SCATTERING THEORY IN QUANTUM MECHANICS AND ASYMPTOTIC COMPLETENESS

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I - INTRODUCTION

It is a rather difficult task to describe the status of scattering theory in Quantum mechanics as it appears today. First the mathematics of such a theory are of interest for themselves in the sense that the fundamental question of this theory, namely unitary equivalence of two self-adjoint operators is a problem which existed even before its implications for the existence of the S-matrix were recognized. Once this was done the models furnished by the second order elliptic differential operators of Quantum mechanics have attracted a variety of mathematicians interested by the connection between this type of existence and uniqueness problem with Cauchy data at "infinity" and the above mentioned unitary equivalence problems. So it is not so surprising that the outcome of all those efforts for physics consists mostly in the conclusion, which everyone here will easily believe, that for a particle decreasing faster than $|x|^{1-E}$, $E > 0$, the S-matrix exists and is unitary. The literature on this type of problem and those related to it (self-adjointness for more and more singular potentials, elimination of positive energy bound-states and singular continuous spectrum, etc...) is quite abundant and I need to apologize in advance for my very incomplete list of results and references.

The second reason which makes my task difficult is that you certainly want to know if simple proofs of asymptotic completeness have been found for N-particle systems and if such proofs give some insight on a general method applicable to quantum field theory models. In other words, is there a transparent and reasonably short extension to N-particle systems of the (beautiful !) work of L.D. Faddeev in 1963 [22]. Unfortunately the most recent and complete results I know are contained in a two hundred pages manuscript of I. Segal [31] whose results will be described later.

One should not draw pessimistic conclusions from this fast and critical description. First because the powerful methods developed by mathematicians in the last decade provide intermediate results which certainly are as important as the well accepted unitarity of the S-matrix and contribute greatly to our understanding
of N-particle scattering processes. Second the methods are still evolving and new techniques are in progress originating from other branches of P.D.E. mathematics. It can be expected also that present developments in analysis on infinite dimensional spaces, like those presented by Albeverio at this conference [51] will through the use of Feynman path integrals also clarify the undoubtedly very deep connection between quantum and classical scattering theory which I strongly believe (and will try to show at the end of this lecture) is the ingredient necessary to make decisive and fundamental progress in Scattering theory.

II - THE MATHEMATICAL PROBLEMS OF QUANTUM SCATTERING THEORY

The basic operator to be analysed is the "S-matrix":

\[ S = \sum_{\pm}^* (J^*H, H_0) \sum_{-} (J^*H, H_0) \]

where the "wave-operators" \( \sum_{\pm} (J^*H, H_0) \) are defined as:

\[ \sum_{\pm} (J^*H, H_0) = \lim_{t \to \pm \infty} e^{iHt} J^e^{-iHt} P_{\text{ac}}(H_0) \quad (11.1) \]

Here \( H_0 \) and \( H \) are the hamiltonians for the free and interacting systems, \( J \) being an "identification operator" between the Hilbert spaces describing respectively the free and interacting states. The projection operator \( P_{\text{ac}}(H_0) \) selects those states whose \( H_0 \)-spectral measure is Lebesque - absolutely continuous.

Properties of wave-operators, in particular intertwining between the absolutely continuous parts of \( H_0 \) and \( H \) are reviewed in [1].

Obviously wave-operators exist only under certain very restrictive conditions on the pair \( \{ H_0, H \} \). One of them is the celebrated Kato-Birman theorem ([5]) asserting that if the difference \( (H-\zeta)^{-1} J(J(H_0-\zeta)) \), \( \zeta \in \sigma(H_0) \cap \sigma(H) \), is trace-class, then not only the wave-operators exist but they are complete, i.e. their range is the absolutely continuous subspace of \( H \) (which we will denote by \( \sum_{\pm} (H) \)). This implies unitarity of \( S \) (but it is not equivalent to it).

The applicability of this criteria to one body-Schrödinger operators with local potentials (in which case \( J = 1 \) ) was at the origin of the work of Kuroda which culminated in [2] after some various kinds of improvements.

The non-direct applicability of Kato-Birman theorem to many-particle systems if one merely takes \( J = 1 \), was already implicit in the earliest formulation of multi-channel scattering theory [3]. Here in fact one expects that the continuous part of \( H \) is a direct sum of operators, each of them being unitarily equivalent.
to the Hamiltonian of some free system but in which some particles are bound by interparticle forces. The basic reason of this fact is that the total potential does not decay in every direction of the configuration space of the many-particle systems. So for each asymptotic partition $\mathcal{A}$ of the $n$-particle system into composite fragments there should exist the channel wave-operators:

$$\Omega_{\mathcal{A}} = \lim_{E \to 2\pi i} e^{iHt} e^{-iH_0 t} \rho^\mathcal{A}$$

where $H_0$ is the kinetic energy operator for this system of non-interacting composite particles and $\rho^\mathcal{A}$ is the projection operator on the corresponding states.

The $S$-matrix is then defined as:

$$S_{\mathcal{A}/\mathcal{B}} = (\Omega_{\mathcal{A}}^\dagger)^* \Omega_{\mathcal{B}}$$

Its unitarity is a consequence of the asymptotic completeness relation:

$$\bigoplus \text{Range} e^{-i\Omega_{\mathcal{A}}^\dagger} = \mathcal{M}_{\text{ac}}(H)$$

which is one of the main problems to be discussed below. As we will see later multichannel scattering theory can be reformulated in the two Hilbert space formalism described at the beginning of this chapter with a suitable choice of $H_0$ and $\Gamma$.

