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# RELATIVISTIC FLUIDS IN SPHERICALLY SYMMETRIC SPACE *

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## ABSTRACT

Some of McVittie and Wiltshire's (1977) solutions of Walker's (1935) isotropy conditions for relativistic perfect fluid spheres are generalized. Solutions are spherically symmetric and conformally flat.

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## I. INTRODUCTION

For a spherically symmetric metric we have

$$ds^2 = e^{2\lambda}\, d\eta^2 - e^{2\mu}\,(d\xi^2 + f^2\, d\theta^2 + f^2 \sin^2\theta\, d\phi^2) \quad , \tag{1.1}$$

where $f$ is a function of $\xi$ alone and $\lambda$ ,$\mu$ are functions of $\xi$ and $\eta$ . The Einstein equation for a perfect fluid reduces to

$$e^{2(\mu-\lambda)}\,(\,4K_3^2 - 2K_1K_4) = K_1K_2 \quad , \tag{1.2}$$

where

$$\left.\begin{aligned}
K_1 &= \mu'' + \lambda'' + \lambda'^2 - \mu'^2 - 2\lambda'\mu' - \frac{f'}{f}(\mu'+\lambda') + \frac{f''}{f^2} - \frac{f'^2}{f^2}\frac{1}{f^2} \,,\\
K_2 &= \mu'' - \lambda'' + \mu'^2 - \lambda'^2 + \frac{f'}{f}(3\mu'-\lambda') + \frac{f''}{f} + \frac{f'^2}{f^2} - \frac{1}{f^2}\,,\\
K_3 &= \dot{\mu}' - \dot{\lambda}\dot{\mu} \,,\\
K_4 &= \ddot{\mu} - \dot{\lambda}\dot{\mu} \,,
\end{aligned}\right\} \tag{1.3}$$

$$\mu' \equiv \frac{\partial\mu}{\partial\xi} \,,\quad \dot{\mu} \equiv \frac{\partial\mu}{\partial\eta} \,,\quad \dot{\lambda} \equiv \frac{\partial\lambda}{\partial\eta} \,,\quad \lambda' \equiv \frac{\partial\lambda}{\partial\xi} \text{ and so on.}$$

$$(x^1,x^2,x^3,x^4) \equiv (\xi,\theta,\phi,\eta)$$

provided that pressure $p$, density $\rho$ and velocity $v^\mu$ of the fluid are given by

$$8\pi p = G_2^2 \quad , \quad 8\pi\rho = e^{-2\mu}K_1 - G_4^4 \quad , \tag{1.4}$$

$$\left.\begin{aligned}
e^{2\mu}(v^1)^2 &= \frac{e^{-2\mu}\,K_1^2}{4e^{-2\lambda}\,K_3^2 - e^{-2\mu}\,K_1^2}\\
e^{2\lambda}(v^4)^2 &= \frac{4e^{-2\lambda}\,K_3^2}{4e^{-2\lambda}\,K_3^2 - e^{-2\mu}\,K_1^2}
\end{aligned}\right\} \tag{1.5}$$

$$v^2 = 0 \,,$$
$$v^3 = 0 \,,$$

where $G_\mu{}^\nu \equiv R_\mu{}^\nu - \frac{1}{2}\,\delta_\mu{}^\nu R$ , $R_{\mu\nu}$ being the Ricci tensor. (1.2) is the isotropy condition of Walker (1935). This equation can be simplified if we note that there exists a choice of co-ordinates that keeps the form of (1.1) intact but sets $u' = 0$ and hence by (1.4), $K_1 = 0$. (1.2) then merely means $K_3 = 0$. Such co-ordinates are known as co-moving co-ordinates and such solutions have been obtained by many authors, e.g. Bonnor and Faulkes (1967), Cahill and Taub (1971). However, all such solutions have not been found and obviously the solutions found are those that are expressible in terms of simple functions. Since a solution that cannot be expressed in terms of simple functions in a co-moving system might be expressible in terms of simple functions in some other co-ordinate system, say a non-co-moving one, McVittie and Wiltshire have felt that there is a need to look into solutions of (1.1) in a non-co-moving system as well, i.e. when

and
$$\left.\begin{array}{l} K_1 \neq 0 \\ K_3 \neq 0 \,. \end{array}\right\} \qquad \text{(I)}$$

