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EIGÉNSTATES OF COMPLEX LINEAR COMBINATIONS OF J_1, J_2, J_3 FOR ANY REPRESENTATION OF SU(2)

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ABSTRACT : The states which minimalize the uncertainty relation $\Delta J_1 \ \Delta J_2 \geqslant \frac{1}{2}|\langle J_3 \rangle|$ are eigenstates of complex linear combinations of J_1 and J_2 (S. Rushin and Y. Ben-Aryeh, 1976). This kind of states is shown to have a very simple geometrical interpretation in the constellation formalism. A detailed description is given in the present paper.

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1. INTRODUCTION

Whenever SU(2) or the rotation group occurs in Physics, two kinds of states are utilized, namely the old standard discrete states* $|j\,m\rangle$ with $m = -j, -j+1, -j+2, \cdots, +j$ and the continuous spin coherent states $|j\,\theta\,\varphi\rangle$ (or equivalently $|j\,z\rangle$) where (θ,φ) labels points of the ordinary two-dimensional sphere S_2 and z is the image of (θ,φ) in the stereographic projection of S_2 on the complex line. The spin coherent states have been introduced by Radcliffe[1] and are known in Quantum Optics as Bloch states. The use of the word coherent is justified not only by the close analogy of their mathematical properties** with those of the ordinary Schrödinger-Glauber coherent states[2,3] but also by their interpretation in Quantum Optics[4,5].

It has been shown in [6] that any state of spin j can be represented with the aid of S_2 as a geometrical being : a constellation of order $2j$. States $|j\,m\rangle$ and $|j\,z\rangle$ are only special kinds of constellations ($|j\,m\rangle$ is of apparent order 1 or 2, $|j\,z\rangle$ of apparent order 1). The advantage of the constellation concept is to visualize a state as a geometrical object on S_2 and to see very easily how the rotation group acts on it : just rotate the sphere. Some applications have been made with the aid of this new tool, especially the classification of kinds of spin states[6] (that is the classification of orbits of the rotation group on spin states), the quantization of the three-dimensional harmonic oscillator[7], the relationship between the Radcliffe-Bloch sphere and the Poincaré sphere[3]. Here we present a new application.

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* in Quantum Optics they are also referred to Dicke states.

** From the formal point of view, one must underline that the parametrization by a complex number is common to both kinds of states ; the only difference is that $z = \infty$ does correspond to a spin c.s. but has no interpretation in ordinary c.s.

The application we are dealing with has been suggested by a recent work of S. Rushin and Y. Ben-Arieh[8]. These Authors started from the property that Radcliffe-Bloch states minimalize the uncertainty relation

$$\Delta J_1 \, \Delta J_2 \geq \frac{1}{2} |\langle J_3 \rangle| \qquad (1)$$

and generalized this property by proving that all states which minimalize (1) are eigenstates of complex linear combinations of J_1 and J_2. Moroever, any given such linear combination, except $J_1 \pm i J_2$ has $2j+1$ eigenstates of eigenvalues $-j, -j+1, \ldots, +j$. The operators $J_1 \pm i J_2$ only have one eigenstate, a Radcliffe-Bloch state. All these properties are quite simple to describe in the constellation formalism. In the next Section, we will recall the constellation description of a spin state. In Section 3, we will show that any operator of the form $\vec{J}.\vec{F}$ (where $\vec{F}$ is a complex vector, i.e. an element of the Lie algebra of the complex rotation group) can be associated with a constellation of order 2. Finally, in Section 4, we will find out the constellation associated with eigenstates of a given element $\vec{J}.\vec{F}$.

2. Let us recall here some results of the work [6] where the constellation concept was introduced (the word itself has first been introduced in [3]). It is well known that a spinor up to a complex factor, can be written as

$$\psi = \begin{vmatrix} 1 \\ z \end{vmatrix} \sim \begin{vmatrix} \lambda \\ \lambda z \end{vmatrix} \qquad \text{with} \qquad \lambda \in \mathbb{C} - \{0\} \qquad (2)$$

provided we include the possibility of having $z = \infty$, which corresponds to the case $\lambda = 0$, $\lambda z \neq 0$. The stereographic projection from the South pole* associates with ψ the point (θ, φ) on S_2 such that

* The South pole is chosen in order to recover the usual orientation of the 3-dimensional space with the usual Pauli matrices and the usual spherical coordinates on the sphere.

$$\psi = \begin{vmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{vmatrix} \sim \begin{vmatrix} 1 \\ \operatorname{tg}\frac{\theta}{2}\, e^{i\varphi} \end{vmatrix} \tag{3}$$

Since a state is defined up to a factor (or, equivalently, normalized and defined up to a phase), any $\frac{1}{2}$- spin state is uniquely characterized by one point on S_2. Such a point will be called a constellation of order 1. The states $\begin{vmatrix}1\\0\end{vmatrix}$ (North pole) and $\begin{vmatrix}0\\1\end{vmatrix}$ (South pole) are eigenstates of $J_3 = \frac{1}{2}\sigma_3$ with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$, respectively.