### III - STATIONARY METHODS

#### The one-body problem

A quantum mechanical particle in a potential $V$ is described by the elliptic second order differential operator on $L^2(\mathbb{R}^3)$:

$$H = -\Delta + V$$

There is an abundant literature on the definition of $H$ as a self-adjoint operator when $V$ is a real function. There is an almost exhaustive summary of results in [4] culminating with the condition that the positive part of $V$ is $L^1_{\text{loc}}(\mathbb{R}^3)$ and the negative part is $\Delta$-bounded; then $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. For scattering by singular potentials or obstacles the Hamiltonian has to be defined in terms of forms; this kind of problem is investigated in particular in [5], [6]. The abstract stationary scattering theory tries to construct the wave-operators from their stationary form which reads in the one-body case:
where $E_0(\cdot)$ is the spectral family of $H_0$.

This form is derived from the Abel limit method and spectral theory (see e.g. [?]).

It leads to the usual expression for scattering amplitudes

$$\mathcal{F}(k, k') = \tilde{V}(k-k') + \hbar \imath \lim_{\epsilon \to 0} \left( \phi_{k'}(H-k^2-\imath \epsilon)^{-1} \phi_k \right)$$

where $\phi_k(x) = V(x) e^{-ikx}$ and $k^2 = k^2$.

The point is that the limits as $\epsilon \to 0$ of the integrands in (III.2) for example do not exist even as unbounded operators on $L^2(\mathbb{R}^3)$. However their weak limits do for certain states and this leads naturally to the introduction of auxiliary spaces such that these limits exist as bounded operators between those spaces.

A natural choice of auxiliary spaces is suggested by the elliptic character of our operators; in particular the weighted Sobolev spaces

$$H_{m,m'}^{\epsilon} = \left\{ \varphi(x) : (1+|x|^2)^{m/2} \mathcal{D}^\epsilon \varphi \in L^2, 0 \leq |\epsilon| \leq m \right\}$$

play a fundamental role in Agmon's analysis [?]. Notice that if one allows fractional derivatives, this class of spaces is invariant by Fourier transform. One has the following standard properties:

**Proposition 1** (trace theorem in Sobolev spaces [?])

If $\Omega$ is a bounded domain of $\mathbb{R}^N$ with $C^\infty$ boundary $\Gamma$, then for any $m > \frac{1}{2}$ and any $m' \in \mathbb{R}$ there exists a bounded linear map

$$\mathcal{I} : H_{m,m'}^{\epsilon}(\mathbb{R}^N) \to \mathcal{H}_{m-m',m'}(\Gamma)$$

such that

$$\mathcal{I} \varphi = \varphi \mid_{\Gamma}$$

The other result is some kind of Sobolev inequality stated in [?]:

**Proposition 2**

For all $m' > \frac{1}{2}$ and all $\kappa > 0$ there exists a constant $C$ depending only on $m'$ and $\kappa$ such that

$$\| \varphi \|_{H_{m,m'}^{\epsilon}} \leq C \| (\Delta + \kappa) \varphi \|_{H_{m'-m',m'}^{\epsilon}}$$

(III.4)
for all \( \varphi \in \mathcal{X}_{2,0}(\mathbb{R}) \) and \( \kappa^{-1} < |z| < \kappa \).

The first result has the obvious consequence that any \( \varphi \) in \( \mathcal{D}(\lambda + x^2) \) has for \( \lambda > \frac{1}{4} \) a square integrable trace on the energy spheres \( \mathbb{R}^+ = \mathbb{E}, \mathbb{E} \neq 0 \), which gives us a very convenient framework to work at fixed energy. The second gives a precise meaning to the boundary values of the free resolvant on the spectrum \( \mathbb{R}^+ \) of \( -\Delta \). That such properties remain valid for some perturbations of the Laplacian is of course of prime importance as can be seen from (III.2) or from the subsequent expression (III.3) for scattering amplitudes. This problem is most conveniently analysed in the general context of "smooth operator" techniques, a concept introduced by T. Kato [10] which is now recognized as a standard tool in perturbation theory for the continuous spectrum.

**Definition**

Let \( H \) be a self-adjoint operator on the Hilbert space \( \mathcal{H} \) and \( A \) a densely defined closed operator from \( \mathcal{H} \) to another Hilbert space \( \mathcal{H}' \) with \( \mathcal{Q}(A) \supseteq \mathcal{D}(H) \). Then \( A \) is said to be \( H \)-smooth if one of the following (common) values is finite:

\[
\begin{align*}
a) & \quad \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} \| \chi (H-A+i\varepsilon)^{-1} \varphi \|^2 \, d\lambda \\
b) & \quad \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} \| A (H-A+i\varepsilon)^{-1} \varphi \|^2 \\
c) & \quad \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} \| A \exp(-iHt) \varphi \|^2 \\
d) & \quad \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} \| (A^* \varphi, (H-A+i\varepsilon)^{-1} (H-A-i\varepsilon)^{-1} A \varphi) \|^2 \\
\end{align*}
\]

There is also a notion of "locally smooth" operator : \( A \) is \( H \)-smooth on the Borel set \( I \subseteq \mathbb{R} \) if \( A \in \mathcal{D}(I) \) is \( H \)-smooth.

