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CONSTRUCTING QUANTUM FIELDS IN A FOCK SPACE USING A NEW PICTURE OF QUANTUM MECHANICS *

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ABSTRACT

For any conventional non-relativistic quantum theory of a finite number of degrees of freedom we construct a picture which we call "the scattering picture", combining the "nice" properties of both the interaction and the Heisenberg pictures, and show that in the absence of bound states, the theory could be formulated in terms of a free Hamiltonian and an effective potential. We generalize the equations thus derived to the relativistic case and show that, given a Poincaré invariant self-adjoint operator D densely defined on a Fock space, there exists an interacting field which is asymptotically free and has as the scattering matrix the non-trivial operator $S = e^{iD}$, provided that D annihilates the vacuum and the one-particle states. Crossing relations could easily be imposed on D, but apart from a few comments, the problem of analyticity of S is left open.

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INTRODUCTION

In a conventional non-relativistic quantum theory (of a finite number of degrees of freedom) there exist three well known equivalent pictures: the Schrödinger picture, the Heisenberg picture and the interaction picture. However, trying to meet the requirements of relativistic invariance, the Schrödinger picture was found to be a "bad" picture [1], because it treats the time co-ordinate in a way essentially different from the way in which it treats the other spatial co-ordinates [2]. Also, the interaction picture was found not to exist [2]. So we were left with one picture, i.e. the Heisenberg picture. But all attempts to construct a non-trivial relativistic quantum theory in four-dimensional space-time in the Heisenberg picture were not successful [3].

The problem of constructing a non-trivial quantum field theory has two aspects: the kinematical aspect, which is the construction of the underlying Hilbert space with the unitary representation of the Poincaré group, and the dynamical aspect, which is the construction of the quantum field itself as a local covariant operator-valued distribution. However, the two aspects are closely related to each other, to the extent that solving one of them might be equivalent to solving the other.

We think that this situation exists because the axioms of quantum field theory are too restrictive [2]. Trying to see what a general quantum field theory, not necessarily satisfying the Wightman axioms [2], might look like, we had to study first the non-relativistic analogue of a quantum theory having a finite number of degrees of freedom. We have found that for such a theory one may construct a picture, which we call "the scattering picture", having the following properties.

i) To every element φ in a subspace of the Hilbert space, called the subspace of scattering states, which in the absence of bound states is equal to the whole space, there corresponds another element in the Hilbert space denoted by φ_{in}.

ii) To every Heisenberg operator $A(t)$ there corresponds a scattering picture operator denoted by $\hat{A}(t)$ satisfying $\hat{A}(t) = e^{itH_0/\hbar}\,\hat{A}(0)\,e^{-itH_0/\hbar}$, where H_0 is the free Hamiltonian.

iii) If $\varphi \to \varphi_{in}$, $\psi \to \psi_{in}$ and $A(t) \to \hat{A}(t)$ in the above correspondence, then

$$\langle \varphi , A(t)\psi \rangle = \langle \varphi_{in} , \hat{A}(t)\, \psi_{in} \rangle \quad .$$

iv) There exists a unitary operator S such that for any bounded operator $\hat{A}(t)$ in the scattering picture there correspond two operators $A_{in}(t)$ and $A_{out}(t)$ satisfying the conditions:

a) $A_{in}(t) = e^{itH_0/\hbar} A_{in}(0)\, e^{-itH_0/\hbar}$,

b) $A_{out}(t) = e^{itH_0/\hbar} A_{out}(0)\, e^{-itH_0/\hbar}$,

c) $A_{out}(t) = S^{-1} A_{in}(t) S$,

d) $w - \lim_{t \to -\infty} \{\hat{A}(t) - A_{in}(t)\} = 0$,

e) $w - \lim_{t \to +\infty} \{\hat{A}(t) - A_{out}(t)\} = 0$,

where "w-lim" means the weak limit.

v) In the absence of bound states one may introduce an "effective potential" $F(t)$ satisfying $F(t) = e^{itH_0/\hbar} F e^{-itH_0/\hbar}$ such that the integral $\int_{-\infty}^{\infty} F(t)\, dt = D$ exists, and such that the equation of motion of an observable in the scattering picture is given by $\hat{A}(t) = U^{-1}(t)\, A_0(t)\, U(t)$, where $U(t) = \exp\left\{ i \int_{-\infty}^{t} F(t')\, dt' \right\}$ and $A_0(t)$ is the equation of motion in the interaction picture, i.e. the "free" observable. The S matrix is given by $S = e^{iD}$.

One may easily note that the scattering picture has the "mathematics" of the Heisenberg picture of a free theory, yet it contains all the information about scattering. The states are time-independent as in the Heisenberg picture, and the observables transform under the free Hamiltonian as in the interaction picture. Hence, using the scattering picture instead of the Heisenberg picture, one can disentangle kinematics from dynamics, loosing nothing, except of course the direct information about bound states, if there are any.

