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Al.B.Zamolodchikov

ON THE STRUCTURE OF NON-LOCAL CONSERVATION LAWS IN THE TWO-DIMENSIONAL NON-LINEAR σ-MODEL

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Замолодчиков Ал.Б. **E2 - 11485**

О структуре нелокальных законов сохранения в двумерной нелинейной σ -модели

Показано, что предположение о специальном мультипликативном характере действия нелокальных зарядов в пространстве асимптотических состояний нелинейной σ- модели приводит к уравнениям факторизации для двухчастичных матричных элементов рассеяния, а также в значительной степени определяет вид этих зарядов на асимптотических состояниях. Сохранение зарядов оказывается совместным с факторизованной S -матрицей нелинейной σ -модели. Показано также, что факторизованная S -матрица солитонов модели sine-Gordon совместима с подобным семейством законов сохранения.

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On the Structure of Non-Local Conservation Laws in the Two-Dimensional Non-Linear σ-Model

The non-local conserved charges are supposed to satisfy a special multiplicative law in the space of asymptotic states of the non-linear σ -model. This supposition leads to factorization equations for two-particle scattering matrix elements and determines to some extent the action of these charges in the asymptotic space. Their conservation turns out to be consistent with the factorized S -matrix of the non-linear σ -model. It is shown also that the factorized sine-Gordon S-matrix is consistent with a similar family of conservation laws.

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1. INTRODUCTION

The authors of a recent paper /1/ have discovered in the classical version of the two-dimensional non-linear σ- model a one-parameter family of non-local conserved charges. Let N-component field $n^a(x)$, $a=1,2,\dots,N$ satisfy classical equations of this model:

$$\begin{gathered} \partial_\mu^2 n^a + \omega n^a = 0 \\ \sum_{a=1}^{N} (n^a)^2 = 1 . \end{gathered} \tag{1.1}$$

One can easily verify that in this case the matrix

$$Q(x,y|w) = P \exp \frac{w}{1-w^2} \int_x^y (\epsilon_{\mu\nu} J_\nu - w J_\mu)\, dx_\mu , \tag{1.2}$$

where $J_\mu(x)$ are matrices with components

$$J_\mu^{ab} = n^a \partial_\mu n^b - n^b \partial_\mu n^a \tag{1.3}$$

and the symbol P means ordering of matrix multiplication along the integration contour in (1.2) from x to y , for any w does not depend on this contour. Therefore, under appropriate boundary conditions at infinity the following family of charges is conserved:

$$Q^{ab}(w) = Q^{ab}(-\infty,\infty|w). \tag{1.4}$$

The integration contour in (1.4) is taken along the spatial axis. This family leads to the infinite sequence of conserved charges Q_n^{ab}

$$Q^{ab}(w) = \sum_{n=0}^{\infty} Q_n^{ab} w^n. \tag{1.5}$$

The authors of Ref./1/ have noted also that being non-local, these charges are not additive, instead they obey the following multiplicative law: let the field $n(x)$ configuration form at any moment two well-separated waves: $n(x) = n_1(x)$ at $x \leq A$: $n(x) = n_2(x)$ at $x \geq B$ and $n(x) = Const$ at $A \leq x \leq B$. Then

$$Q^{ab}(n(x), w) = \sum_{c=1}^{N} Q^{ac}(n_1(x), w)\, Q^{cb}(n_2(x), w). \tag{1.6}$$

In Ref./2/ the quantum analog $\tilde{Q}_2^{ab}$ of the antisymmetric part of the classical charge Q_2^{ab} from (1.5) has been constructed in the quantum non-linear σ -model. It has been shown that the conservation of this charge forbids multiple production in the model and forces certain relations between two-particle scattering matrix elements. These relations turn out to coincide with the factorization equations of the multiparticle non-linear σ -model S-matrix /3/.

In the present paper we assume an existence of the family of conserved charges $Q^{ab}(w)$ in the quantum non-linear σ -model. We suppose also, that they satisfy a certain multiplicative law which is a quantum version of Eq.(1.6). In Sec. 2 it is shown that these suggestions lead to factorization equations for two-particle matrix elements and fix to some extent the form of these charges in the asymptotic space of scattering. In Sec.2 it is demonstrated also that the factorized S-matrix is consistent with this family of conserved charges.