As an example let us consider Agmon's class of short-range potentials satisfying:

1) \( V(\chi) = O(\chi^2(1+\chi^2)^{\varphi}) \), \( \varphi > \frac{1}{2} \), \( \chi \chi \to \infty \)

2) A local regularity assumption : \( V \in L^2_{0,\infty}(\mathbb{R}^3) \) ensuring in particular that \( H \) is self-adjoint.

Then Proposition 1 implies that for any compact interval \( I \subseteq \mathbb{R} \setminus \{0\} \), the operator \( A = |V|^{\frac{1}{2}} \) is \( \bar{A} \)-smooth on \( I \). In view of the next theorems one would like
to know that \( A \) also is \( -\Delta + V \) smooth. Actually the key results of the theory of smooth-operators are

**Theorem 1** ([13])

If \( A \) is \( H \) smooth on \( I \subset \mathbb{R} \) then \( E(I) \text{Range} A \) is contained in \( M_{\text{loc}}(H) \).

This gives a criteria for absolute continuity for \( H \) on certain intervals. This result is complemented by a theorem giving sufficient-conditions for unitary equivalence:

**Theorem 2** (Kato [10])

Let \( H_0 \) and \( H \) be self-adjoint operator on Hilbert spaces \( \mathcal{H}_0 \) and \( \mathcal{H} \) respectively. Let \( J \) be a bounded operator from \( \mathcal{H}_0 \) to \( \mathcal{H} \) taking the domain of \( H_0 \) into the domain of \( \mathcal{H} \). Then if

\[
HJ - JH_0 = A^* A_0
\]

where \( A \) is \( H_0 \) smooth and \( A \) is \( H \) smooth, the wave-operators \( \Omega_r(J, H, H_0) \) exist and are complete.

The same is true if \( HJ - JH_0 \) is a sum of such operators.

There is an obvious local version of this theorem if \( A_0 \) and \( A \) are simply locally smooth on some interval.

In order to apply these results to short-range potentials one can use the factorisation \( V = A^* A_0 \), with \( A = |V|^{1/2} \) and \( A_0 = \arg\sqrt{|V|^{1/2}} \), and tries to solve:

\[
A(H - z)^{-1} A^* = A(H_0 - z)^{-1} A^* - A(H - z)^{-1} A^* A_0 (H_0 - z)^{-1} \quad (\text{III.6})
\]

From (III.5d) what is needed in order to show that \( A \) is locally \( H \) smooth is \( H_0 \) smoothness of \( A_0 \), and bounded invertibility of \( A + G(\lambda \pm i0) \) with:

\[
G(\lambda \pm i0) = \lim_{\varepsilon \to 0} A_0 (H_0 - \lambda \pm i\varepsilon)^{-1} \quad (\text{III.7})
\]

By Proposition 2 and Rellich compactness criteria Agmon shows that the limit (III.7) exists and defines a compact operator for almost all \( \lambda \neq 0 \). So Fredholm alternative can be used for such points to solve (III.6). In addition he shows that the null set of singular points where \( (I + G(\lambda \pm i0)) \gamma = 0 \) has a non trivial solution coincides with the set of eigenvalues of \( H \) thus excluding singular continuous
spectrum. Actually under extra assumptions one can also exclude positive energy bound states so that \( \mathcal{H} \) has absolutely continuous spectrum on \( \mathbb{R}^+ \) and is unitarily equivalent to \(-\Delta\) when restricted to \( \mathcal{M}_{\text{AC}}(\mathcal{H}) \).

Other results using smooth operator techniques are those of T. Kato \([10]\), \([11]\). Kato gives uniform estimates for (III.7) which are stronger than those derived from (III.4) in the sense that they extend up to the threshold. This is a consequence of a stronger decay condition \( V(x) = O(1|x|^{-2-\varepsilon}) \) on the potentials which allows to get a compact limit for (III.7) up to \( \lambda = 0 \). Notice that this implies finiteness of the discrete spectrum of \( \mathcal{H} \), and T. Kato and K. Yajima \([12]\) in the weak coupling case, which apply also to non symmetric perturbations.

R. Lavine \([13]\) uses commutator techniques to prove directly \( H \) -smoothness of a very general class of multiplicative operators \( \mathcal{A} \). The work of Kuroda \([2]\) does not explicitly use the smoothness concept although a decomposition \( \sqrt{x} \mathcal{A}^* \mathcal{A}_0 \) with \( \mathcal{A} = \langle 1 + |x|^2 \rangle^{m/2} \) for \( m > \frac{1}{2} \) is used which in view of Propositions 1 and 2 plays a basic role in solving (III.6). Finally I want to mention the important series of articles by M. Schecter (see e.g. \([14]\)) generalizing most of the existing results to elliptic operators of arbitrary order and to a class of pseudo-differential operators.

Concerning scattering by long-range potentials \( V(x) = C (1 + |x|^2)^{-m} \), \( m \leq \frac{1}{2} \), the most recent approaches using adaptations of the above framework are those of Pinchuk \([15]\) and Kitada \([16]\). It is well-known that for such potentials the usual wave-operators do not exist and a new definition is needed \([17]\), \([18]\).

To get a crude idea of this fact in the stationary framework one can notice that the boundary value of \( \mathcal{R}^\lambda \) for the operator \( \mathcal{U}(\lambda) \) is just the trace on \( \mathcal{R}^2 \lambda \) of vectors in \( \mathcal{D}^{m_0}((\mathcal{H}^2)\lambda) \) with \( m \leq \frac{1}{2} \) and this trace is not continuous in this case (see \([9]\) theorem 9.5, ch. 1).

