is sometimes called the electromagnetic momentum. The resulting Hamiltonian is easily shown to be

$$H(q, p, t) = e\phi + c[(\vec{p} - e\vec{A})^2 + m_0^2c^2]^{\frac{1}{2}} \tag{15}$$

where the p_k have been expressed in vector form.

3. Some Properties of the Hamiltonian

From the Hamiltonian $H(q_k, p_k, t)$ the Hamilton equations of motion are obtained by

$$\dot{q}_k = \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \tag{11}$$

$$\dot{p}_k = \frac{dp_k}{dt} = - \frac{\partial H}{\partial q_k} \tag{12}$$

The Poisson bracket of any two dynamical variables $f(q_k, p_k, t)$ and $g(q_k, p_k, t)$ is defined by

$$\{f, g\} \equiv \sum_k \left\{ \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right\} \tag{16}$$

One sees that

$$\{f, f\} \equiv 0$$

$$\{g, f\} = - \{f, g\}$$

The time derivative of f

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left\{ \frac{\partial f}{\partial q_k} \dot{q}_k + \frac{\partial f}{\partial p_k} \dot{p}_k \right\}$$

$$= \frac{\partial f}{\partial t} + \{f, H\} \qquad \text{(from (11), (12) and (16))} \tag{17}$$

In particular, if $f = H$ we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{18}$$

If H does not depend explicitly on time it is a constant of the motion. Any invariant of the motion not containing t explicitly, has a vanishing Poisson bracket with H.

4. Canonical Transformations

The form of the equations is preserved in transforming between coordinate systems (q_k, p_k) and (Q_k, P_k). The necessary and sufficient condition for a transformation to be canonical is

$$\left[\sum_k P_k\, dQ_k - H_1(Q_k,\ P_k,\ t)dt\right] - \left[\sum_k p_k\, dq_k - H(q_k,\ p_k,\ t)dt\right] = dG \qquad (19)$$

where dG is a total differential. This follows from Hamilton's variational principle

$$\delta\int_{t_1}^{t_2} L\ dt = \delta\int_{t_1}^{t_2} \left[\sum_k p_k\, \dot{q}_k - H(q_k,\ p_k,\ t)\right] dt = 0 \qquad (20)$$

and equation (6). The canonical transformation

$$Q_k = Q_k(q_1,\ q_2\ \ldots .\ p_1,\ p_2\ \ldots .\ t) \qquad (21)$$

$$P_k = P_k(q_1,\ q_2\ \ldots .\ p_1,\ p_2\ \ldots .\ t) \qquad (22)$$

is derived from the generating function G which can have one of four forms:

$$G_1(q,\ Q,\ t): \qquad p_k = \frac{\partial G}{\partial q_k}\ ; \qquad P_k = -\frac{\partial G}{\partial Q_k} \qquad (23a)$$

$$G_2(q,\ P,\ t): \qquad p_k = \frac{\partial G}{\partial q_k}\ ; \qquad Q_k = \frac{\partial G}{\partial P_k} \qquad (23b)$$

$$G_3(Q,\ p,\ t): \qquad P_k = -\frac{\partial G}{\partial Q_k}\ ; \qquad q_k = -\frac{\partial G}{\partial p_k} \qquad (23c)$$

$$G_4(p,\ P,\ t): \qquad q_k = -\frac{\partial G}{\partial p_k}\ ; \qquad Q_k = \frac{\partial G}{\partial P_k} \qquad (23d)$$

For all forms the new Hamiltonian is

$$H_1(Q_k,\ P_k,\ t) = H(q_k,\ p_k,\ t) + \frac{\partial G}{\partial t} \qquad (24)$$

5. Integral Invariants, Liouville's Theorem

The action integral

$$\int_{t_1}^{t_2} L\ dt \qquad (25)$$

is invariant under a canonical transformation. So also are the phase space integral invariants (Poincaré invariants):

$$J_1 = \iint \sum_k dp_k \, dq_k \tag{26a}$$

$$J_2 = \iiiint \sum_{k,i} dp_i \, dp_k \, dq_i \, dq_k \tag{26b}$$

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$$J_n = \int \dots \int dp_1 \dots dp_n \, dq_1 \dots dq_n \tag{27}$$

where the integrals are taken over any arbitrary phase space submanifold of appropriate dimension (2, 4 ... 2n respectively). For a system with n degrees of freedom, the invariance of J_n is Liouville's theorem.