In Sec. 3 we show that the factorized S-matrix of sine-Gordon solitons is also consistent with a family of conserved charges with close properties.

2. NON-LINEAR σ - MODEL. CONSERVED CHARGES

The particle spectrum of the non-linear σ - model consists of N-component multiplet of massive particles which is transformed by the vector representation of the isotopic group O(N) /3,6/. Suppose, that the conserved charges $Q^{ab}(w)$ are transformed as tensors under orthogonal transformations and their actions in the asymptotic space satisfy the following multiplicative law: let $|\theta_1 c_1; \theta_2 c_2; \dots; \theta_k c_k, \mathrm{out}\rangle$ be a k-particle out-state, θ_i and c_i being i-th particle rapidity and the isotopic index: $\theta_1 < \theta_2 < \dots < \theta_k$. As $t \to \infty$, the spatial coordinates of particles x_i are arranged in the order of rapidity growth $x_1 < x_2 < \dots < x_k$. Then*)

$$Q^{ab}(w)|\theta_1 c_1; \dots; \theta_k c_k, out\rangle = \sum_{a_1, \dots a_{k-1}=1}^{N} Q^{aa_1}(w)|\theta_1 c_1\rangle Q^{a_1 a_2}(w)|\theta_2 c_2\rangle \cdots Q^{a_{k-1} b}(w)|\theta_k c_k\rangle . \quad (2.1)$$

If $t \to -\infty$ and in-states are under consideration the spatial coordinates of particles and their rapidities are arranged in the inverse order. Therefore, if again $\theta_1 < \theta_2 < \theta_3 < \dots < \theta_k$ one writes:

$$Q^{ab}(w)|\theta_1 c_1; \dots \theta_k c_k, in\rangle = \sum_{a_1, \dots a_{k-1}=1}^{N} Q^{a_1 b}(w)|\theta_1 c_1\rangle Q^{a_2 a_1}(w)|\theta_2 c_2\rangle \cdots Q^{a a_{k-1}}(w)|\theta_k c_k\rangle . \quad (2.2)$$

Taking into account the isotopic covariance of the charge $Q^{ab}(w)$ and its conservation one has in general:

$$Q^{ab}(w)|\theta, c\rangle = f_1(w,\theta)\delta^{ab}|\theta, c\rangle + f_2(w,\theta)\delta^{ac}|\theta, b\rangle + f_3(w,\theta)\delta^{cb}|\theta, a\rangle \quad (2.3)$$

with indeterminate functions f_1, f_2 and f_3. The two-particle scattering matrix has the following general form:

---

*) Charges $Q^{ab}(w)$ are supposed to be normalized so that

$$Q^{ab}(w)|0\rangle = \delta^{ab}|0\rangle .$$

$$
{}_{c_1c_2}S_{c_1'c_2'}(\theta_{12}) = \langle \theta_1 c_1 ; \theta_2 c_2, out \mid \theta_1' c_1' ; \theta_2' c_2', in \rangle =
$$

$$
= (2\pi)^2 \delta(\theta_1 - \theta_1') \delta(\theta_2 - \theta_2') \Big[\delta^{c_1c_2} \delta^{c_1'c_2'} \sigma_1(\theta_{12}) + \tag{2.4}
$$

$$
+ \delta^{c_1c_1'} \delta^{c_2c_2'} \sigma_2(\theta_{12}) + \delta^{c_1c_2'} \delta^{c_2c_1'} \sigma_3(\theta_{12}) \Big],
$$

where $\theta_{12} = \theta_2 - \theta_1$ and $\theta_2 > \theta_1 ;\ \theta_2' > \theta_1'$.