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**N-body systems**

The Hamiltonian for \( n \)-interacting particles is given by:

\[
\mathcal{H} = -\sum_{\ell=1}^{N} A_\ell + \sum_{j<i}^{N} V_{ij}
\]  

\[\text{(III.8)}\]

where the \( V_{ij} \) are the two-body potentials. From translational invariance, we can get rid of the degrees of freedom associated to the center of mass of the systems so that the Hilbert space for this system is \( \mathcal{H}_c = L^2(\mathbb{R}^{3(N-1)}) \).

The lack of decay of \( \sqrt{V} \) in all directions of configuration space makes the one-body method not directly applicable; in particular even for \( \sqrt{V} \geq 0 \) the
operator $\mathcal{G}(\mathcal{Q})$ of (III.7) never has compact boundary values on $\mathbb{R}$. One can get rid of this difficulty in many ways which will be described below. But for the moment we simply notice that if the couplings are strong enough, two-particles systems will have bound states and then the continuous part of is no more unitarily equivalent to the kinetic energy operator $H_0 = -\sum \Delta_i$. Rather one expects that it is unitarily equivalent to a direct sum $\bigoplus H^\chi$, where $H^\chi$ is the kinetic energy operator for a 'channel' $\chi$ built with the $n$ initial particles, but some of these particles being bound together. So let us introduce

$$\mathcal{H}_0 = \bigoplus \mathcal{H}^\chi$$

where $\mathcal{H}^\chi$ is the Hilbert space of states for the "composite particles" in channel $\chi$; it is identified as the subspace of vectors in $\mathcal{H}$ which are tensor products of wave-functions for those composite particles (solutions of the bound-state problem for subsystems).

One defines the identification operator $\mathcal{I}: \mathcal{H}_0 \to \mathcal{H}$ as

$$\mathcal{I}(q^\chi) = \sum \chi q^\chi$$

and the "free" hamiltonian

$$\tilde{H}_0 = \bigoplus \tilde{H}_0^\chi$$

where $\tilde{H}_0^\chi$ is the kinetic energy operator for particles in channel $\chi$ (with a suitable subtractive constant $-E_\chi$, the sum of binding energies for composite particles). The wave-operator

$$\Omega_{\pm} (\mathcal{I}; H, \tilde{H}_0) = \lim \frac{1}{\epsilon} \exp \left[ \frac{i}{\hbar} H \epsilon \right] \mathcal{I} \exp \left[ \frac{i}{\hbar} \tilde{H}_0 \epsilon \right]$$

can be seen to reduce in each channel subspace $\mathcal{H}^\chi$ to the wave-operators $\Omega_{\pm}^\chi$ introduced in (11.2). Asymptotic completeness is equivalent to

$$\text{Range} (\Omega_{\pm} (\mathcal{I}; H, \tilde{H}_0)) = M_{\text{ac}} (H)$$

A stationary form of this wave-operator can be written as in (III.2) for the one-body case. We will see that this reduction to a one-body like problem does not make the problems as simple. In particular, most authors have to make a stronger assumption on the decay of the potential than was necessary for the one-body problem namely $V(x) = O(x^{1/2})^{-1/2} \sim \epsilon$, $x > \epsilon$; this is due to two reasons; first this condition guarantees a finite number of channels. Second any energy sphere $\mathcal{E} = E$ contains zero energy thresholds for two-body relative
kinetic energies \((E_i - E_j)^2\) and to generalize Propositions 1 and 2 for such
degenerate hypersurfaces one needs a stronger decay for \(V_{ij}\). I emphasize that
those decay assumptions for \(V_{ij}\) which are done in most works on many-body
systems are technically convenient but by no way essential; this is shown by
Mourre [20] who has succeeded in handling those threshold singularities for poten-
tials having the decay \(O(\lambda^n)\), \(n > 1\), and being repulsive at infinity.
for boundary values of two-body resolvents.

Now we come to a fast description of the methods used to handle \(N\)-body
problems. In the one-channel case which corresponds to \(O(\lambda)\) potentials with
sufficiently small coupling constant there is a direct generalization of smooth-
operator techniques as shown by Iorio and O'Carrol [21] following a remark of
Kato [10]. The idea is to consider a new Hilbert space

\[
\mathcal{H} = \bigoplus_{\mathbb{N}} \mathcal{H}_a, \quad \mathcal{H}_a = L^2(\mathbb{R}^{2N-2a}) \otimes a,
\]

where the sum is over pairs of particles. The mapping

\[
\mathcal{J}_a^* \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_a)
\]

and the matrix operator \([V] = \left[ V_{ab} \delta_{ab} \right]\) are then introduced and allow
to rewrite the second resolvent equation as

\[
(H - \xi)^{-1} = (H_0 - \xi)^{-1} - (H_0 - \xi)^{-1} \mathcal{J}_a \left[ V \right] \mathcal{J}_a^* (H - \xi)^{-1}
\]

Writing \(A = \left[ V_{ab} \delta_{ab} \right] \mathcal{J}_a^*\) and \(A_0 = \left[ V_{ab} \right] \mathcal{J}_a^*\), one obtains
equation (III.6). Using suitable sets of Jacobi coordinates one shows that for
weak coupling \(A_0\) and \(A\) are respectively \(H_0\) and \(H\)-smooth so that the
wave-operators exist and are complete by theorem 2.