The relativistic version of the equation of motion of an observable in the scattering picture in the absence of bound states can easily be guessed. The underlying space is the Fock space with its "free" representation of the Poincaré group. If $\{G(x);\, x \in \mathbb{R}^4\}$ is an integrable covariant self-adjoint operator-valued function over space-time, then the integral $\int_{y \in \Gamma_-(x)} G(y)\, d^4y$, where $\Gamma_-(x)$ is the past cone of the point $x \in \mathbb{R}^4$, exists and is a covariant self-adjoint operator-valued function of $x \in \mathbb{R}^4$, which converges strongly to zero as x goes to the infinite past along a timelike direction, and converges strongly to $D = \int_{y \in \mathbb{R}^4} G(y)\, d^4y$ as x goes to the infinite future along a timelike direction. The general equation of a field operator is: $\varphi(x) = U^{-1}(x)\, \varphi_0(x)\, U(x)$, where $\varphi_0(x)$ is the free field and $U(x) = \exp\left\{ i \int_{y \in \Gamma_-(x)} G(y)\, d^4y \right\}$. This is the equation of an asymptotically free covariant field, where $\varphi_{in} = \varphi_0(x)$ and $\varphi_{out} = S^{-1}\, \varphi_0(x)\, S$, and $S = e^{iD}$.

One could safely say that in fact this is a theory of the phase matrix: For given a Poincaré invariant self-adjoint operator D which annihilates the vacuum and the one-particle states, one could easily construct the scattering matrix $S = e^{iD}$ and an effective potential density $\{G(x)\,;\, x \in \mathbb{R}^4\}$ such that $D = \int G(x)\, d^4x$, from which the interacting field could be derived. The field theory thus constructed satisfies all the Wightman axioms [2] except microcausality. If we try to satisfy this requirement too, then, as a consequence of Haag's theorem [2] we must take $D = 0$. On the other hand, the S matrix thus constructed satisfies all the required axioms [4] except crossing and analyticity, which are consequences of microcausality [5),6)].

One could think that such a theory cannot be physical because it violates causality. Yet, it is believed that the S matrix could have macrocausal aspects without the fields being microcausal [7]. The various consequences of microcausality are discussed below.

Spin and statistics

The correct relation between spin and statistics could be derived by assuming that the asymptotic fields are microcausal [8),9)]. According to an argument due to Doebner [10], it makes sense to talk about the commutativity of local asymptotic observables for a spacelike separation, because we only measure these observables. However, the spin of a particle during the interaction might not be a meaningful concept.

Crossing and PCT

It is not difficult at all to choose among the possible phase matrices those operators which satisfy the requirements of crossing and PCT, and regard the rest as being non-physical.

Analyticity

This is the hardest part of the construction. It is not easy to tell what choice of the phase matrix would give an analytic S matrix. To a person who is interested in the theory itself, this would be the right point to tackle, while to a person who is interested in numerical results, microcausality is a tool by which the number of unknowns in the theory is reduced to a few numerical parameters, which can be fixed by comparison with experiments. It is not difficult at all to suggest for the D matrix schemes with this property, but this still leaves open the question of the analyticity of the S matrix.

PART I

THE SCATTERING PICTURE OF QUANTUM MECHANICS

1. The theory of scattering

This section is a revision of a work by Jauch [11], presented in a slightly different notation and included for the sake of completeness.

Axiom I (quantum assumption)

$\mathcal{H}$ is a complete Hilbert space and H_0 (the free Hamiltonian) and H (the total Hamiltonian) are self-adjoint operators, densely defined on $\mathcal{H}$, each having a dense set of analytic vectors [12].

It follows directly from axiom I, that because both H_0 and H have dense sets of analytic vectors, the series $\exp(itH_0/\hbar) = \sum_{n=0}^{\infty} \frac{1}{n!}(itH_0/\hbar)^n$ and the series $\exp(itH/\hbar) = \sum_{n=0}^{\infty} \frac{1}{n!}(itH/\hbar)^n$ converge strongly on their respective sets of analytic vectors for every $t \in \mathbb{R}$. Since $e^{itH_0/\hbar}$ and $e^{itH/\hbar}$ are bounded operators, they can be extended by continuity to all elements of $\mathcal{H}$.

Defining $V(t) = e^{itH_0/\hbar} e^{-itH/\hbar}$, we can write the equation satisfied by the wave function of any system in the interaction picture in the form:

$$\psi_I(t) = V(t)\,\psi \quad ; \quad \psi_I(0) = \psi \in \mathcal{H}. \qquad (1)$$

Denote by $\mathcal{H}_+$ and $\mathcal{H}_-$ the sets of vectors $\varphi \in \mathcal{H}$ for which the strong limits $s-\lim_{t\to+\infty} V(t)\varphi$ and $s-\lim_{t\to-\infty} V(t)\varphi$ exist, respectively. $\mathcal{H}_+$ and $\mathcal{H}_-$ are closed subspaces of $\mathcal{H}$. If $\varphi_{in} = s-\lim_{t\to-\infty} V(t)\varphi$, $\varphi \in \mathcal{H}_-$ and $\varphi_{out} = s-\lim_{t\to+\infty} V(t)\varphi$, $\varphi \in \mathcal{H}_+$, then the maps $W_-\varphi = \varphi_{in}$ and $W_+\varphi = \varphi_{out}$, called the wave operators, are unitary transformations from their domains $\mathcal{H}_-$ and $\mathcal{H}_+$ into their ranges $W_-(\mathcal{H}_-) = \mathcal{H}^{in}$ and $W_+(\mathcal{H}_+) = \mathcal{H}^{out}$, respectively.

If I_A denotes the identity map on an arbitrary set A, then the unitarity of W_- and W_+ can be expressed in the form:

$$W_-^{\dagger} W_- = I_{\mathcal{H}_-} \; ; \; W_+^{\dagger} W_+ = I_{\mathcal{H}_+} \; ; \; W_- W_-^{\dagger} = I_{\mathcal{H}^{in}} \; ; \; W_+ W_+^{\dagger} = I_{\mathcal{H}^{out}} . \qquad (2)$$

In order to get more information, we must postulate something about the domains and ranges of the wave operator. On a physical basis, one can say that any free state may undergo some scattering process, and the outcomes of all scattering processes cover the full range of free states. Hence, we introduce the following axiom about the ranges of the wave operators.