One can verify directly that the conservation of charges $Q^{ab}(w)$ in 2 → 2 scattering leads to the following relations:

$$
\left[f_1(\theta) f_2(\theta') + f_3(\theta) f_1(\theta') \right] \sigma_1(\theta - \theta') = f_3(\theta) f_2(\theta') \sigma_2(\theta - \theta') \tag{2.5a}
$$

$$
\left[f_1(\theta) f_2(\theta') - f_2(\theta) f_1(\theta') \right] \sigma_3(\theta - \theta') = f_2(\theta) f_2(\theta') \sigma_2(\theta - \theta') \tag{2.5b}
$$

$$
\left[f_3(\theta) f_1(\theta') - f_1(\theta) f_3(\theta') \right] \sigma_3(\theta - \theta') = f_3(\theta) f_3(\theta') \sigma_2(\theta - \theta') \tag{2.5c}
$$

$$
\left[f_3(\theta) f_2(\theta') - f_2(\theta) f_3(\theta') \right] \sigma_3(\theta - \theta') = \Big[f_2(\theta) f_1(\theta') + f_1(\theta) f_3(\theta') +
$$

$$
+ f_2(\theta) f_2(\theta') + N f_2(\theta) f_3(\theta') + f_3(\theta) f_3(\theta') \Big] \sigma_1(\theta - \theta') + \tag{2.5d}
$$

$$
+ f_2(\theta) f_3(\theta') \sigma_2(\theta - \theta'),
$$

where for the sake of brevity the arguments w near f's are omitted.

The general solution of Eqs. (2.5a-c) depends on three parameters λ , γ_1 and γ_2 :

$$\sigma_3(\theta) = -i\frac{\lambda}{\theta}\sigma_2(\theta)\ ;\ \sigma_1(\theta) = \frac{\lambda}{\gamma_1+\gamma_2-i\theta}\sigma_2(\theta) \quad (2.6)$$

$$f_2(\theta) = \frac{\lambda}{\gamma_1+i\theta} f_1(\theta)\ ;\quad f_3(\theta) = \frac{\lambda}{\gamma_2-i\theta} f_1(\theta). \quad (2.7)$$

Eq.(2.5d) leads to the following relation between parameters:

$$\gamma_1+\gamma_2 = -\frac{\lambda}{2}(N-2). \quad (2.8)$$

Eqs.(2.6) and (2.8) coincide with the multiparticle S-matrix factorization equations of the nonlinear σ -model /3/. Considering the crossing-symmetry relations

$$\begin{aligned} \sigma_2(\theta) &= \sigma_2(i\pi-\theta) \\ \sigma_1(\theta) &= \sigma_3(i\pi-\theta) \end{aligned} \quad (2.9)$$

one obtains

$$\lambda = \frac{2\pi}{N-2}\ ;\quad \gamma_1+\gamma_2 = -\pi \quad (2.10)$$

so that only one parameter $\gamma_1 = \frac{i\lambda}{w'}$; $\gamma_2 = -\pi - \frac{i\lambda}{w'}$ remains undetermined and must be in some connection with the parameter w of the corresponding charge. Finally Eq.(2.3) acquires the form:

$$Q^{ab}(w)|\theta,c\rangle = f_1(w,\theta)\left[\delta^{ab}|\theta,c\rangle - \right. \quad (2.11)$$

$$- \frac{iw'}{1+\frac{\theta w'}{\lambda}} \delta^{ac} |\theta, b\rangle + \frac{iw'}{1-\frac{w'}{\lambda}(i\pi-\theta)} \delta^{bc} |\theta, a\rangle \Big].$$

To make sure that the conservation laws considered are consistent with the factorized multiparticle S-matrix it is convenient to use its algebraic representation /3/. In this way the states are represented by products of non-commutative symbols $A_c(\theta)$, corresponding to particles. The arrangement of the symbols in these products corresponds to that of particles along the spatial axis and their rearrangements (pair collisions) can be carried out by means of:

$$A_{c_1}(\theta_1)A_{c_2}(\theta_2) = \delta^{c_1c_2}\sigma_1(\theta_{12})\sum_{c=1}^{N} A_c(\theta_2)A_c(\theta_1) + \tag{2.12}$$

$$+ \sigma_2(\theta_{12})A_{c_2}(\theta_2)A_{c_1}(\theta_1) + \sigma_3(\theta_{12})A_{c_1}(\theta_2)A_{c_2}(\theta_1),$$

with the amplitudes σ_1, σ_2 and σ_3 defined by (2.4). In these terms Eqs.(2.1) and (2.2) can be replaced by:

$$Q^{ab}(w)A_{c_1}(\theta_1)\cdots A_{c_k}(\theta_k) = \sum_{a_1,\ldots a_{k-1}=1}^{N} Q^{aa_1}(w)A_{c_1}(\theta_1)Q^{a_1a_2}(w)A_{c_2}(\theta_2)\cdots Q^{a_{k-1}b}(w)A_{c_k}(\theta_k) \tag{2.13}$$

while Eq.(2.3) takes the form:

$$Q^{ab}(w)A_c(\theta) = f_1(w,\theta)\delta^{ab}A_c(\theta) + f_2(w,\theta)\delta^{ac}A_b(\theta) + f_3(w,\theta)\delta^{bc}A_a(\theta). \tag{2.14}$$

Using these formulas it is easy to show that if Eqs.(2.5) are true, the operation of pair transposition of neighbouring symbols is commutative with the action of the conserved charges $Q^{ab}(w)$.

3. CHARGES CONSERVED IN THE QUANTUM SINE-GORDON SCATTERING

The quantum sine-Gordon model is equivalent to the massive Thirring model /7/ and therefore possesses the hidden internal symmetry O(2)=U(1). The mass spectrum of this model contains both O(2)-charged particles, namely, massive soliton (A) and corresponding antisoliton ($\bar{A}$), and a number of their neutral bound states /8/. The total sine-Gordon S-matrix is factorized /4,5/ and its algebraic representation /4/ with non-commutative symbols $A(\theta)$ and $\bar{A}(\theta)$ corresponding to the soliton and antisoliton, respectively, will be used throughout all the following consideration. The commutation relations for these symbols introduce two-particle soliton amplitudes S , S_T and S_R :

$$\begin{aligned} A(\theta_1)A(\theta_2) &= S(\theta_{12})\,A(\theta_2)A(\theta_1) \\ \bar{A}(\theta_1)\,\bar{A}(\theta_2) &= S(\theta_{12})\bar{A}(\theta_2)\,\bar{A}(\theta_1) \\ A(\theta_1)\,\bar{A}(\theta_2) &= S_T(\theta_{12})\bar{A}(\theta_2)A(\theta_1)+S_R(\theta_{12})A(\theta_2)\bar{A}(\theta_1). \end{aligned} \tag{3.1}$$

Suppose that in the model under consideration there is a one-parameter family of conserved charges $Q^{ab}(w)$ which are second rank O(2)-tensors and satisfy the multiplicative law, described in the previous sectio.. (see Eqs.(2.1) and (2.2)). Their action on a one-particle state has the form of Eq.(2.14) where real components of solitons A_a ; a =1,2 are related with charged solitons by:

$$\begin{aligned} A &= A_1 + i A_2 \\ \bar{A} &= A_1 - i A_2 . \end{aligned} \tag{3.2}$$

The second rank O(2)-tensor Q^{ab} is reducible; it is convenient to extract its irreducible components:

$$
\begin{aligned}
Q_0 &= Q^{11} + Q^{22} \\
Q_1 &= i\left(Q^{12} - Q^{21}\right) \\
Q_2 &= \tfrac{1}{2}\left(Q^{11} - Q^{22} + iQ^{12} + iQ^{21}\right) \\
Q_{-2} &= \tfrac{1}{2}\left(Q^{11} - Q^{22} - iQ^{12} - iQ^{21}\right).
\end{aligned}
\tag{3.3}
$$

Then Eqs.(2.14),(3.2) and (3.3) result in

$$
\begin{aligned}
Q_0(w)A(\theta) &= F_0(w,\theta)A(\theta) \\
Q_0(w)\bar{A}(\theta) &= F_0(w,\theta)\bar{A}(\theta) \\
Q_1(w)A(\theta) &= F_1(w,\theta)A(\theta) \\
Q_1(w)\bar{A}(\theta) &= -F_1(w,\theta)\bar{A}(\theta) \\
Q_2(w)A(\theta) &= 0 \\
Q_2(w)\bar{A}(\theta) &= F_2(w,\theta)A(\theta) \\
Q_{-2}(w)A(\theta) &= F_2(w,\theta)\bar{A}(\theta) \\
Q_{-2}(w)\bar{A}(\theta) &= 0 ,
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
F_0(w,\theta) &= 2f_1(w,\theta) + f_2(w,\theta) + f_3(w,\theta) \\
F_1(w,\theta) &= f_2(w,\theta) - f_3(w,\theta) \\
F_2(w,\theta) &= f_2(w,\theta) + f_3(w,\theta).
\end{aligned}
\tag{3.5}
$$