To treat real multi-channel systems one needs refinements of the above method
since then \(J\) is no more the identity operator and according to (III.1) and Theo-
rem 2 what one needs to show is \(H J - J H_0 = A_0 \mathcal{J}_a\) where \(A_0\) is \(H_0\)-smooth
and \(A\) is \(H\)-smooth. This unfortunately cannot be done since it would imply
by Theorem 2 that \(A = \lim_{\mathbb{N}} e^{iH_0 \mathcal{J}_a} \mathcal{J}_a^* e^{-iH_0 \mathcal{J}_a} \mathcal{J}_a^* \mathcal{J}_a (\mathcal{J}_a)\) exists; this leads to
a contradiction since one can show from prime principles that only the weak limit
exists (or the strong Abel limit). One is then led to use different techniques.
One of them is based on the celebrated Faddeev-Yakubovsky equations ([22], [23])
which we describe here below in the case \(N = 3\). One of the advantages of Faddeev
equations is to incorporate the solutions of the two-body problems into three-body
equations. Then one can separate out in the kernel of Faddeev equations the different
kind of singularities coming from two-body bound-states or two-body continuum.

Faddeev equations can be derived from the multiple collision expansion of Watson [27], which is a special way to perform partial summations of the Born series; one gets for the transition operator, related to the resolvent by

$$ T(\tilde{z}) = V + V (H - \tilde{z})^{-1} V $$

$$(H - \tilde{z})^{-1} = (H_0 - \tilde{z})^{-1} + (H_0 - \tilde{z})^{-1} T(\tilde{z}) (H_0 - \tilde{z})^{-1}$$

the expansion

$$ T(\tilde{z}) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} T_{\alpha\beta}(\tilde{z})(H_0 - \tilde{z})^{-1} T_{\beta\alpha}(\tilde{z})... (H_0 - \tilde{z})^{-1} T_{\beta\alpha}(\tilde{z})$$

where $T_{\alpha\beta}(\tilde{z})$ is the two-body transition operator for the pair $\alpha_\beta$ of particles.

Introducing $M_{\alpha\beta}(\tilde{z})$ as the sum of contributions in (III.14) with $\alpha_\alpha = \alpha, \alpha_\beta = \beta$ we get finally

$$ M_{\alpha\beta}(\tilde{z}) = T_{\alpha\beta}(\tilde{z}) \sum_{\lambda} T_{\beta\lambda}(\tilde{z})(H_0 - \tilde{z})^{-1} M_{\beta\lambda}(\tilde{z}) $$

Those equations are most conveniently treated using the following factorization technique suggested by R. Newton [24]:

$$ T_{\alpha\beta} (\tilde{z}) = [V_{\alpha}]_{\beta}^{\alpha} T_{\alpha\beta} (\tilde{z}) $$

Then the equation for $(H - \tilde{z})^{-1}$ takes the form

$$(H - \tilde{z})^{-1} = \sum_{\alpha, \beta} [V_{\alpha}]_{\beta}^{\alpha} \sum_{\alpha, \beta} [V_{\alpha}]_{\beta}^{\alpha} L_{\alpha\beta}(\tilde{z})(H_0 - \tilde{z})^{-1}$$

where the operators $L_{\alpha\beta}$ satisfy the equations:

$$ L_{\alpha\beta}(\tilde{z}) = \sum_{\alpha, \beta} [V_{\alpha}]_{\beta}^{\alpha} \sum_{\alpha, \beta} [V_{\alpha}]_{\beta}^{\alpha} L_{\alpha\beta}(\tilde{z})(H_0 - \tilde{z})^{-1} $$

whose kernel is obviously connected.

Another advantage of Eq.(III.16) is that it shows more explicitly the singularity structure of $(H - \tilde{z})^{-1}$. Apart from the possible pole singularities coming from eventual non trivial solutions of the homogeneous equation associated to (III.17), $(H - \tilde{z})^{-1}$ will have the singularities of $\sum (H^0_0 - \tilde{z})^{-1}$ as expected from asymptotic completeness.

Equations (III.17) are solvable using two-Hilbert space and smoothness techniques as shown by Combescure and Giniere [25]. It is unfortunate that for
the reasons given above their proof of asymptotic completeness is indirect and
requires, as Faddeev did [22], the stationary form of wave-operators and regularity
properties of the Green's function, instead of a direct use of Theorem 2.

- Other methods and results for the N-body problem

For the three-body problem there is an alternative approach by L. Thomas
[26] using spectral integrals (J. Howland [40] and K. Yajima [28]). The four body
case has been investigated by K. Hepp [25], G.A. Hagedorn [41], along the lines
of Faddeev [22]. He also proves asymptotic completeness for N-body problems with
repulsive potentials; this case is also investigated using commutator techniques
by R. Lavine [13] and P. Ferrero, C. de Farris and D. Robinson [6], this last
paper containing a general discussion of singular potentials, in particular hard­
corees. The most recent results on the N-body problem have been derived by I. Sigal
[31]; Sigal does not use explicitly N-body equations but constructs directly
regularizers for the operators $H-\frac{1}{\lambda}$, $\text{Im} \frac{1}{\lambda} \neq 0$, i.e. a family $\mathcal{R}(\lambda)$ of
bounded linear operators such that

$$\mathcal{R}(\lambda) (H-\frac{1}{\lambda})^{-1} = \mathcal{A}(\lambda)$$

where $\mathcal{A}(\lambda)$ is a Fredholm operator. Of course there are many ways to do this
and a suitable choice allows a well-behaved continuation of $\mathcal{R}$ and $\mathcal{A}$ up to
the real axis. Sigal's analysis is quite long but presents in turn the advantage
of being complete in the sense that no a-priori assumption is made on the spectra
of Hamiltonians for subsystems. Under the usual $|\lambda|^{-\xi}$, $\xi > 0$, decay condition
for two-body potentials Sigal shows:

a) Except for non dense set of values of the coupling constants the discrete
spectrum is finite. If furthermore the potentials are dilatation analytic this is
also valid for resonances. The exceptional values correspond to the sudden appear­
ance of an infinity of bound states for some values of the coupling constants;
this occurs if and only if at least two subsystems have simultaneously a quasi
bound-state at their continuum threshold. It can be shown that in this case exchange
forces (not direct forces) between quasi bound-states are long-range hence responsible
for the Efimov effect.