Any state of spin j is characterized by a constellation of order 2j. By this we mean a set of $2j$ complex numbers $\{z_1, z_2, \cdots, z_{2j}\}$ not necessarily distinct*. The number of distinct values is called the apparent order of the constellation*. It is clear that through a stereographic projection, the 2j complex numbers are replaced by 2j points on S_2. The relationship between constellations of order 2j and states of spin j is the following one. Let

$$|\psi\rangle = S_0|j\rangle + S_1|j-1\rangle + S_2|j-2\rangle + \cdots\cdots + S_{2j}|-j\rangle \tag{4}$$

be such a state. The constellation associated with $|\psi\rangle$ is given by the relations :

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* A more sophisticated definition would be as follows : a constellation of order 2j is a mapping f of S_2 on $\mathbb{N}_+$ (the set of non negative integers) such that $f^{-1}(\mathbb{N}_+ - \{0\})$ is finite and the sum of the images is $2j$. See also [3] for another definition. The definition given here is the original one [6].

$$\left\{\begin{array}{l} S_0 = 1 \\ S_1 = \dfrac{z_1 + z_2 + z_3 + \cdots + z_{2j}}{\sqrt{2j}} \\ S_2 = \dfrac{z_1 z_2 + z_1 z_3 + \cdots + z_{2j-1} z_{2j}}{\sqrt{j(2j-1)}} \\ - - - - - - - - - - \\ S_p = \binom{2j}{p}^{-\frac{1}{2}} \sum \left(z_{i_1} z_{i_2} \cdots z_{i_p} \right) \\ - - - - - - \\ S_{2j} = z_1 z_2 \cdots z_{2j} \,. \end{array}\right. \tag{5}$$

where the summations are made on all combinations. If two (or more) of the z_k's are equal, the constellation is said to be degenerate. In particular, if all the z_k's are equal, the apparent order is equal to one.

3. Let us now consider an element of the complex Lie algebra, say $\vec{\sigma}.\vec{F}$ Since we are interested in its eigenstates (in the representation of spin j), we do not have to distinguish between $\vec{\sigma}.\vec{F}$ and $\lambda\vec{\sigma}.\vec{F}$ where λ is an arbitrary non-zero complex number. In other words we are only interested in the class $[\vec{F}]$ of elements $\vec{\sigma}.\vec{F}$ such that[x]

$$[\vec{F}] = \left\{ \vec{F} \sim \lambda \vec{F} \;\middle|\; \lambda \neq 0 \right\} \tag{6}$$

that is a state of spin 1 or, equivalently, a constellation of order 2. It is clear that constellations of order 2 are of two types, those which are of apparent order 2 (non degenerate) and those of apparent order equal to 1 (degenerate). In order to classify these constellations, let us associated with $[\vec{F}]$ the following 2x2 matrix symmetric in z_1, z_2 .

[x] We are dealing with the projective complex Lie algebra of SO(3,C).

$$\sigma[\vec{F}] \doteq \begin{vmatrix} z_1+z_2 & -2 \\ 2z_1z_2 & -(z_1+z_2) \end{vmatrix} = (z_1+z_2)\sigma_3 + z_1z_2\sigma_- - \sigma_+ \tag{7}$$

$$\det \sigma[\vec{F}] = (z_1-z_2)^2 \tag{8}$$

The associated constellation is $\{z_1, z_2\}$. We note that $\det \sigma[\vec{F}]$ is zero if and only if the constellation is degenerate $(z_1 = z_2)$. We also note that if $\vec{F}$ is real, $\sigma[\vec{F}]$ is Hermitian (up to a factor) and $\{z_1, z_2\}$ correspond to opposite points on the sphere : $z_1\bar{z}_2 + 1 = 0$. The proof is as follows : Take the Hermitian conjugate of (7) :

$$\sigma[\vec{F}]^\dagger = (\bar{z}_1+\bar{z}_2)\sigma_3 + \bar{z}_1\bar{z}_2\,\sigma_+ - \sigma_- = \bar{z}_1\bar{z}_2\left[\left(\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}\right)\sigma_3 + \sigma_+ - \frac{1}{\bar{z}_1\bar{z}_2}\sigma_-\right]$$