In order to obtain such solutions of (1.2), McVittie and Wiltshire have assumed

$$\lambda = \alpha + \Psi \ , \quad \mu = \beta + \psi \ , \qquad (1.6)$$

where

$$\alpha = \alpha(z), \ \beta = \beta(z) \ , \ \Psi = \Psi(\eta), \ \psi = \psi(\eta), \quad z = h(\xi) + g(\eta) \ , \qquad (1.7)$$

h and g being two functions. Then they consider two different cases

$$\text{Case (I)} \quad g = 0 \ , \qquad (1.8)$$

$$\text{Case (II)} \quad \psi = ag \ , \quad a \text{ being a constant} \ . \qquad (1.9)$$

For both these cases some particular solutions have been obtained by the two authors; although none has been completely solved. In the present work, we shall try to generalize some of their solutions.

Of the various solutions obtained by McVittie and Wiltshire, there is one which seems to deserve a special mention. It is Case (II) with $a = 0$ and $\alpha = \beta$. In this case, the solution obtained is conformally flat; the function $z$ takes a special form, i.e. $z = \xi^2 - \eta^2$, and the metric itself, when expressed in standard conformally flat form, is a function of $z$. Moreover, it can be shown that the fluid is irrotational and barotropic (i.e. $p$ and $\rho$ are functionally related) and that pressure, density and velocity potentials are functions of $z$. The function $z$ thus attains a very special role. This will be one of the cases which we shall try to generalize and



see how many of these special roles are retained by generalization. However, we discuss Case (I) first.

## II. SOLUTIONS FOR g = 0

In view of (1.7), without loss of generality, one can set

$$z = \xi \ . \qquad (2.1)$$

(2.1) then reduces to

$$e^{2(\beta-\alpha+\psi-\Psi)}\left\{4(\dot\psi\alpha')^2 - 2(\ddot\psi-\dot\psi\dot\Psi)\,K_1\right\} - K_1K_2 = 0 \qquad (2.2)$$

In view of (1.7) and (2.1), $\alpha$, $\beta$, $K_1$ and $K_2$ are functions of $\xi$ only. (2.2) has been completely solved by McVittie and Wiltshire under further assumptions

$$\beta = \alpha - \ln f \qquad (2.3)$$

and

$$K_1 = 2p\alpha'^2 \ , \qquad (2.4)$$

$p$ being a constant.

Here we attempt to solve (2.2) for the case when (2.3) holds, but not (2.4), i.e.

$$K_1 \neq 2p\alpha'^2 \qquad \text{(II)}$$

for any constant $p$. From (1.1), (1.6), (1.7) and (2.1) we note that by a co-ordinate transformation of the form $\eta' = \int e^{2(\Psi-\psi)}\,d\eta$, one can, without loss of generality, set

$$\Psi = \psi \ . \qquad (2.5)$$

Since $\alpha$, $\beta$, $K$ are functions of $\xi$ only, (2.3) and (II) remain intact by this transformation. (2.2) then reduces to

$$e^{2(\beta-\alpha)}\left\{4(\dot\psi\alpha')^2 - 2(\ddot\psi-\dot\psi^2)\,K_1\right\} - K_1K_2 = 0 \ . \qquad (2.6)$$

Differentiating (2.6) with respect to $\eta$,

$$4\alpha'^2\, e^{2(\beta-\alpha)} \frac{d}{d\eta}(\dot\psi^2) - 2K_1\, e^{2(\beta-\alpha)} \frac{d}{d\eta}(\ddot\psi-\dot\psi^2) = 0 \ . \qquad (2.7)$$