b) In case there is no bound-state or quasi-bound states at thresholds asymptotic
completeness holds.

- Spectral transformations

Among other results on N-body systems we can mention those linked to
dilation analyticity. This method introduced originally by Bottino, Longoni and
Regge in 1962 to study analyticity of scattering amplitudes and Regge poles for non relativistic one-body systems turns out to be also very fruitful for the analysis of N-body systems. We summarize here shortly the description given in [32].

Consider the linear group:

$$L_N = \left\{ (\tau, \lambda) ; \tau \in \mathbb{R}^{3(n-1)} , \lambda \in \mathbb{R}^+ \right\}$$

with the group law

$$(\tau, \lambda) \ast (\tau', \lambda') = (\lambda^{1/2} \tau + \tau' ; \lambda \lambda')$$

and its representation on

$$L^2(\mathbb{R}^{3(n-1)}) :$$

$$\mathcal{F}^{-1}(\mathcal{U}(\tau, \lambda) \phi)(\rho) = \lambda^{-3(n-1)/2} \mathcal{F}^{-1}(\phi)(\lambda^{-1/2} \rho + \overline{M} \tau)$$

where $\mathcal{F}^{-1}$ denotes the Fourier transform and $\overline{M}$ is the mass matrix.

The main interest of this group for our purpose is that the family of operators

$$H_0(\tau, \lambda) = \mathcal{U}(\tau, \lambda) H_0 \mathcal{U}^{-1}(\tau, \lambda)$$

has for any channel $\lambda$ an analytic continuation in the parameters $\tau, \lambda$ with spectrum

$$\sigma(h(\tau, \lambda)) = E_\lambda + \lambda^{-2} \left\{ \xi \in \mathbb{C} ; \Re \xi \geq -\frac{\xi}{2} \frac{\overline{M} \xi}{\sigma} \right\}$$  \quad (III.18)

and

$$\text{Im} \, \xi \leq 2 \left( \sigma \frac{\overline{M} \xi}{\sigma} \sigma \right)^2$$

where $\sigma = \text{Im} (h \overline{\lambda})$ and $\overline{\lambda}$ is the mass-matrix for composite particles in channel $\lambda$. The set (III.18) is the interior of a parabola whose axis makes an angle $-2 \tan \frac{\lambda}{2}$ with $\mathbb{R}^+$. 

Now if one tries to analyse along the lines of Hunziker [33] the spectrum of the analytic continuation of $\mathcal{U}(\tau, \lambda) H \mathcal{U}^{-1}(\tau, \lambda)$ (which exists if two-body potentials are local and dilation analytic [34]) one finds that the essential spectrum of this continuation is exactly $\bigcup \sigma(h(\tau, \lambda))$. Notice that if this is to be expected from the eventual unitary equivalence of the absolutely continuous part of $H$ and $\mathcal{U}^{-1} H \mathcal{U}$, it is obtained here from prime principles and not from scattering theory; in fact the above result is true even if wave-operators do not exist! In case they do, this result in turn suggests the existence of an analytic
continuation for their kernels in momentum spaces which should be strongly related to analyticity properties of scattering amplitudes. So it is not surprising that the main outcome of this approach is for such properties; in the case of two-body elastic or inelastic amplitudes this has been shown e.g. by J.H. Combes [32] and A. Tip [35]. Other results include spectral properties of $H$ in particular absence of continuous singular spectrum [34] and of positive energy bound-states. Dilation analyticity technique also appears as very useful for resonance energy calculation, resonances showing up in this approach as complex isolated eigenvalues of $H(\omega, \lambda)$ for $\text{Im} \lambda \neq 0$.

**IV - TIME-DEPENDENT METHODS**

These methods try to solve directly the existence problem for wave-operators without recourse to resolvent methods. They are based on the integral representation.

$$e^{iHt} \int e^{-iH_0 \tau} = \int e^{iH_0 \tau}(HJ-JH_0) e^{-iH_0 \tau} \, d\tau \quad (IV.1)$$

which plays a basic role in the proof of Theorem 2.

For single channel scattering ($J = 1$) one of the most familiar approach to the existence problem for the $S$-matrix is Dyson's perturbation expansion obtained by iterating (IV.1). For repulsive interactions [29] or in the case of weak coupling ([21], [36]) this expansion can be shown to converge to a unitary operator. This last property is obviously lost when none of the above conditions is satisfied and Dyson's expansion is not a good tool then to show completeness.