By comparaison with (7), one gets the conditions for $\sigma[\vec{F}]$ to be Hermitian up to a factor :

$$\begin{cases} z_1+z_2 = -\left(\dfrac{1}{\bar{z}_1}+\dfrac{1}{\bar{z}_2}\right) \\[2ex] z_1z_2 = -\dfrac{1}{\bar{z}_1\bar{z}_2} \end{cases}$$

It is a simple matter to prove that it is equivalent to $z_1\bar{z}_2 + 1 = 0$
The associated constellation will be said to be <u>real</u>[*].

Let us now state the following theorem.

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[*] Note that the conditions for two spin $\frac{1}{2}$ states to be orthogonal is that the associated constellations are opposite on the sphere. The concept of real constellation will be generalized later.

Theorem 1

The eigenstates of $\sigma[\vec{F}]$ associated with the nondegenerate constellation $\{z_1, z_2\}$ are the constellations $\{z_1\}$ and $\{z_2\}$. The eigenstates are orthogonal (then opposite on the sphere) if and only if $\{z_1, z_2\}$ is real. If $z_1 = z_2$, there is a single eigenstate, namely $\{z_1\}$.

The proof is quite easy. In fact, if $z_1 \neq z_2$

$$\sigma[\vec{F}] \begin{vmatrix} 1 \\ z_1 \end{vmatrix} = (z_2 - z_1) \begin{vmatrix} 1 \\ z_1 \end{vmatrix}$$

$$\sigma[\vec{F}] \begin{vmatrix} 1 \\ z_2 \end{vmatrix} = -(z_2 - z_1) \begin{vmatrix} 1 \\ z_2 \end{vmatrix}$$

and the eigenstates are orthogonal : $\begin{vmatrix} 1 & \bar{z}_1 \end{vmatrix} \begin{vmatrix} 1 \\ z_2 \end{vmatrix} = \bar{z}_1 z_2 + 1 = 0$ if and only if $\sigma[\vec{F}]$ is Hermitian up to a factor. If now $z_1 = z_2 = z$, the matrix $\sigma[\vec{F}]$ reads $\begin{vmatrix} z & -1 \\ z^2 & -z \end{vmatrix}$ and has determinant zero. The only eigenstate is $\begin{vmatrix} 1 \\ z \end{vmatrix}$ with eigenvalue zero.

It is of interest to consider the following particular cases :

a) $z_1 = 0$, $z_2 = \infty$ $\qquad \sigma[\vec{F}] \simeq \dfrac{1}{1 - \frac{z_1}{z_2}} \begin{vmatrix} 1 + \frac{z_1}{z_2} & -\frac{2}{z_2} \\ 2 z_1 & -\left(1 + \frac{z_1}{z_2}\right) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \sigma_3$

If we remember that σ_3 generates rotations around the third axis and that to be an eigenstate of σ_3 means to be invariant under a rotation around the third axis, the fact that $\{0\}$ and $\{\infty\}$ are both invariant constellations under those rotations is quite obvious on Figure 1. Similarly

b) $z_1 = 1$, $z_2 = -1$ $\qquad \sigma[\vec{F}] = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = \sigma_1$

c) $z_1 = i$, $z_2 = -i$. $\qquad \sigma[\vec{F}] = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} = \sigma_2$

Again, in these two cases, the interpretation is immediately obtained from Fig. I

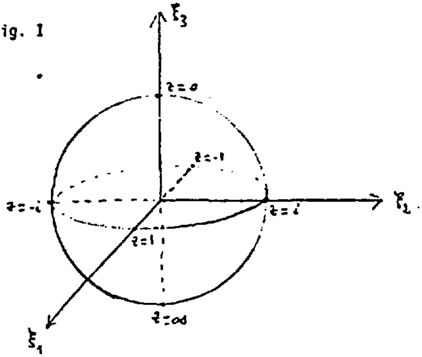


Fig. 1

It is a simple exercize to generalize the above results : if $\{z_1\}$ and $\{z_2\}$ are opposite on the sphere, they are eigenstates of the generator of rotations around the corresponding diameter, in the spin $\frac{1}{2}$ representation.

We are now going to generalize the above results for any spin representation.