6. Linear Oscillator

The energies are:

kinetic $\quad T = \frac{m}{2} \dot{q}^2 \quad$ (m = mass)

potential $\quad U = \frac{k}{2} q^2 \quad$ (k = spring constant)

The Lagrangian is:

$$L(q, \dot{q}) = T - U = \frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2$$

and the canonical momentum:

$$p = \frac{\partial L}{\partial \dot{q}} = m \, \dot{q}$$

whence $\dot{q} = \frac{p}{m}$. Using this to replace $\dot{q}$ in T, the Hamiltonian becomes:

$$H = T + U = \frac{p^2}{2m} + \frac{k \, q^2}{2}$$

and the Hamilton equations of motion are:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \text{and} \quad \dot{p} = - \frac{\partial H}{\partial q} = - k \, q$$

These can be expressed as one second-order equation

$$\ddot{q} = \frac{\dot{p}}{m} = -\frac{k}{m} q \quad \text{or} \quad \ddot{q} + \omega^2 q = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

We can transform to a new co-ordinate system of action-angle variables using a canonical transformation with the generating function

$$G(q, Q) = \frac{\sqrt{k\,m}}{2} q^2 \cot Q \tag{28}$$

Then using (23a) one has

$$p = \frac{\partial G}{\partial q} = \sqrt{k\,m} \cdot q \cot Q$$

and

$$P = -\frac{\partial G}{\partial Q} = \frac{\sqrt{k\,m}}{2} q^2 \operatorname{coseq}^2 Q$$

whence

$$q^2 = \frac{2P \sin^2 Q}{\sqrt{k\,m}} \quad \text{and} \quad p^2 = 2P \sqrt{k\,m} \cos^2 Q$$

Also,

$$H_1 = H + \frac{\partial G}{\partial t} = H \qquad (\text{since } \frac{\partial G}{\partial t} = 0),$$

so

$$H_1(Q, P) = P \sqrt{\frac{k}{m}} \cos^2 Q + P \sqrt{\frac{k}{m}} \sin^2 Q = P \sqrt{\frac{k}{m}} = \omega P$$

The transformed Hamilton equations of motion are then

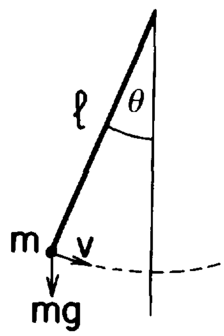
$$\dot{P} = -\frac{\partial H_1}{\partial Q} = 0 \; ; \quad (P = \text{constant and } H_1 \text{ is cyclic in } Q)$$

and

$$\dot{Q} = \frac{\partial H_1}{\partial Q} = \omega \; ; \quad (Q = \omega t + \phi \text{ and } q = \sqrt{\frac{2P}{m\omega}} \sin(\omega t + \phi))$$

This type of canonical transformation is useful in some nonlinear problems to transform away the linear part of the motion and treat the nonlinear part by perturbation theory.

7. Simple Pendulum



$$U = m\,g\,\ell(1 - \cos\theta)$$

$$T = \frac{m\,v^2}{2} = \frac{m\,\ell^2\,\dot{\theta}^2}{2}$$

$$L(\theta, \dot{\theta}) = T - U = \frac{m\,\ell^2\,\dot{\theta}^2}{2} - m\,g\,\ell(1 - \cos\theta)$$

$$p = \frac{\partial L}{\partial \dot{\theta}} = m\,\ell^2\,\dot{\theta}, \quad \text{so} \quad \dot{\theta} = \frac{p}{m\,\ell^2}$$