Another well-known and very ancient tool is the Cook's method ([37]) which is based on the observation

$$\left\| \int T e^{iH_0 \tau}(HJ-JH_0) e^{-iH_0 \tau} \psi \right\| d\tau < \infty \quad (IV.2)$$

So if

$$\left\| \int_T^\infty (HJ-JH_0) e^{-iH_0 \tau} \psi \right\| d\tau < \infty$$

for some $T$ and for a dense set of vectors $\psi$, the wave-operators exist. This method has been adapted recently by Schecter [38] and Simon [39] to handle situations where

$$HJ-JH_0 = A \times A_0$$

and

$$\int_T^\infty || A_0 \psi \cdots e^{-iH_0 \tau} \psi \right\| d\tau < \infty$$

for a dense set of $\psi$'s.
In this new form the theorem allows to prove existence of wave-operators when two-body potentials satisfy

\[(1 + |x|^2)^{\frac{m}{2}} \mathcal{V}(x) \in L^p \]

for \(0 < p \leq 2\), \(m > -\frac{1}{p}\).

One disadvantage of Cook's method is that since decay properties of $e^{-itH_0}$ play a fundamental role it does not work usually to show "completeness" since decay properties of $e^{-it\mathcal{H}}$ are usually unaccessible by known methods.

Cook's method has been adapted by many authors ([17], [18], [42], [43]) to treat long-range forces. There it is well-known that modified wave-operators have to be defined

\[\mathcal{W}(t) = \lim_{\tau \to \pm \infty} e^{iH\tau} e^{-i\mathcal{W}(t)}\]

where $\mathcal{W}(t)$ satisfies the Hamilton-Jacobi equation:

\[\nabla(\frac{\partial \mathcal{W}}{\partial \tau}) + \rho^l - \frac{\partial \mathcal{W}(t)}{\partial E} = 0\]

One can write $\mathcal{W}(t) = \rho^l + \zeta(t)$; clearly $\zeta(t)$ is a divergent phase when $\nabla$ decays more slowly than the Coulomb potential at infinity.

This general formulation of time-dependent scattering theory is due to L. Hörmander [43]; he uses stationary phase methods to prove convergence in (IV.2). His methods apply actually to more general elliptic systems than those obtained from the Laplacian in $\mathbb{R}^n$ and have been used in [42] by Berti and Collet to study perturbations of pseudo-differential operators. Hörmander's method makes an optimal use of Cook's theorem since it covers all existence theorems previously known for both short-range and long-range potentials (modulo an adaptation to singular potentials as done by Simon [36]). A proof of asymptotic completeness along the lines of Hörmander would require informations about the symbol of the operator $e^{it\mathcal{H}}$; let us hope they will be provided by further progress in pseudo-differential operator theory.

I would like to emphasize that the success of Hörmander's method can be traced to its semi-classical aspect in the sense that the stationary phase method is a special way to single-out from the quantum dynamics the contribution of classical orbits. A trivial example is the free evolution for $H_0 = -\Delta$

\[(e^{-itH_0}\psi)(x) = (2\pi)^{-\frac{3}{2}} \int e^{i(x\cdot p - \rho^2 p^2}\phi(p)} dp\]
which immediately gives for $X = \rho t$

$$e^{iH_0 t} \rho (\rho t) = e^{i\rho t} (e^{iH_0 t}\rho) = e^{i\rho t} + O(t^{3/2}).$$

This is a precise indication, already known for acoustical scattering (see e.g. [45]), that energy density is carried away for large time along bicharacteristics of the partial differential operator, i.e. in our case along classical trajectories. The result (IV.4) is known in quantum scattering literature as the "cone theorem" of Dollard [46] (see also [47]). For any cone $\mathcal{E}$ in configuration space, the probability that particles are in $\mathcal{E}$ tends for large time to the probability that asymptotic momenta are in $\mathcal{E}$. This theorem was mostly used in studies about the connection between observed and theoretical cross-sections [47] and partly motivated the promising, but apparently abandoned, Algebraic scattering theory (see e.g. [48]) in which those asymptotic momenta were the basic objects to be studied instead of wave-operators.

Before discussing further those deep interrelations between quantum and classical dynamics, let me recall that the main difficulties with the stationary methods originated from threshold singularities which just express the fact that some particles may have arbitrarily small velocities, so that the asymptotic regions can be reached in an arbitrarily large time. It is quite disappointing that so much work has to be done to get rid of these arbitrarily small sets in momentum space. This is typically a consequence of the undeterminacy principle and such difficulties do not show up in classical scattering; here one has simple proofs of asymptotic completeness by W. Hunziker [49] and B. Simon [50]. As an example, one has [50]:

\textbf{Theorem 3}

\begin{itemize}
  \item [a)] $\Delta \rho \cdot \nabla V_{ij}(x) < \infty$
  \item [b)] For some $R_1 > 0$, $\lambda_1 > 2$ and $C_1 > 0$
  \hspace{1cm} $|\nabla V_{ij}(x)| < C_1 |x|^{-\lambda_1}$ \quad if $|x| > R_1$
  \item [c)] For some $R_2 > 0$, $\lambda_2 > 2$ and $C_2 > 0$
  \hspace{1cm} $|\nabla V_{ij}(x) - \nabla V_{ij}(y)| < C_2 |x|^{-\lambda_2}$ \quad if $|x|, |y| \leq R_2$
\end{itemize}

Then for each $X_0 \in \mathbb{R}^{3N}$ with $\rho \neq \rho'$ for $i \neq j$, there exists one and only one solution $X(t), t \in \mathbb{R}$, of classical equations of motion.
such that

\[ |X(t) - X_t - P| + |X(t) - P| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \]

Many channel systems are also investigated by W. Hunziker [49]. The question is whether one can use those results as an input in attempts to prove asymptotic completeness in the quantum theory. The first idea which comes to mind is to use Feynman path integrals techniques [51] with infinite trajectories, which remains to be studied along the lines of Albeverio-Høegh-Krohn.