Since $\alpha$, $\beta$, $K_1$ are functions of $\xi$, $\psi$ is a function of $\eta$, (2.7) can hold only if

$$\dot{\psi}^2 = \text{constant} \quad , \tag{2.8}$$

$$\ddot{\psi} - \dot{\psi}^2 = \text{constant} \quad . \tag{2.9}$$

However, it is easy to note that (2.9) follows from (2.8) and one can, without loss of generality or violation of previous conditions, set

$$\psi = \eta \ , \tag{2.10}$$

and (2.6) reduces

$$e^{2(\beta-\alpha)} \ (4\alpha'^2 + 2K_1) = K_1K_2 \quad . \tag{2.11}$$

Putting (1.6), (1.7), (2.1) and (2.3) into (1.1) we note that a co-ordinate transformation of the form

$$\xi' = \int \frac{d\xi}{f^2}$$

makes

$$\beta = \alpha \quad , \quad f = 1 \quad . \tag{2.12}$$

Since such a transformation does not involve $\eta$, this will keep (2.5) and (2.10) intact, as well as all other previous equations applicable here. Inequality (II), however, may no longer hold. But that is immaterial, because once (2.5) and (2.10) remain intact, we no longer need (II). From (1.3), (1.6), (1.7), (2.1), (2.11) and (2.12) we get

$$6\alpha'' - 2\alpha'^2 + 3 = 0 \ , \tag{2.13}$$

which can easily be integrated to get $\alpha$ explicitly in terms of $\xi$. Thus the metric takes the form

$$ds^2 = e^{2(\alpha+\eta)} \ [d\eta^2 - d\xi^2 - d\Omega^2] \quad , \tag{2.14}$$

where $\alpha = \alpha(\xi)$ is given by (2.13).

## III. SOLUTIONS FOR $\psi = ag$

In this case, McVittie and Wiltshire have further assumed

$$\frac{f''}{f} - \frac{f'^2}{f^2} + \frac{1}{f^2} = 0 \quad , \tag{3.1}$$



$$h'' - \frac{f'h'}{f} = bh'^2 \tag{3.2}$$

and

$$h'' = b_1h'^2 + b_2 \quad , \tag{3.3}$$

$b$, $b_1$, $b_2$ being constants.

With these, the isotropy condition reduces to

$$2k^2\dot{g}^2 e^{2(\beta-\alpha+ag-\Psi)}\left(\beta_{zz}-\alpha_z\beta_z-a\alpha_z\right)\left\{K_5-(2a+b)\alpha_z\right\}$$
$$-K_1\left\{2e^{2(\beta-\alpha+ag-\Psi)}(\beta_z+a)(\ddot{g}-\dot{g}\dot{\Psi})+2\left[b_2(2\beta_z-\alpha_z)+\frac{f''}{f}\right]\right.$$
$$\left.+k^2\left[K_5+(2b_1-b)(2\beta_z-\alpha_z)\right]\right\} = 0 \ , \tag{3.4}$$

where $K_1$ takes the form

$$K_1 = h'^2\left\{\alpha_{zz} + \beta_{zz} + \alpha_z^2 - \beta_z^2 - 2\alpha_z\beta_z + b(\alpha_z + \beta_z)\right\} \tag{3.5}$$

and

$$K_5 = \beta_{zz} - \alpha_{zz} + \beta_z^2 - \alpha_z^2 - b\beta_z \ . \tag{3.6}$$

(3.1), (3.2) and (3.3) have been completely solved by McVittie and Wiltshire to get the following four sets of solutions:

$$\left.\begin{array}{l}
\text{Solution (1)} \ f = \xi \quad h = b_2\xi^2/2 + N_2 \quad b = 0, \quad b_1 = 0 \ , \\
\text{Solution (2)} \ f = \xi \quad h = -\frac{1}{b_1} \ \ln(N_3\xi) \ b = 2b_1 \neq 0, \quad b_2 = 0 \ , \\
\text{Solution (3)} \ f = \frac{1}{n_1}\sin(n_1\xi) \ , \quad h = -\frac{1}{b_1}\ln\left\{N_3 \sin\left(\frac{n_1\xi}{2}\right)\right\} \ , \\
\text{Solution (4)} \ f = \frac{1}{n_1}\sinh(n_1\xi) \ , \quad h = -\frac{1}{b_1}\ln\left\{N_3 \sinh\left(\frac{n_1\xi}{2}\right)\right\} \ , \\
\qquad b = 2b_1 \neq 0 \ , \qquad n_1^2 = -4b_1b_2 \ .
\end{array}\right\} \tag{3.7}$$