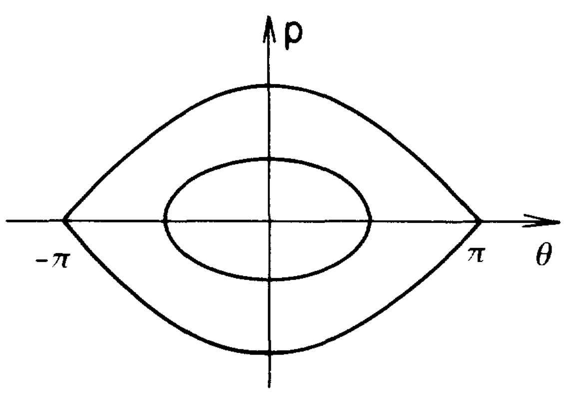
$$H(\theta, p) = T + U = \frac{p^2}{2m\,\ell^2} + m\,g\,\ell(1 - \cos\theta)$$

$$\dot{p} = -\frac{\partial H}{\partial \theta} = -\,m\,g\,\ell\,\sin\theta$$

$$\ddot{\theta} = \frac{\dot{p}}{m\,\ell^2} = -\,\frac{g}{\ell}\sin\theta \qquad \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

The Hamiltonian can be written

$$H(\theta, p) = \frac{p^2}{2m\,\ell^2} + 2m\,g\,\ell\,\sin^2\frac{\theta}{2} = \text{constant} \qquad (\text{since } \frac{dH}{dt} = \frac{\partial H}{\partial t} = 0).$$



Trajectories of constant H in (θ, p) phase plane.

This example is closely analogous to the problem of phase oscillations in a synchrotron or storage ring.

8. Relativistic Hamiltonian with Electromagnetic Fields

The relativistic Lagrangian for a single particle in an electromagnetic field is, using vector notation

$$L = - m_0 c^2 \sqrt{1 - \beta^2} + e\vec{A} \cdot \vec{v} - e\phi \qquad (29)$$

where $\vec{A}$ is the vector potential and ϕ the scalar potential.

Note that $L \neq T - U$ relativistically. However, the canonical momenta p_k are still obtained from

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = m_0 \gamma \, v_k + eA_k \qquad (30)$$

but are no longer the same as the mechanical momenta $m_0\gamma \, v_k$ in the presence of a velocity-dependent potential $\vec{A}$.

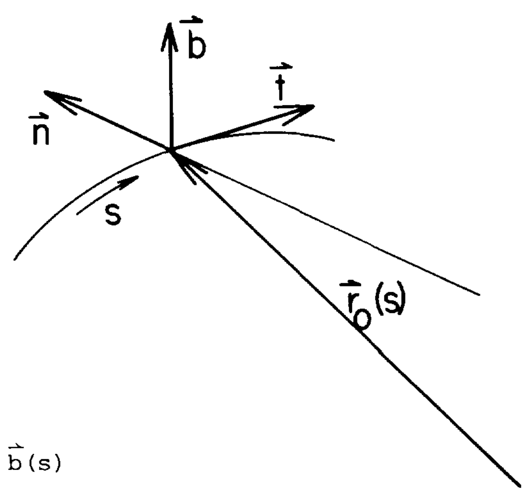
The relativistic single-particle Hamiltonian in the electromagnetic field becomes

$$H(\vec{q}, \vec{p}, t) = e\phi + c[(\vec{p} - e\vec{A})^2 + m_0{}^2c^2]^{\frac{1}{2}} \qquad (31)$$

Since $(\vec{p} - e\vec{A})^2 = (m_0 c \, \beta \, \gamma)^2$, $H = e\phi + m_0 c^2 \, \gamma$ has the value of the total energy, including the rest energy.

9. Acceleration to an Accelerator Orbit

We take as reference curve $\vec{r}_0(s)$ and express any neighbouring point $\vec{r}(x, y, s)$ in terms of the Frenet unit vectors



$\vec{t}(s)$ = tangent

$\vec{n}(s)$ = normal

$\vec{b}(s)$ = binormal

$$\vec{r}(x, y, s) = \vec{r}_0(s) + x\,\vec{n}(s) + y\,\vec{b}(s)$$

The following properties hold

$$\frac{d\vec{r}_0}{ds} = \vec{t} \qquad \frac{d\vec{t}}{ds} = \kappa\,\vec{n}$$

$$\frac{d\vec{n}}{ds} = \tau\,\vec{b} - \kappa\,\vec{t} \qquad \frac{d\vec{b}}{ds} = -\,\tau\,\vec{n} \tag{32}$$

where $\kappa = \frac{1}{\rho}$ is the curvature and τ the torsion of the reference curve.