Axiom II (asymptotic completeness)

$$\mathcal{H}^{in} = \mathcal{H}^{out} = \mathcal{H}.$$

Also, if something goes into the scattering region, something should come out of it, and nothing comes out of the scattering region unless something goes into it. Hence, we introduce the following axiom about the domains of the wave operators.

Axiom III (unitarity)

$$\mathcal{H}_+ = \mathcal{H}_- = \mathcal{H}^{sc} .$$

To the free Hamiltonian H_0, all elements of $\mathcal{H}$ represent free states, while to the total Hamiltonian H, the space $\mathcal{H}$ is decomposed into two orthogonal subspaces: the space $\mathcal{H}^{sc}$ of scattering states and the space $\mathcal{H}^{b}$ of bound states. These two spaces are invariant under H. On the other hand, H decompoes $\mathcal{H}$ into two orthogonal subspaces: the space $\mathcal{H}^{c}$ of continuous spectrum and the space $\mathcal{H}^{d}$ of discrete spectrum. Usually, axiom III is strengthened by requiring $\mathcal{H}^{sc} = \mathcal{H}^{c}$.

With the aid of axioms II and III, Eq.(2) takes the simple form:

$$W_\pm^\dagger W_\pm = I_{\mathcal{H}^{sc}} \; ; \quad W_\pm W_\pm^\dagger = I \quad . \tag{3}$$

It follows directly from definitions of the wave operator that

$$e^{itH_0/\hbar} W_\pm = W_\pm e^{itH/\hbar} \quad ; \; t \in \mathbb{R} \tag{4}$$

or, equivalently, if $\mathcal{D}_0$ and $\mathcal{D}$ are the domains of H_0 and H, respectively, then

$$H = W_\pm^\dagger H_0 W_\pm \quad \text{on} \quad \mathcal{D} \cap \mathcal{H}^{sc} = W_\pm^\dagger(\mathcal{D}_0) \, . \tag{5}$$

Recalling the definitions of φ_{in} and φ_{out} for any $\varphi \in \mathcal{H}^{sc}$ and defining the S matrix by the equation $S\varphi_{in} = \varphi_{out}$, it follows that $S = W_+ W_-^\dagger$. Also, if we write $U(t) = e^{itH_0/\hbar} W_-^\dagger e^{-itH_0/\hbar}$ defined on $\mathcal{H}$ and ranging in $\mathcal{H}(t) \subseteq \mathcal{H}$, then, for any $t \in \mathbb{R}$, $U(t)$ is a unitary transformation of $\mathcal{H}$ into $\mathcal{H}(t)$ with the property

$$s - \lim_{t \to -\infty} U(t) = I \quad , \quad s - \lim_{t \to +\infty} U(t) = S \quad . \tag{6}$$

2. The scattering picture

We derive in this section the scattering picture for a general quantum system satisfying the three axioms of the previous section. We note first that because $U(0) = W_-^\dagger$, we have $\mathcal{H}(0) = \mathcal{H}^{sc}$, and if E^{sc} is the projection operator on $\mathcal{H}^{sc}$, then $E(t) = e^{itH_0/\hbar} E^{sc} e^{-itH_0/\hbar}$ is the projection operator on $\mathcal{H}(t)$.

Let $\varphi \in \mathcal{H}$ and A be an operator (densely) defined on $\mathcal{H}$, and denote by the subscripts S, H and I, the Schrödinger, Heisenberg and interaction pictures, respectively. Now, for any $t \in \mathbb{R}$ we have the well-known equation of motion

$$\begin{aligned} A_S(t) &= A \qquad &;\quad \varphi_S(t) &= e^{-itH/\hbar}\varphi \\ A_H(t) &= e^{itH/\hbar} A \, e^{-itH/\hbar} &;\quad \varphi_H(t) &= \varphi \\ A_I(t) &= e^{itH_0/\hbar} A \, e^{-itH_0/\hbar} &;\quad \varphi_I(t) &= V(t)\varphi \end{aligned} \tag{7}$$

provided we fix $A_S(0) = A_H(0) = A_I(0) = A$ and $\varphi_S(0) = \varphi_H(0) = \varphi_I(0) = \varphi$. The equivalence of the three pictures is established by the relation

$$\langle \varphi_S(t), A_S(t)\psi_S(t) \rangle = \langle \varphi_H(t), A_H(t)\psi_H(t) \rangle = \langle \varphi_I(t) A_I(t) \psi_I(t) \rangle , \tag{8}$$

where $\langle \; , \; \rangle$ denotes the scalar product in $\mathcal{H}$.

We shall write, for simplicity, $A(t)$ instead of $A_H(t)$, $A_0(t)$ instead of $A_I(t)$ and substitute for $\varphi_H(t)$ its value φ.