Straightforward calculations show that the conservation of charges in two-particle collisions (3.1) requires three relations to be satisfied:

$$2F_2(\theta)F_2(\theta')S_T(\theta-\theta') = [F_0(\theta)F_1(\theta')-F_1(\theta)F_0(\theta')]S_R(\theta-\theta') \qquad (3.6a)$$

$$[F_0(\theta)+F_1(\theta)]F_2(\theta')S_T(\theta-\theta')+F_2(\theta)[F_0(\theta')-F_1(\theta')]S_R(\theta-\theta') = \qquad (3.6b)$$

$$= [F_0(\theta)-F_1(\theta)]F_2(\theta')S(\theta-\theta')$$

$$[F_0(\theta)+F_1(\theta)]F_2(\theta')S_R(\theta-\theta')+F_2(\theta)[F_0(\theta')-F_1(\theta')]S_T(\theta-\theta') = \qquad (3.6c)$$

$$= F_2(\theta)[F_0(\theta')+F_1(\theta')]S(\theta-\theta').$$

The general solution of this system depends on three parameters and has the form:

$$\frac{S(\theta)+S_T(\theta)}{S_R(\theta)} = \frac{\operatorname{sh}\lambda(\theta+\delta)}{\operatorname{sh}\lambda\delta}$$

$$\frac{S(\theta)-S_T(\theta)}{S_R(\theta)} = \frac{\operatorname{ch}\lambda(\theta+\delta)}{\operatorname{ch}\lambda\delta}, \qquad (3.7)$$

where λ and δ are parameters, and

$$\frac{F_0(\theta)}{F_2(\theta)} = \frac{\operatorname{sh}\lambda(\theta+\eta)}{\operatorname{sh}\lambda\delta}$$

$$\frac{F_1(\theta)}{F_2(\theta)} = \frac{\operatorname{ch}\lambda(\theta+\eta)}{\operatorname{ch}\lambda\delta}, \qquad (3.8)$$

where η is the third parameter, somehow connected with the charge parameter w. Eqs.(3.7) are equivalent to the factorization equations of the multiparticle sine-Gordon soliton S-matrix /4,5,9/. Therefore the factorized sine-Gordon S-matrix is consistent with the conservation laws of the type considered.

I would like to thank Alexander Zamolodchikov, B.Z.Kopeliovich and Prof.L.I.Lapidus for interesting discussions.

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$$Q^{ab}(w)|\theta_1 c_1;\dots;\theta_k c_k, out\rangle=\sum_{a_1,\dots a_{k-1}=1}^{N} Q^{aa_1}(w)|\theta_1 c_1\rangle Q^{a_1a_2}(w)|\theta_2 c_2\rangle\cdots Q^{a_{k-1}b}(w)|\theta_k c_k\rangle . \quad (2.1)$$

If $t\to-\infty$ and in-states are under consideration the spatial coordinates of particles and their rapidities are arranged in the inverse order. Therefore, if again $\theta_1<\theta_2<\theta_3<\dots<\theta_k$ one writes:

$$Q^{ab}(w)|\theta_1 c_1;\dots\theta_k c_k, in\rangle=\sum_{a_1,\dots a_{k-1}=1}^{N} Q^{a_1b}(w)|\theta_1 c_1\rangle Q^{a_2a_1}(w)|\theta_2 c_2\rangle\cdots Q^{aa_{k-1}}(w)|\theta_k c_k\rangle . \quad (2.2)$$

Taking into account the isotopic covariance of the charge $Q^{ab}(w)$ and its conservation one has in general:

$$Q^{ab}(w)|\theta c\rangle=f_1(w,\theta)\delta^{ab}|\theta c\rangle+f_2(w,\theta)\delta^{ac}|\theta,b\rangle+f_3(w,\theta)\delta^{cb}|\theta,a\rangle \quad (2.3)$$

with indeterminate functions f_1, f_2 and f_3. The two-particle scattering matrix has the following general form:

---

*) Charges $Q^{ab}(w)$ are supposed to be normalized so that

$$Q^{ab}(w)|0\rangle = \delta^{ab}|0\rangle .$$

$$
{}_{c_1c_2}S_{c_1'c_2'}(\theta_{12}) = \langle \theta_1 c_1; \theta_2 c_2, out \mid \theta_1' c_1'; \theta_2' c_2', in \rangle =
$$

$$
= (2\pi)^2 \delta(\theta_1 - \theta_1')\delta(\theta_2 - \theta_2')\left[\delta^{c_1c_2}\delta^{c_1'c_2'}\sigma_1(\theta_{12}) + \right. \tag{2.4}
$$

$$
\left. + \delta^{c_1c_1'}\delta^{c_2c_2'}\sigma_2(\theta_{12}) + \delta^{c_1c_2'}\delta^{c_2c_1'}\sigma_3(\theta_{12})\right],
$$

where $\theta_{12} = \theta_2 - \theta_1$ and $\theta_2 > \theta_1$; $\theta_2' > \theta_1'$.

One can verify directly that the conservation of charges $Q^{ab}(w)$ in 2 → 2 scattering leads to the following relations:

$$
[f_1(\theta)f_2(\theta') + f_3(\theta)f_1(\theta')]\sigma_1(\theta-\theta') = f_3(\theta)f_2(\theta')\sigma_2(\theta-\theta') \tag{2.5a}
$$

$$
[f_1(\theta)f_2(\theta') - f_2(\theta)f_1(\theta')]\sigma_3(\theta-\theta') = f_2(\theta)f_2(\theta')\sigma_2(\theta-\theta') \tag{2.5b}
$$

$$
[f_3(\theta)f_1(\theta') - f_1(\theta)f_3(\theta')]\sigma_3(\theta-\theta') = f_3(\theta)f_3(\theta')\sigma_2(\theta-\theta') \tag{2.5c}
$$

$$
[f_3(\theta)f_2(\theta') - f_2(\theta)f_3(\theta')]\sigma_3(\theta-\theta') = [f_2(\theta)f_1(\theta') + f_1(\theta)f_3(\theta') +
$$

$$
+ f_2(\theta)f_2(\theta') + N f_2(\theta)f_3(\theta') + f_3(\theta)f_3(\theta')]\sigma_1(\theta-\theta') + \tag{2.5d}
$$

$$
+ f_2(\theta)f_3(\theta')\sigma_2(\theta-\theta'),
$$

where for the sake of brevity the arguments w near f's are omitted.

The general solution of Eqs. (2.5a-c) depends on three parameters λ , γ_1 and γ_2 :

$$\sigma_3(\theta) = -i\frac{\lambda}{\theta}\sigma_2(\theta)\ ;\ \sigma_1(\theta) = \frac{\lambda}{\gamma_1+\gamma_2-i\theta}\sigma_2(\theta) \tag{2.6}$$

$$f_2(\theta) = \frac{\lambda}{\gamma_1+i\theta}f_1(\theta)\ ;\ f_3(\theta) = \frac{\lambda}{\gamma_2-i\theta}f_1(\theta). \tag{2.7}$$

Eq.(2.5d) leads to the following relation between parameters:

$$\gamma_1+\gamma_2 = -\frac{\lambda}{2}(N-2). \tag{2.8}$$

Eqs.(2.6) and (2.8) coincide with the multiparticle S-matrix factorization equations of the nonlinear σ -model /3/. Considering the crossing-symmetry relations

$$\begin{aligned} \sigma_2(\theta) &= \sigma_2(i\pi-\theta) \\ \sigma_1(\theta) &= \sigma_3(i\pi-\theta) \end{aligned} \tag{2.9}$$

one obtains

$$\lambda = \frac{2\pi}{N-2}\ ;\quad \gamma_1+\gamma_2 = -\pi \tag{2.10}$$

so that only one parameter $\gamma_1 = \frac{i\lambda}{w'}$; $\gamma_2 = -\pi - \frac{i\lambda}{w'}$ remains undetermined and must be in some connection with the parameter w of the corresponding charge. Finally Eq.(2.3) acquires the form:

$$Q^{ab}(w)|\theta,c\rangle = f_1(w,\theta)\left[\delta^{ab}|\theta,c\rangle - \right. \tag{2.11}$$

$$-\frac{iw'}{1+\frac{\theta w'}{\lambda}}\delta^{ac}|\theta,b\rangle+\frac{iw'}{1-\frac{w'}{\lambda}(i\pi-\theta)}\delta^{bc}|\theta,a\rangle\Big].$$