Let me describe here a strongly related method which has not been worked out completely yet but is suggestive of such possibilities. It is inspired from a work of Maslov [52] which deals with the $\hbar \rightarrow 0$ limit of quantum dynamics. One can try to use scaling arguments to transform the $\hbar \rightarrow 0$ limit into a $t \rightarrow \infty$ limit; I prefer instead to reformulate Maslov's idea in the context of one-body scattering theory.

Our aim is to analyse the connection between the solutions of Hamilton equations

\[
\begin{cases}
\frac{dX}{dt} = P(t) \\
\frac{dP}{dt} = -\nabla V(X(t))
\end{cases}
\]

and Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = (-\Delta + V) \psi \]

in the limit of large times. So it is natural to look at solutions of (I) with Cauchy data at $t = 0$. From Theorem 3, we know that there exists a $2n$ parameter of such solutions ($n$ is the space dimension here). We denote by $\mathcal{G}$ the subclass of trajectories such that:

\[ |X(t) - X_t - P| + |\dot{X}(t) - P| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \]

(The fact that we get rid of the $n$ parameters associated to initial positions can be traced to the fact that only asymptotic momenta are measured in a scattering experiment). Let us make the following regularity assumption on trajectories in $\mathcal{G}$:

Assumption

For fixed $T > T_0$, and almost all $x \in \mathbb{R}^n$, there exists a unique $X(T, T_0, x) \in \mathcal{G}$ such that $X(T, T_0, x) = x$.
In other words through each point in $\mathbb{R}^n$, there is one and only one trajectory in $G$ passing through $X$ at time $T$. In other words the situation depicted on the figure is not allowed.

A similar statement has to be made (and is proved) in [52]. Since trajectories in $G$ are parametrized by an asymptotic momentum we have then a one to one mapping

$$\Phi(T) : X \in \mathbb{R}^n \rightarrow P \in \mathbb{R}^n$$

where $P$ is the asymptotic direction of $X(x, t)$.

Example: $V = 0$. Then for $T \neq 0$, one has $\Phi(T)X = \frac{X}{T}$ and $X(x, t) = \frac{x}{t}$. We now define the classical action associated to this family of trajectories as the solution of the Hamilton-Jacobi equation

$$\frac{1}{2} \left( \nabla S(x, t) \right)^2 + V(x) + \frac{\partial S}{\partial t} (x, t) = 0$$

which is pointwise asymptotic as $t \rightarrow \infty$ to the free action $S_0(x, t) = \frac{1}{2} \frac{x^2}{t}$. Finally let $\tilde{\Psi}(x, t)$ be a solution of (II) with $\psi \in \mathcal{M}_{\text{loc}}(\mathbb{R})$ (so that under reasonable conditions on $V$, $\psi(x, t) \rightarrow 0$ pointwise). Let us define

$$\tilde{\psi}(P, t) = \sqrt{\psi(t)}^{-1} e^{i S(x, t)} \psi(x, t)$$

where $\psi(t) = \det | \frac{\partial \Phi(t)}{\partial x} |$, $P = \Phi(t) X$. Notice that with classical dynamics instead of (II) $\tilde{\psi}(P, t)$ would be constant along classical trajectories, i.e.

$$\tilde{\psi}(\Phi(t)X, t) = \tilde{\psi}(P)$$

So if $\hbar = 1$ one has instead [52]

$$i \frac{\partial \tilde{\psi}}{\partial t}(P, t) = -\frac{1}{\sqrt{\psi(t)}} \Delta_P \sqrt{\psi(t)} \tilde{\psi}(P, t)$$

where $\Delta_P$ is the Laplacian in curvilinear coordinates $P$. Equation (IV.7) then incorporates all the diffusion effects due to quantum dynamics. It just describes what remains of the Schrödinger equation once you have subtracted the classical dynamics. The relevance of this equation for physical purposes lies in the following remark. First one expects that $H_0(t) = \frac{1}{\sqrt{\psi(t)}} \Delta_P \sqrt{\psi(t)}$ converges to zero
in strong generalized sense as \( t \to \infty \). As an example for \( V = 0 \), one has 
\[
-H_{0}(t) = \Delta / t^{2}.
\]
Accordingly as \( t \to \infty \), \( \Psi(\rho, t) \) should tend to limits \( \Psi(\rho, \pm \infty) \). In view of the cone theorem one has for any cone \( C \)
\[
\lim_{t \to \infty} \int_{C} |\psi(x, t)|^{2} \, dx = \lim_{t \to \pm \infty} \int_{C} |\psi(\rho, t)|^{2} \, d\rho.
\]
Since \( \Phi(t) \) is expected to converge to the affine mapping \( \Phi(t) x = x / t \)
one obtains finally
\[
\lim_{t \to \pm \infty} \int_{E} |\psi(x, t)|^{2} \, dx = \int_{E} |\psi(\rho, \pm \infty)|^{2} \, d\rho.
\]
Since the probability to be in \( E \) at large times is directly related to the scattering cross sections \([47]\) equation (IV.7) gives a complete description of the scattering experiment. Of course, all those arguments are very formal and mathematical details will be published later. But I hope they are enough convincing that both from a mathematical and physical point of view classical mechanics are more than just the \( \hbar = 0 \) limit of quantum mechanics.
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