Let $\varphi, \psi \in \mathcal{H}^{sc}$, then $W_-\varphi = \varphi_{in}$, $W_-\psi = \psi_{in}$, or $\varphi = W_-^\dagger \varphi_{in}$, $\psi = W_-^\dagger \psi_{in}$ and

$$\begin{aligned} \langle \varphi, A(t)\psi \rangle &= \langle W_-^\dagger \varphi_{in}, e^{itH/\hbar} A \, e^{-itH/\hbar} W_-^\dagger \psi_{in} \rangle \\ &= \langle e^{-itH/\hbar} W_-^\dagger \varphi_{in}, A \, e^{-itH/\hbar} W_-^\dagger \psi_{in} \rangle \quad \text{(and using (4) we get)} \\ &= \langle W_-^\dagger e^{-itH_0/\hbar} \varphi_{in}, A W_-^\dagger e^{-itH/\hbar} \psi_{in} \rangle \\ &= \langle W_-^\dagger e^{-itH_0/\hbar} \varphi_{in}, E^{sc} A W_-^\dagger e^{-itH/\hbar} \psi_{in} \rangle \\ &\quad + \langle W_-^\dagger e^{-itH_0/\hbar} \varphi_{in}, (I - E^{sc}) A W_-^\dagger e^{-itH/\hbar} \psi_{in} \rangle \, . \end{aligned} \tag{9}$$

But $e^{-itH_0/\hbar}\varphi_{in} \in \mathcal{H}$ and $E^{sc} A W_-^\dagger e^{-itH_0/\hbar}\psi_{in} \in \mathcal{H}^{sc}$, hence, by unitarity of $W_-^\dagger$ (between $\mathcal{H}$ and $\mathcal{H}^{sc}$) we get

$$\begin{aligned} &\langle W_-^\dagger e^{-itH_0/\hbar} \varphi_{in}, E^{sc} A W_-^\dagger e^{-itH_0/\hbar} \psi_{in} \rangle \\ &\quad = \langle e^{-itH_0/\hbar} \varphi_{in}, W_- E^{sc} A W_-^\dagger e^{-itH_0/\hbar} \psi_{in} \rangle \\ &\quad = \langle \varphi_{in}, e^{itH_0/\hbar} U^\dagger_{(0)} E(0) A_0(0) U(0) e^{-itH_0/\hbar} \psi_{in} \rangle \\ &\quad = \langle \varphi_{in}, U^\dagger(t) E(t) A_0(t) U(t) \psi_{in} \rangle \, . \end{aligned} \tag{10}$$

Also, $W_-^\dagger e^{-itH_0/\hbar} \varphi_{in} \in \mathcal{H}^{sc}$ and $(I - E^{sc}) A W_-^\dagger e^{-itH_0/\hbar} \psi_{in} \in \mathcal{H}^b = (\mathcal{H}^{sc})^\perp$, hence

$$\langle W_-^\dagger e^{-itH_0/\hbar} \varphi_{in}, (I - E^{sc}) A W_-^\dagger e^{-itH_0/\hbar} \psi_{in} \rangle = 0 \, . \tag{11}$$

Hence, from (9) to (11) we get

$$\langle \varphi, A(t)\psi \rangle = \langle \varphi_{in}, U^\dagger(t) E(t) A_0(t) U(t) \psi_{in} \rangle \quad . \tag{12}$$

Using (12), we define the scattering picture operator

$$\hat{A}(t) = U^\dagger(t)\ E(t)\ A_0(t)\ U(t) \tag{13}$$

satisfying

$$\hat{A}(t) = e^{itH_0/\hbar}\ \hat{A}(0)\ e^{-itH_0/\hbar} \tag{14}$$

and the scattering picture state $\varphi_{in} = W_-\varphi$, whenever $\varphi \in \mathcal{H}^{sc}$. Hence, we have shown that to every Heisenberg operator A(t), there corresponds a scattering picture operator $\hat{A}(t)$ such that for any $\varphi, \psi \in \mathcal{H}^{sc}$ we have

$$\langle \varphi, A(t)\ \psi \rangle = \langle \varphi_{in}, \hat{A}(t)\ \psi_{in} \rangle . \tag{15}$$

Going back to Eq.(13) we have

$$\begin{aligned} \hat{A}(t) &= e^{itH_0/\hbar} W_- E^{sc} A W_-^\dagger e^{-itH_0/\hbar} \\ &= W_- e^{itH/\hbar} E^{sc} A e^{-itH/\hbar} W_-^\dagger \\ &= W_- E^{sc} e^{itH/\hbar} A e^{-itH/\hbar} E^{sc} W_-^\dagger \\ &= W_- E^{sc} A(t) E^{sc} W_-^\dagger . \end{aligned} \tag{16}$$

Hence, $W_- : \mathcal{H}^{sc} \to \mathcal{H}$ defines a unitary equivalence between $E^{sc}\ A(t)\ E^{sc}$, the restriction of A(t) as a quadratic form to $\mathcal{H}^{sc} \times \mathcal{H}^{sc}$, and $\hat{A}(t)$. The inverse of (16) is given by

$$E^{sc}\ A(t)\ E^{sc} = W_-^\dagger\ \hat{A}(t)\ W_- \quad . \tag{17}$$

Although Eq.(16) defines the scattering picture operator $\hat{A}(t)$ in terms of the Heisenberg operator A(t), Eq.(17) tells us that A(t) cannot be recovered from $\hat{A}(t)$ except on $\mathcal{H}^{sc} \times \mathcal{H}^{sc}$.

3. The S matrix

The S matrix in the interaction picture is defined as the unitary transformation which relates the asymptotic behaviour of a scattering state at $t = +\infty$ to its asymptotic behaviour at $t = -\infty$. This gives the expression $S = W_+ W_-^\dagger$.

In the scattering picture, the states are time-independent. So we should be able to derive the S matrix from the observables.

We shall prove that $w - \lim_{t \to -\infty} \{\hat{A}(t) - A_0(t)\} = 0$ and $w - \lim_{t \to +\infty} \{\hat{A}(t) - S^{-1} A_0(t) S\} = 0$, whenever A is a bounded operator.