To make sure that the conservation laws considered are consistent with the factorized multiparticle S-matrix it is convenient to use its algebraic representation /3/. In this way the states are represented by products of non-commutative symbols $A_c(\theta)$, corresponding to particles. The arrangement of the symbols in these products corresponds to that of particles along the spatial axis and their rearrangements (pair collisions) can be carried out by means of:

$$A_{c_1}(\theta_1)A_{c_2}(\theta_2)=\delta^{c_1c_2}\sigma_1(\theta_{12})\sum_{c=1}^{N}A_c(\theta_2)A_c(\theta_1)+ \tag{2.12}$$

$$+\sigma_2(\theta_{12})A_{c_2}(\theta_2)A_{c_1}(\theta_1)+\sigma_3(\theta_{12})A_{c_1}(\theta_2)A_{c_2}(\theta_1),$$

with the amplitudes σ_1, σ_2 and σ_3 defined by (2.4). In these terms Eqs.(2.1) and (2.2) can be replaced by:

$$Q^{ab}(w)A_{c_1}(\theta_1)\cdots A_{c_k}(\theta_k)=\sum_{a_1,\dots a_{k-1}=1}^{N}Q^{aa_1}(w)A_{c_1}(\theta_1)Q^{a_1a_2}(w)A_{c_2}(\theta_2)\cdots Q^{a_{k-1}b}(w)A_{c_k}(\theta_k) \tag{2.13}$$

while Eq.(2.3) takes the form:

$$Q^{ab}(w)A_c(\theta)=f_1(w,\theta)\delta^{ab}A_c(\theta)+f_2(w,\theta)\delta^{ac}A_b(\theta)+f_3(w,\theta)\delta^{bc}A_a(\theta). \tag{2.14}$$

Using these formulas it is easy to show that if Eqs.(2.5) are true, the operation of pair transposition of neighbouring symbols is commutative with the action of the conserved charges $Q^{ab}(w)$.

3. CHARGES CONSERVED IN THE QUANTUM SINE-GORDON SCATTERING

The quantum sine-Gordon model is equivalent to the massive Thirring model /7/ and therefore possesses the hidden internal symmetry O(2)=U(1). The mass spectrum of this model contains both O(2)-charged particles, namely, massive soliton (A) and corresponding antisoliton ($\bar{A}$), and a number of their neutral bound states /8/. The total sine-Gordon S-matrix is factorized /4,5/ and its algebraic representation /4/ with non-commutative symbols $A(\theta)$ and $\bar{A}(\theta)$ corresponding to the soliton and antisoliton, respectively, will be used throughout all the following consideration. The commutation relations for these symbols introduce two-particle soliton amplitudes S , S_T and S_R :

$$
\begin{aligned}
&A(\theta_1)A(\theta_2) = S(\theta_{12})\,A(\theta_2)A(\theta_1)\\
&\bar{A}(\theta_1)\,\bar{A}(\theta_2) = S(\theta_{12})\bar{A}(\theta_2)\,\bar{A}(\theta_1) \qquad (3.1)\\
&A(\theta_1)\,\bar{A}(\theta_2) = S_T(\theta_{12})\bar{A}(\theta_2)A(\theta_1) + S_R(\theta_{12})\,A(\theta_2)\bar{A}(\theta_1).
\end{aligned}
$$

Suppose that in the model under consideration there is a one-parameter family of conserved charges $Q^{ab}(w)$ which are second rank O(2)-tensors and satisfy the multiplicative law, described in the previous sectio. (see Eqs.(2.1) and (2.2)). Their action on a one-particle state has the form of Eq.(2.14) where real components of solitons A_a ; a =1,2 are related with charged solitons by:

$$
\begin{aligned}
A &= A_1 + i A_2\\
\bar{A} &= A_1 - i A_2 \,.
\end{aligned} \qquad (3.2)
$$