The canonically-conjugate momenta $\vec{p}$ in the co-ordinate system $\vec{r}$ are obtained from the generating function

$$\begin{aligned} G_2(\vec{r}, \vec{P}) &= \vec{P} \cdot \vec{r} \\ &= \vec{P} \cdot [\vec{r}_0(s) + x\,\vec{n}(s)\;y\,\vec{b}(s)] \end{aligned} \tag{33}$$

which generates the identity transformation

$$p_k = \frac{\partial G}{\partial r_k} = P_k \tag{34a}$$

$$Q_k = \frac{\partial G}{\partial P_k} = r_k \tag{34b}$$

and since $\frac{\partial G}{\partial t} = 0$, the Hamiltonian is unchanged. From (33) and (34) we have, using (32)

$$\begin{aligned} p_s &= \frac{\partial G}{\partial s} \\ &= \vec{P} \cdot \left[\frac{d\vec{r}_0}{ds} + x\,\frac{d\vec{n}}{ds} + y\,\frac{d\vec{b}}{ds}\right] \\ &= \vec{p} \cdot [\vec{t} + x(\tau\,\vec{b} - \kappa\,\vec{t}) - y\,\tau\,\vec{n}] \\ &= \vec{p} \cdot [(1 - \kappa\,x)\vec{t} + \tau(x\,\vec{b} - y\,\vec{n})] \end{aligned} \tag{35}$$

If we now define the reference curve to be plane, the torsion τ vanishes. Taking also the other canonical momenta from (34a) we have

$$\begin{aligned} p_x &= \vec{p} \cdot \vec{n} \\ p_y &= \vec{p} \cdot \vec{b} \\ p_s &= \vec{p} \cdot \vec{t}(1 - \kappa\,x) \end{aligned} \tag{36}$$

The vector potential $\vec{A}$ transforms similarly

$$A_x = \vec{A} \cdot \vec{n}$$
$$A_y = \vec{A} \cdot \vec{b} \qquad (37)$$
$$A_s = \vec{A} \cdot \vec{t}(1 - \kappa\, x)$$

One notes that p_s, A_s are generally <u>not</u> the components in the tangential direction of the reference curve.

The Hamiltonian (31) can now be written explicitly in terms of the canonical momenta (36) and the vector potential components (37). We choose the Coulomb gauge for Maxwell's equations to make $\phi = 0$ and obtain

$$H(q_k,\ p_k,\ t) = c\left[(p_x - eA_x)^2 + (p_y - eA_y)^2 + \left\{\frac{p_s - eA_s}{1 - \kappa\, x}\right\}^2 + m_0{}^2c^2\right]^{\frac{1}{2}} \qquad (38)$$

where the A_k are functions of position and time in general.

For a cyclic accelerator it is convenient to use s as the independent variable instead of t, since the fields arising from $\vec{A}$ can readily be expanded in Fourier components periodic around the circumference. In equation (20), Hamilton's principle can be re-expressed with the change of independent variable to s [3,4] as

$$\delta \int_{s_1}^{s_2} \left[p_x \frac{dx}{ds} + p_y \frac{dy}{ds} - H \frac{dt}{ds} + p_s\right] ds = 0 \qquad (39)$$

where we now consider the new Hamiltonian to be

$$- p_s(x,\ a,\ t,\ p_x,\ p_y,\ - H,\ s) = F \quad \text{say}, \qquad (40)$$

and where (t, - H) is a new pair of canonically-conjugate variables. The satisfying of Hamilton's variational principle (29) guarantees the validity of the transformation, and the previous results may be taken over with the appropriate changes in notation.