To prove the first limit, we note that

$$\begin{aligned} \hat{A}(t) - A_0(t) &= U^\dagger(t) E(t) A_0(t) U(t) - A_0(t) \\ &= U^\dagger(t) E(t) A_0(t) \{U(t) - I\} + \{U^\dagger(t) E(t) - I\} A_0(t) . \end{aligned} \tag{18}$$

Let $f, g \in \mathcal{H}$, then

$$\begin{aligned} \langle f, \{\hat{A}(t) - A_0(t)\} g \rangle &= \langle f, U^\dagger(t) E(t) A_0(t) \{U(t) - I\} g \rangle \\ &\quad + \langle f, \{U^\dagger(t) E(t) - I\} A_0(t) g \rangle \\ &= \langle A_0^\dagger(t) U(t) f, \{U(t) - I\} g \rangle \\ &\quad + \langle \{U(t) - I\} f, A_0(t) g \rangle . \end{aligned} \tag{19}$$

Hence

$$\begin{aligned} |\langle f, \{\hat{A}(t) - A_0(t)\} g \rangle| &\leq |\langle A_0^\dagger(t) U(t) f, \{U(t) - I\} g \rangle| \\ &\quad + |\langle \{U(t) - I\} f, A_0(t) g \rangle| \\ &\leq \| A_0^\dagger(t) U(t) f \| \, \| \{U(t) - I\} g \| + \| \{U(t) - I\} f \| \, \| A_0(t) g \| \\ &\leq \| A^\dagger \| \, \| f \| \, \| \{U(t) - I\} g \| + \| \{U(t) - I\} f \| \, \| A \| \, \| g \| . \end{aligned} \tag{20}$$

But because $s - \lim_{t \to -\infty} U(t) = I$, it follows from (20) that $w - \lim_{t \to -\infty} \{\hat{A}(t) - A_0(t)\} = 0$. The proof of the other limit follows the same steps. Hence, the S matrix derivable in this picture is defined on all $\mathcal{H}$ and is identical with the S matrix derived in the interaction picture. However, it is known [13] that if A is a bounded operator on $\mathcal{H}$, then the Heisenberg operator A(t) converges weakly on $\mathcal{H}^{sc} \times \mathcal{H}^{sc}$ to an in-operator as $t \to -\infty$ and to an out-operator as $t \to +\infty$, related to each other by a different S matrix given by $\tilde{S} = W_-^\dagger W_+$. If $\varphi, \psi \in \mathcal{H}^{sc}$, then

$$\langle \varphi, \hat{S}\psi \rangle = \langle \varphi, W_-^\dagger W_+ \psi \rangle = \langle W_-^\dagger \varphi_{in}, W_-^\dagger W_+ W_-^\dagger \psi_{in} \rangle$$

$$= \langle W_-^\dagger \varphi_{in}, W_-^\dagger S \psi_{in} \rangle = \langle \varphi_{in}, S\, \psi_{in} \rangle \, . \tag{21}$$

which is the correct formula for transformation from the Heisenberg picture on $\mathcal{H}^{sc}$ to the scattering picture on $\mathcal{H}$.

4. Pure scattering and effective potentials

We study in this section the special case of pure scattering, which means the absence of bound states. Mathematically, this means $\mathcal{H}^{sc} = \mathcal{H}$ or equivalently $E^{sc} = I$. Hence, the wave operators $W_\pm$ become unitary operators on $\mathcal{H}$, and $U(t)$ becomes, for any $t \in \mathbb{R}$, a unitary transformation of $\mathcal{H}$ into itself. In this case, the scattering picture becomes equivalent to any other picture, and the time dependence of an operator in this picture takes the form:

$$\hat{A}(t) = U^{-1}(t)\, A_0(t)\, U(t) \quad . \tag{22}$$

To derive the other pictures from the scattering picture, we introduce the following table:

Picture	States	Observables
Scattering	φ	$A(t)$
Interaction	$\varphi_I(t) = U(t)\varphi$	$A_I(t) = U(t)\ A(t)\ U^{-1}(t)$
Heisenberg	$\varphi_H(t) = U(0)\varphi$	$A_H(t) = U(0)\ A(t)\ U^{-1}(0)$
Schrödinger	$\varphi_S(t) = U(0)\ e^{-itH_0/\hbar}\varphi$	$A_S(t) = U(0)\ A(0)\ U^{-1}(0)$.

We now proceed to analyse the family $\{U(t) : t \in \mathbb{R}\}$. Since $W_-^\dagger$ is a unitary operator, it follows that there exists a unique resolution of the identity on the interval $[0, 2\pi]$ such that [14]

$$W_-^\dagger = \int_0^{2\pi} e^{i\lambda}\, dE_\lambda \quad . \tag{23}$$

Using this we define the bounded self-adjoint operator

$$A = \int_0^{2\pi} \lambda\ \ dE_\lambda \quad . \tag{24}$$

Setting $A(t) = e^{itH_0/\hbar} A\, e^{-itH_0/\hbar}$, it follows that $e^{iA(t)} = e^{itH_0/\hbar} e^{iA} e^{-itH_0/\hbar} = e^{itH_0/\hbar} W_-^\dagger\, e^{-itH_0/\hbar} = U(t)$. We also set $F(t) = \frac{i}{\hbar}[H_0, A(t)]$, from which it follows