The second rank O(2)-tensor Q^{ab} is reducible; it is convenient to extract its irreducible components:

$$
\begin{aligned}
Q_0 &= Q^{11} + Q^{22} \\
Q_1 &= i\left(Q^{12} - Q^{21}\right) \\
Q_2 &= \tfrac{1}{2}\left(Q^{11} - Q^{22} + iQ^{12} + iQ^{21}\right) \\
Q_{-2} &= \tfrac{1}{2}\left(Q^{11} - Q^{22} - iQ^{12} - iQ^{21}\right).
\end{aligned}
\tag{3.3}
$$

Then Eqs.(2.14),(3.2) and (3.3) result in

$$
\begin{aligned}
Q_0(w)A(\theta) &= F_0(w,\theta)A(\theta) \\
Q_0(w)\bar{A}(\theta) &= F_0(w,\theta)\bar{A}(\theta) \\
Q_1(w)A(\theta) &= F_1(w,\theta)A(\theta) \\
Q_1(w)\bar{A}(\theta) &= -F_1(w,\theta)\bar{A}(\theta) \\
Q_2(w)A(\theta) &= 0 \\
Q_2(w)\bar{A}(\theta) &= F_2(w,\theta)A(\theta) \\
Q_{-2}(w)A(\theta) &= F_2(w,\theta)\bar{A}(\theta) \\
Q_{-2}(w)\bar{A}(\theta) &= 0 \quad ,
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
F_0(w,\theta) &= 2f_1(w,\theta) + f_2(w,\theta) + f_3(w,\theta) \\
F_1(w,\theta) &= f_2(w,\theta) - f_3(w,\theta) \\
F_2(w,\theta) &= f_2(w,\theta) + f_3(w,\theta).
\end{aligned}
\tag{3.5}
$$

Straightforward calculations show that the conservation of charges in two-particle collisions (3.1) requires three relations to be satisfied:

$$2F_2(\theta)F_2(\theta')S_T(\theta-\theta') = [F_0(\theta)F_1(\theta')-F_1(\theta)F_0(\theta')]S_R(\theta-\theta') \qquad (3.6a)$$

$$[F_0(\theta)+F_1(\theta)]F_2(\theta')S_T(\theta-\theta')+F_2(\theta)[F_0(\theta')-F_1(\theta')]S_R(\theta-\theta')= \qquad (3.6b)$$
$$=[F_0(\theta)-F_1(\theta)]F_2(\theta')S(\theta-\theta')$$

$$[F_0(\theta)+F_1(\theta)]F_2(\theta')S_R(\theta-\theta')+F_2(\theta)[F_0(\theta')-F_1(\theta')]S_T(\theta-\theta')= \qquad (3.6c)$$
$$=F_2(\theta)[F_0(\theta')+F_1(\theta')]S(\theta-\theta').$$

The general solution of this system depends on three parameters and has the form:

$$\frac{S(\theta)+S_T(\theta)}{S_R(\theta)} = \frac{\operatorname{Sh}\lambda(\theta+\delta)}{\operatorname{Sh}\lambda\delta}$$
$$\frac{S(\theta)-S_T(\theta)}{S_R(\theta)} = \frac{\operatorname{Ch}\lambda(\theta+\delta)}{\operatorname{Ch}\lambda\delta}, \qquad (3.7)$$

where λ and δ are parameters, and

$$\frac{F_0(\theta)}{F_2(\theta)} = \frac{\operatorname{Sh}\lambda(\theta+\eta)}{\operatorname{Sh}\lambda\delta}$$
$$\frac{F_1(\theta)}{F_2(\theta)} = \frac{\operatorname{Ch}\lambda(\theta+\eta)}{\operatorname{Ch}\lambda\delta}, \qquad (3.8)$$

where η is the third parameter, somehow connected with the charge parameter w . Eqs.(3.7) are equivalent to the factorization equations of the multiparticle sine-Gordon soliton S-matrix /4,5,9/. Therefore the factorized sine-Gordon S-matrix is consistent with the conservation laws of the type considered.

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