For each of these solutions of (3.1), (3.2) and (3.3), solutions of (3.4) have been obtained by McVittie and Wiltshire under various simplifying assumptions, e.g. taking solution (1), together with assumptions

$$a = 0 \ , \quad \alpha = \beta \quad \text{and} \quad \beta_{zz} - \beta_z^2 \neq 0 \ ,$$

(3.4) have been completely solved by them. However, in the present work, we shall see that if $\alpha = \beta$ is assumed, then solution (1) is the only solution of (3.1), (3.2) and (3.3) that gives solution of (3.4). (3.4) will then be solved completely for both $a = 0$ and $a \neq 0$. This is as follows. Using (3.5), (3.6) and (3.8), one can reduce (3.4) to

$$AU + BV + C + Dh'^2 = 0 \quad , \tag{3.9}$$

where

$$\left.\begin{aligned} A &= (a+b)\alpha_z(\alpha_{zz} - \alpha_z^2 - a\alpha_z)\,, \quad B = (\alpha_z + a)(\alpha_{zz} - \alpha_z^2 + b\alpha_z) \\ C &= (b_2\alpha_z - n_1^2 k)(\alpha_{zz} - \alpha_z^2 + b\alpha_z)\,, \quad D = -\frac{b}{2}(\alpha_{zz} - \alpha_z^2 + b\alpha_z)\alpha_z \end{aligned}\right\} \tag{3.10}$$

and

$$\left.\begin{aligned} &U \equiv \dot{g}^2\, e^{2(ag-\Psi)}\,, \quad V \equiv (\ddot{g} - \dot{g}\dot{\Psi})\, e^{2(ag-\Psi)} \quad , \\ &\text{where we note that} \\ &V = \frac{1}{2}\frac{dU}{dg} - aU \end{aligned}\right\} \tag{3.11}$$

and $k = 0$ for solutions (I) and (II), $k = 1$ for solution (III), $k = -1$ for solution (IV), where from (3.7) we have noted that $b = 2b_1$ holds for all four solutions in (3.7) and $\frac{f''}{f} = -n_1^2 k$. Also, it is to be noted in (3.11) that A, B, C are functions of $z$; U and V are functions of $\eta$.

We shall now prove that $D = 0$. If possible, let $D \neq 0$. Treating $\eta$ as a constant, differentiating (3.9) with respect to h,

$$A_z U + B_z V + C_z + D_z h'^2 + 2Dh'' = 0 \quad . \tag{3.12}$$

From (3.3), (3.9) and (3.12)

$$[A(D_z + bD) - A_z D]\, U + [B(D_z + bD) - B_z D] + [C(D_z + bD) - (C_z + 2b_2 D)D] = 0\,,$$

which can be rewritten as

$$E(D_z + bD) - E_z D = 0 \quad , \tag{3.13}$$

where



$$E \equiv AU + BV + C - \frac{2b_2}{b}\, D \quad , \tag{3.14}$$

$$E_z \equiv \left.\frac{\partial E}{\partial z}\right|_{\text{treating } \eta \text{ as a constant.}}$$

and in view of (3.10), $b \neq 0$ follows from $D \neq 0$. For $D \neq 0$, (3.13) can be integrated with respect to $z$, treating $\eta$ as a constant to give

$$E = \Phi D e^{b_1 z} \quad , \tag{3.15}$$

where $\Phi$ is some function of $\eta$.