$$A(t) - A(t_0) = \int_{t_0}^{t} F(t')\ dt' \ . \tag{25}$$

We show in the appendix to Part I that under mild mathematical restrictions, the strong limit of $A(t)$ as $t \to -\infty$ is zero. Hence $A(t) = \int_{-\infty}^{t} F(t')\ dt'$ and thus

$$U(t) = \exp\left\{ i \int_{-\infty}^{t} F(t')\ dt' \right\} \ . \tag{26}$$

It is because of (26) that we call $F(t)$ the effective potential. The left-hand side of (26) converges strongly to S as $t \to +\infty$. Hence, one would expect that the integral $D = \int_{-\infty}^{\infty} F(t)\ dt$ should exist. We call this quantity the phase matrix. However, the prescription $S \to D$ such that $e^{iD} = S$ is by no means unique. For, if N is a non-bounded operator densely defined on $\mathcal{H}$ with a dense set of analytic vectors [12], then the operator $U = e^{iN}$ exists and is unitary. Hence, there exists a unique bounded self-adjoint operator B with spectrum in $[0, 2\pi]$ such that $U = e^{iB}$. Clearly, $N \neq B$. Because of these considerations, we are led to think that the effective potential could be introduced as a fundamental object, satisfying the condition of integrability over the whole real line.

PART II

QUANTUM FIELDS IN A FOCK SPACE

1. The Fock space S matrix

If we consider all stable systems and all systems with lifetimes much longer than the duration of the interaction under consideration as elementary particles, while considering other systems as resonances, then the free states could be described using one single Fock space. Apart from any field-theoretic considerations, the overall transitions could be described using one single unitary operator (called the S matrix) invariant under the Poincaré group. The vacuum state and all one-particle states are eigenstates of the S matrix with eigenvalues on the unit circle $\{z : z \in \mathbb{C} \text{ and } |z| = 1\}$. These multiplicative unimodular Poincaré invariant constants have no physical significance. Hence, it should be assumed that the vacuum state and all one-particle states are eigenstates of the S matrix with unit eigenvalue . This assumption is necessary for the subsequent discussion.

Since the physics of scattering, whether elastic or not, could be described using a single Fock space, we choose the relativistic Fock space as our framework and introduce the following axiom.

Axiom I (relativistic Fock space)

$\mathcal{F}$ is a Fock space and U is a unitary representation of the Poincaré group $E(1,3)$ on $\mathcal{F}$.

Notation

We use natural units in which $\hbar = c = 1$. The four-vector $x = (x_0, \underline{x})$ where $\underline{x} = (x_1, x_2, x_3)$. The inner product $xy = x_0 y_0 - \underline{x}\cdot\underline{y}$ where $\underline{x}\cdot\underline{y} = x_1y_1 + x_2y_2 + x_3y_3$. In particular, $x^2 = xx = x_0^2 - |\underline{x}|^2$. If p is the four-momentum of a particle, then $p^T = (p_0, -\underline{p})$. We often write E_p for p_0 . If f and g are functions of x (and other variables) then

$$f(x,-)\overleftrightarrow{\frac{\partial}{\partial x}}\, g(x,-) = f(x,-)\frac{\partial g}{\partial x}(x,-) - \frac{\partial f}{\partial x}(x,-)\, g(x,-) .$$

Definition 1

$V_m^+ = \{p : p \in \mathbb{R}^4,\ p_0 > 0 \text{ and } p^2 = m^2\}$ is the mass shell of a particle of mass $m \geq 0$.

The Fock space $\mathcal{F}$ is not very suitable for the following discussion. We need a space with nicer properties. We choose a subspace $\mathcal{S}(\mathcal{F}) \subseteq \mathcal{F}$, called the subspace of good vectors, as follows.

On the space $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^{3n})$ of good test functions [2)] of 3n-real variables, we define a sequence of norms as follows: for any $s \in \mathbb{N}$, $f \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^{3n})$

$$|f|_s = \sup_{\substack{x \in \mathbb{R}^{3n} \\ |\alpha|, |\beta| \leq s}} \left| (1+|\underline{x}_1|^2)^{\frac{\alpha_1}{2}} \cdots (1+|\underline{x}_n|^2)^{\frac{\alpha_n}{2}} \frac{\partial^{\beta_1^1}}{\partial x_1^1}\frac{\partial^{\beta_1^2}}{\partial x_1^2}\frac{\partial^{\beta_1^3}}{\partial x_1^3} \cdots \frac{\partial^{\beta_n^1}}{\partial x_n^1}\frac{\partial^{\beta_n^2}}{\partial x_n^2}\frac{\partial^{\beta_n^3}}{\partial x_n^3} f(x) \right| ,$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $|\beta| = \beta_1^1 + \beta_1^2 + \beta_1^3 + \cdots + \beta_n^1 + \beta_n^2 + \beta_n^3$.

Consider for simplicity the case of a single scalar neutral particle and define $\mathcal{F}^{(n)}$ as the subspace of n particles. Denote by E_n the projection operator on $\mathcal{F}^{(n)}$. $\mathcal{S}(\mathcal{F}^{(0)}) = \mathcal{F}^{(0)}$ and $\mathcal{S}(\mathcal{F}^{(m)})$ for $n \in \mathbb{N}^*$ is the subspace of $\mathcal{F}^{(n)}$ consisting of vectors with good wave functions in momentum space. Now, if $\psi \in \mathcal{F}^{(n)}$ with $\psi = \int_p^{(n)} \hat{\psi}(p_1, p_2, \ldots, p_n)\,|p_1, p_2, \ldots, p_n\rangle$ where $\int_p^{(n)}$ denotes $\int \frac{d^3p_1}{2E_{p_1}} \int \frac{d^3p_2}{2E_{p_2}} \cdots \int \frac{d^3p_n}{2E_{p_n}}$, then $\|\psi\|_s = |\tilde{\psi}|_s$.