4. EIGENCONSTELLATIONS OF $\vec{J}.\vec{F}$ IN REPRESENTATION OF SPIN j

Theorem 2

Given a constallation $\{z_1, z_2\}$ associated with the matrix $\sigma[\vec{F}]$, the eigenconstellations of its representative in the representation of spin j are the constellations (of order $2j$) of the form $\{z_1, z_1, \dots z_1, z_2, z_2, \dots z_2\}$
If $z_1 \neq z_2$, they are $2j+1$ in number and of apparent order 2 or 1.
If $z_1 = z_2$, there is a single constellation of apparent order 1 (Bloch constellation).

This simple result is obtained as follows. Let us denote by $J[\vec{F}]$ the representative of $\sigma[\vec{F}]$ in representation of spin j

$$J[\vec{F}] = (z_1 + z_2)\, J_3 + z_1 z_2 J_- - J_+$$

It is a simple calculation to verify that the state

$$|\psi_\lambda\rangle = \sum_{m=-j}^{+j} \sum_{k=0}^{j-m} \left[\frac{(j+m)!(j-m)!}{(2j)!}\right]^{\frac{1}{2}} \frac{(j-\lambda)!(j+\lambda)!}{k!(j-\lambda-k)!(j-m-k)!(\lambda+m+k)!}\; z_1^{j-m-k} z_2^{k} |m\rangle$$

is an eigenstate of $J[\vec{F}]$

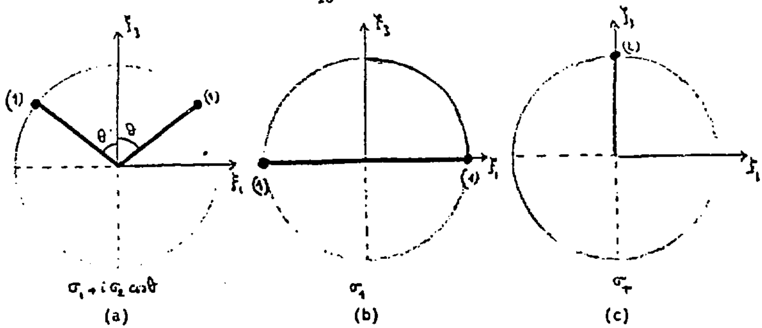
$$J[\vec{F}]|\psi_\lambda\rangle = \lambda(z_2 - z_1)\,|\psi_\lambda\rangle$$

Moreover with a small combinatory calculation, it is not difficult to show that the state $|\psi_\lambda\rangle$ is represented by the constellation $\{z_1, z_1, \dots z_1, z_2, \dots z_2\}$ where the multiplicity of z_1 is 2λ and the multiplicity of z_2 is $2j - 2\lambda$. Moreover if z_2 becomes equal to z_1, all $|\psi_\lambda\rangle$ collapse in a single state and the corresponding eigenvalue is zero. It is this case which corresponds to Radcliffe-Bloch states.

In order to illustrate our final theorem, let us give the constellations associated with the Rushin-Ben-Aryeh states, i.e. eigenconstellations of linear combinations of J_1 and J_2. Since these states minimalize the uncertainty relations associated with the frame $O\xi_1\xi_2\xi_3$. It is clear that they must have some geometrical relationship with the three axes. The result is represented on Fig. 2. We note that the ξ_3 axis is a symmetry axis for the constellation ; the $\xi_1 \xi_3$ plane is also a symmetry plane. A greater symmetry is present when

i) $\vec{F}$ is real (maximum symmetry, Fig. 3b)

ii) $\vec{F}$ is singular ($\det \sigma[\vec{F}] = 0$, Fig. 3c).



<u>Fig. 2</u> : constellations $\sigma[\vec{F}] = \sigma_1 + i\sigma_2 \cos\vartheta$ (multiplicities are in brackets)

In the other cases, the axes ξ_1 , ξ_2 , ξ_3 play <u>distinct</u> privileged roles. In case $\vartheta = 0$ (or π) which is represented on Fig. 3c, the uncertainty relation is minimal not only for the product $\Delta J_1 \, \Delta J_2$ but also for any product of the form

$$\Delta\left(J_1 \cos\varphi - J_2 \sin\varphi\right) \Delta\left(J_1 \sin\varphi + J_2 \cos\varphi\right)$$

5. CONCLUSION

Up to now, the Rushin-Ben-Aryeh states have received no interpretation. If we identify the complex rotation group with the Lorentz group, such states can be interpreted as boosted spin states of the type $|j\, m\rangle$. For example, the state represented in Fig. 2b is an eigenstate of J_1 with $j = 1$, $m = 0$. By boosting it in the ξ_3 direction one gets a state like the one of Fig. 2a. Such an interpretation is merely a curiosity. There is some hope to find more concrete applications in Quantum Optics.

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