Comparing (3.9) with (3.14) and (3.15), and using (1.6),

$$\Phi e^{bg} = e^{-bh}\left\{h'^2 - \frac{2b_2}{b}\right\} \quad . \tag{3.16}$$

Since the left-hand side of (3.16) is a function of $\eta$ only and the right-hand side is a function of $\xi$, they must both be equal to a constant. However, looking into solutions (1), (2), (3) and (4) of (3.7) we see that none can make the right-hand side of (3.16) a constant. Thus, we must have

$$D = 0 \quad . \tag{3.17}$$

Also from (3.5), (3.8) and inequality (I),

$$\left.\begin{aligned} &\alpha_z \neq 0 \\ &\alpha_{zz} - \alpha_z^2 + b\alpha_z \neq 0 \end{aligned}\right\} \tag{III}$$

From (3.10), (3.17) and inequality (III)

$$b = 0 \quad . \tag{3.18}$$

Therefore from (3.7), f and h can only be given by solution (1), i.e.

$$\left.\begin{aligned} f &= \xi\,, \\ h &= \frac{1}{2} b_2 \xi^2 + N_2\,, \end{aligned}\right\} \tag{3.19}$$

and in (3.10)

$$k = 0 \quad . \tag{3.20}$$

From (3.5), (3.19) and inequality (I), we must have $b_2 \neq 0$ and thus it is obvious that, without loss of generality, one can set

$$b_2 = 2 \quad , N_2 = 0 \quad , \tag{3.21}$$

i.e. (3.19) reduces to

$$f = \xi \quad , \quad h = \xi^2 \quad . \tag{3.22}$$

From (3.9) and (3.17),

$$AU + BV + C = 0 \quad , \tag{3.23}$$

where owing to (3.18), (3.20) and (3.21), A, B, C are now given by

$$\left.\begin{aligned} A &= a\alpha_z(\alpha_{zz} - \alpha_z^2 - a\alpha_z), \; B = (\alpha_z + a)\,(\alpha_{zz} - \alpha_z^2)\,, \\ C &= 2\alpha_z\,(\alpha_{zz} - \alpha_z^2)\,. \end{aligned}\right\} \tag{3.24}$$

Eq.(3.23) can be completely solved as follows. From (III), (3.18) and (3.24),

$$C \neq 0 \quad . \tag{IV}$$

We shall also prove that $B \neq 0$ . If possible, let $B = 0$ . Then from (3.10) and (III),

$$\alpha_z + a = 0 \ ,$$

therefore

$$\alpha_{zz} = 0 \quad .$$

These two equations, in view of (3.24), mean

$$A = 0 \ .$$

However, $A = 0$ , $B = 0$, $C \neq 0$ are incompatible with (3.23). Therefore

$$B \neq 0 \quad . \tag{V}$$

From (3.23) and (IV)

$$\frac{A}{B} U + V + \frac{C}{B} = 0 \quad . \tag{3.25}$$

Differentiating (3.25) successively with respect to z (treating $\eta$ as a constant) and with respect to $\eta$ (treating z as a constant),

$$\left(\frac{A}{B}\right)_z \dot{U} = 0 \ .$$



Therefore, either

$$\dot{U} = 0 \quad \text{or} \quad \left(\frac{A}{B}\right)_z = 0 \quad .$$

<u>Case i)</u>

$$\dot{U} = 0 \quad , \quad \text{i.e.} \quad U = \text{constant} = U_0 \ (\text{say}) \quad . \tag{3.26}$$

From (3.11) and (3.26)

$$V = \text{constant} = -\,U_0 \ .$$

(3.23) then reduces to

$$U_0\,(A-B) + C = 0 \quad , \tag{3.27}$$

where A,B,C are given by (3.24). (3.27) is an ordinary second-order differential equation which gives $\alpha$ as a function of z . (3.26) can explicitly be written as

$$\dot{g}^2\, e^{2(ag-\Psi)} = U_0 \quad . \tag{3.28}$$

However, from (1.1), (1.6), (1.7) and (1.9) it is obvious that by a suitable transformation of $\eta$ (where the new $\eta$ is a function of the old $\eta$ only), one can set

$$\Psi = ag \quad . \tag{3.29}$$

From (3.28) one can then set

$$g = \eta\sqrt{U_0} \quad (U_0 > 0) \quad , \tag{3.30}$$

f and h are given by (3.22), z = h + g as in (1.7).