Rewriting (38) in the new form

$$F(x,\ y,\ t,\ p_x,\ p_y,\ -H,\ s) = -\ p_s$$
$$= -\ eA_s - (1 - \kappa\ x)\left[\frac{1}{c^2}\ (H^2 - m_0{}^2c^4) - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{\frac{1}{2}} \qquad (41)$$

where numerically

$$\frac{1}{c^2}\ (H^2 - m_0{}^2c^4) = m_0{}^2c^2(\gamma^2 - 1) = m_0{}^2c^2\ \beta^2\ \gamma^2 = \vec{p}\cdot\vec{p} \qquad (42)$$

is the square of the total momentum.

The Hamilton equations of motion are, following equations (11) and (12)

$$x' = \frac{\partial F}{\partial p_x}\ ; \qquad y' = \frac{\partial F}{\partial p_y}\ ; \qquad t' = -\frac{\partial F}{\partial H} \qquad (43a)$$

$$p_x' = -\frac{\partial F}{\partial x}\ ; \qquad p_y' = -\frac{\partial F}{\partial y}\ ; \qquad H' = \frac{\partial F}{\partial t} \qquad (43b)$$

where the primes denote $\frac{d}{ds}$.

In general the components of the vector potential $\vec{A}$ are functions of all the dynamical variables and of the independent variable s (the distance along the reference trajectory). The curvature κ, which introduces a kinematic nonlinear term (usually small), is generally a function of s. With a given accelerator or storage-ring lattice these terms can be calculated for all the elements (dipoles, quadrupoles, etc.) and expanded as Fourier series in s around the circumference of the machine. The Hamiltonian F in equation (41) can then be expressed as a power series expansion in the dynamical variables up to any order desired. The equations of motion may be obtained from equations (43); however, the Hamiltonian can conveniently be used directly for the study of resonance behaviour by selection of near-resonant terms in the machine periodicities and by judicious approximations. Although these procedures are well defined and straightforward in principle, the calculation of the power-series coefficients is in practice rather laborious for all but the simplest cases.

Certain simplifications can often be made in equation (41). If $\vec{A}$ is not an explicit function of t then $H' = 0$ and H is an invariant of the motion and is called a cyclic variable. This corresponds in (42) to a constant value of $m_0 c\ \beta\ \gamma$; t is an ignorable co-ordinate and the degrees of freedom are reduced from three to two. Such a situation arises for a coasting beam in a storage ring, and as an approximation to the motion in an accelerator on a sufficiently short time scale. Also it is sometimes permissible to approximate bending magnets, quadrupoles, etc. by piecewise constant elements with $A_x = A_y = 0$ except at the ends where they can be expressed as δ-functions.

10. Hamiltonian in a Quadrupole

As a simple example we evaluate the Hamiltonian F inside a quadrupole, assuming purely transverse fields so that $A_x = A_y = 0$. The only component of the vector potential is

$$A_s = \frac{p_0\ K}{2e}\ (x^2 - y^2) \tag{44}$$

where $p_0 = m_0 c\ \beta\ \gamma$ and K is the gradient parameter. The reference orbit is the axis of the quadrupole and the curvature $\kappa = 0$. Then equation (41) simplifies to

$$F(x,\ y,\ p_x,\ p_y) = -\frac{p_0\ K}{2}\ (x^2 - y^2) - [p_0{}^2 - p_x{}^2 - p_y{}^2]^{\frac{1}{2}}$$

The equations of motion (33) then become, assuming $p_0{}^2 >> p_x{}^2 + p_y{}^2$

$$p_x{}' = -\frac{\partial F}{\partial x} = p_0\ K\ x \tag{45a}$$

$$p_y{}' = -\frac{\partial F}{\partial y} = p_0\ K\ y \tag{45b}$$

$$x' = \frac{\partial F}{\partial p_x} = \frac{p_x}{[p_0{}^2 - p_x{}^2 - p_y{}^2]^{\frac{1}{2}}} \approx \frac{p_x}{p_0} \tag{45c}$$

$$y' = \frac{F}{p_y} = \frac{p_y}{[p_0{}^2 - p_x{}^2 - p_y{}^2]^{\frac{1}{2}}} \approx \frac{p_y}{p} \tag{45d}$$

leading to the familiar form of the equations of motion

$$x'' - K\ x = 0 \tag{46a}$$

$$y'' + k\ y = 0$$

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