Definition 2

$\mathcal{S}(\mathcal{F})$ is the set of vectors $f \in \mathcal{F}$ such that for any $r, s \in \mathbb{N}$, $\|f\|_{r,s} = \sum_{n=0}^{\infty} n!\, n^r \|E_n f\|_s^2 < +\infty$. A sequence $\{f_n : n \in \mathbb{N}\}$ of elements of $\mathcal{S}(\mathcal{F})$ converges to zero in the $\mathcal{S}$ topology iff $\lim_{n \in \mathbb{N}} \|f_n\|_{r,s} = 0$ for all $r, s \in \mathbb{N}$.

As we have mentioned in the introduction, we are interested in the case where $S = e^{iD}$ (which is always possible [14)]) with D satisfying certain conditions. We now prove a theorem about a special class of self-adjoint operators on $\mathcal{F}$.

Theorem 1

Let D i) be a self-adjoint operator densely defined on $\mathcal{F}$ and having $\mathcal{S}(\mathcal{F})$ in its domain,

ii) be commuting with $U(\Lambda, a)$ for every $(\Lambda, a) \in E(1,3)$, and

iii) have the space $\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ as a null space,

then there exists a densely defined operator-valued function $\{G(x) : x \in \mathbb{R}^4\}$ such that

a) $G^\dagger(x) = G(x)$ for every $x \in \mathbb{R}^4$,

b) $U(\Lambda, a)\, G(x)\, U^{-1}(\Lambda, a) = G(\Lambda x + a)$, for every $(\Lambda, a) \in E(1,3)$ and

c) $\int G(x)\, d^4x = D$ strongly on $\mathcal{S}(\mathcal{F})$.

Proof

We shall prove the theorem for a theory of a single neutral scalar particle of mass $m > 0$. The generalization to other cases should not be difficult.

Let a be the annihilation operator of $\mathcal{F}$. Since $\mathcal{S}(\mathcal{F})$ belong to the domain of D, there exists a double sequence of distributions $G^{(j,k)} \in \mathcal{S}'_{\mathbb{C}}\{(V_m^{\dagger})^{j+k}\}$; $(j,k) \in \mathbb{N} \times \mathbb{N}$ such that on $\mathcal{S}(\mathcal{F})$ we have

$$D = \sum_{j,k=0}^{\infty} \int_{p'}^{(j)} \int_{p}^{(k)} G^{(j,k)}(p_1', p_2', \ldots, p_j' ; p_1, p_2, \ldots, p_k)\, a^{\dagger}(p_1')\, a^{\dagger}(p_2') \ldots a^{\dagger}(p_j')\, a(p_1)\, a(p_2) \ldots a(p_k), \tag{27}$$

where $\int_p^{(n)} = \int \frac{d^3p_1}{2E_{p_1}} \cdots \int \frac{d^3p_n}{2E_{p_n}}$. Because of Bose statistics, the $G^{(j,k)}$ distribution can be taken to be symmetric in the first j and last k $V_m^{\dagger}$ variables, without any loss of generality. From self-adjointness of D we have

$$G^{*(j,k)}(p_1', p_2', \ldots, p_j' ; p_1, p_2, \ldots, p_k) = G^{(k,j)}(p_1, p_2, \ldots, p_k ; p_1', p_2', \ldots, p_j'). \tag{28}$$

From translational invariance we get

$$G^{(j,k)}(p_1', \ldots p_j' ; p_1, \ldots, p_k) = \delta^{(4)}(p_1' + \cdots + p_j' - p_1 - \cdots - p_k)\, F^{(j,k)}(p_1', \ldots p_j' ; p_1, \ldots, p_k) \tag{29}$$

and Lorentz invariance implies

$$F^{(j,k)}(\Lambda p_1', \ldots, \Lambda p_j' ; \Lambda p_1, \ldots, \Lambda p_k) = F^{(j,k)}(p_1', \ldots, p_j' ; p_1, \ldots, p_k) \quad \text{for any } \Lambda \in SO(1,3). \tag{30}$$

But because $\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ is a null space of D, it follows that

$$F^{(j,0)} = F^{(j,1)} = F^{(0,j)} = F^{(1,j)} = 0 \quad \text{for any } j \in \mathbb{N} . \tag{31}$$

Hence from (27), (29) and (31) we get

$$D = \sum_{j,k=2}^{\infty} \int_{p'}^{(j)} \int_{p}^{(k)} F^{(j,k)}(p_1', \ldots p_j' ; p_1, \ldots p_k)\, a^{\dagger}(p_1') \cdots a^{\dagger}(p_j')\, a(p_1) \cdots a(p_k) , \tag{32}$$

where $\int_{p'}^{(j)} \int_{p}^{(k)} \cdots = \int_{p'}^{(j)} \int_{p}^{(k)} \delta^{(n)}(p_1' + \cdots + p_j' - p_1 - \cdots - p_k)$. The functions $F^{(j,k)}$ are well behaved square integrable functions (because D is an operator and not only a quadratic form).