<u>Case ii)</u>

$$\left(\frac{A}{B}\right)_z = 0, \quad \text{i.e.}\ \frac{A}{B} = \text{constant} \quad .$$

Since U and V are functions of $\eta$ , if in (3.25) $\frac{A}{B}$ is a constant, then $\frac{C}{B}$ , being a function of z , can only be a constant. However, from (3.24) we see that $\frac{C}{B}$ can only be a constant if either $a = 0$ or $\alpha_z$ = constant.

<u>Subcase iia)</u> $a = 0$ .

This is the case that has been worked out by McVittie and Wiltshire. It has been shown that here the metric is the following conformally flat one:

$$ds^2 = e^{2\alpha}\,[d\eta^2 - d\xi^2 - f^2 d\theta^2 - f^2 \sin^2\theta\, d\phi^2] \quad , \tag{3.31}$$

where, as before, $\alpha = \alpha(z)$ , $z$ is given by

$$z = \xi^2 - \eta^2 \quad . \tag{3.32}$$

Subcase iib)

$$\alpha_z = \text{constant} = n_2 \text{ (say)} \quad . \tag{3.33}$$

Here A, B, C are constants and in view of (3.11), (3.23) reduces to an ordinary differential equation in U, which is readily integrable to give U as a function of $\eta$ . $g$ can then be taken as arbitrary function of $\eta$ , to get $\Psi$ from (3.11). However, to simplify the final expressions, we can, as in case i), set

$$\Psi = ag \tag{3.34}$$

without loss of generality.

From (3.11), (3.23), (3.24), (3.33) and (3.34), $g$ is given by

$$\ddot{g} + \frac{2n_2}{a + n_2} = 0 \quad , \tag{3.35}$$

which has trivial solutions. $f$ and $h$ are given by (3.19), $\alpha$ is given as a function of $z$, by (3.33) and $z = h + g$ as in (1.7).

## IV. CONCLUSION

Summarizing, we see that for a metric of the form (1.1) where $\lambda$ and $\mu$ are given by (1.6) and (1.7), if $g = 0$ and (2.3) is satified, then all the solutions of the isotropy condition that are not given by McVittie and Wiltshire are given by a metric of form (2.14) where $\alpha = \alpha(\xi)$ is given by (2.13).

When $\lambda$ and $\mu$ are given by (1.6), (1.7), but $\psi = ag$, then solutions of (1.2) obtained by solving (3.1), (3.2), (3.3) and (3.4) for $\alpha = \beta$ are as follows.

Solutions reduce to three different cases in all of which the metric has spherically symmetric and conformally flat form

$$ds^2 = e^{2(\alpha+ag)} \left(d\eta^2 - d\xi^2 - \xi^2 d\theta^2 - \xi^2 \sin^2\theta \, d\phi^2\right) \quad . \tag{3.36}$$

In Case i) $\alpha$ is given as a function of $z$ by (3.24) and (3.27), $z = \xi^2 + \eta\sqrt{U_0}$ and $g$ is given by (3.30).



An interesting thing to note is that in all three cases, the $z$ = constant curve in $(\xi, \eta)$ plane is a conic section, a parabola in Case i), a hyperbola in Case iia) and a hyperbola or ellipse in Case iib). This suggests that although $z$ was originally introduced to simplify the problem mathematically, $z$ could have some significance that goes beyond it.

In fact, in Case iia), the case which was already solved by McVittie and Wiltshire, one can check from the expressions for velocity in (1.5) that $z$ can be interpreted as a velocity potential, i.e. velocity $v_\mu$ can be written as $v_\mu = \chi z_{,\mu}$, where $\chi$ is some function. Moreover, we can check that p and $\rho$ are functions of $z$ for Case iia) and hence $p$ and $\rho$ are functionally dependent, i.e. the fluid is barotropic. However, such a simple and interesting interpretation of $z$ could be obtained for Case i) and Case iib).

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