Now, define the operator-valued function

$$G(x) = \sum_{j,k=2}^{\infty} \int_{p'}^{(j)} \int_{p}^{(k)} F^{(j,k)}(p_1', \ldots p_j' ; p_1, \ldots, p_k)\, e^{i(p_1' + \cdots + p_j' - p_1 - \cdots - p_k)x}\, a^{\dagger}(p_1') \cdots a^{\dagger}(p_j')\, a(p_1) \cdots a(p_k) . \tag{33}$$

This function meets the requirements of the theorem. ■

We have two comments on the above theorem. The first is that $G(x)$ is not unique, for if we add to $F^{(j,k)}$ in definition (33) a term of the form

$$(p_1' + \cdots + p_j' - p_1 - \cdots - p_k)^2\, F'^{(j,k)}(p_1', \ldots, p_j' ; p_1, \ldots, p_k) \quad ,$$

where $F'^{(j,k)}$ satisfies conditions (28) and (30), then the new function $G'(x)$ also meets the requirements of the theorem. The second comment is about requiring $\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ to be a null space of D. In general, by the hermiticity and Poincaré invariance of D, D could have in addition to expression (32) a term of the form $\alpha + \beta N$ where $N = \int_p a^{\dagger}(p)\, a(p)$; $\alpha, \beta \in \mathbb{R}$. Such a term cannot be obtained by the integration of any Poincaré invariant function, unless $\alpha = \beta = 0$, which means that $\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ should be a null space of D.

2. The interacting fields

In our model of field theory we do not have bound states in the non-relativistic sense, because, as we have explained before, we consider such states either as elementary, and hence include them in the free description, or as resonances, and hence they correspond to peaks in the scattering amplitudes. Hence, the Heisenberg picture and the scattering picture are, in our case, equivalent. However, we prefer the scattering picture for two reasons. First, the equation of motion of any observable is known in general, and could be directly written in terms of an "effective potential density". Second, the S-matrix derivable from the scattering picture has a direct experimental interpretation because it is identical with the S matrix of the interaction picture.

The fundamental object in our construction is the "effective potential density".

Definition 3

For any $x \in \mathbb{R}^4$, $\Gamma_-(x) = \{y : y \in \mathbb{R}^4,\ y_0 < x_0,\ (y-x)^2 > 0\}$ is the past cone of the point $x \in \mathbb{R}^4$.

We state without proof the following theorem.

Theorem 2

For any $n \in \dot{V}_1^\dagger$ and $x \in \mathbb{R}^4$, the function $\begin{cases} \mathbb{R} \to \mathcal{P}(\mathbb{R}^4) \\ \tau \to \Gamma_-(x+\tau n) \end{cases}$,

where $\mathcal{P}(\mathbb{R}^4)$ is the power set of $\mathbb{R}^4$, is increasing. Moreover,

$$\lim_{\tau \to -\infty} \Gamma_-(x+\tau n) = \emptyset \quad \text{and} \quad \lim_{\tau \to +\infty} \Gamma_-(x+\tau n) = \mathbb{R}^4 .$$

We now introduce the following axiom.

Axiom II

$\{G(x) : x \in \mathbb{R}^4\}$ is a densely defined operator-valued function called the effective potential density. It satisfies the following conditions:

i) $G^\dagger(x) = G(x)$ for any $x \in \mathbb{R}^4$.

ii) $U(\Lambda,a)\, G(x)\, U^{-1}(\Lambda,a) = G(\Lambda x + a)$, for any $(\Lambda,a) \in E(1,3)$.

iii) $D = \int G(x)\, d^4x$, called the phase matrix, exists strongly on $\mathcal{F}$.

iv) For any $f \in \mathcal{S}(\mathcal{F})$, $x \in \mathbb{R}^4$ and $n \in V_1^\dagger$ we have

$$\mathcal{S}-\lim_{\tau \to -\infty} U(x+\tau n)\, f = f \quad \text{and} \quad \mathcal{S}-\lim_{\tau \to +\infty} U(x+\tau n)\, f = e^{iD} f ,$$

where $\mathcal{S}$-lim denotes limit in the $\mathcal{S}$ topology and $U = \exp\{i \int_{y \in \Gamma_-(x)} G(y)\, d^4y\}$.

We are now in a position to "define" the S matrix and the interacting field.

Definition 4

The scattering matrix $S = e^{iD}$.

Definition 5

It can easily be verified that the S matrix defined above is unitary and Poincaré invariant, and the interacting field is Poincaré covariant. The family $\{U(x) : x \in \mathbb{R}^4\}$ is a relativistic generalization of the family $\{U(t) : t \in \mathbb{R}\}$ of Part I. To prove this statement, let $x = (t,\underline{x})$, then, in the non-relativistic limit as $c \to \infty$, $\Gamma_-(x) \to \{y : y \in \mathbb{R}^4 \text{ and } y_0 < t\}$.

Hence, $U(x) = U(t,\underline{x}) \to \exp\left\{i \int_{-\infty}^{t} dy_0 \int_{\underline{y} \in \mathbb{R}^n} G(y_0,\underline{y})\, d^3y\right\} = \exp\left\{i \int_{-\infty}^{t} F(t')\, dt'\right\} = U(t)$

where $F(t) = \int d^3x\, G(t,\underline{x})$.

We prove in the appendix to Part II the following important theorem.

Theorem 3

$\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ is a null space of $G(x)$, for any $x \in \mathbb{R}^4$.

Definition 6

Let $f \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^3)$, then the smeared field $\varphi(f,t) = \int_{x_0 = t} d^3x\, \varphi(x)\, f(\underline{x})$.

Theorem 4

For any $f \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^3)$, $w-\lim_{t \to -\infty} \{\varphi(f,t) - \varphi_0(f,t)\} = 0$ and $w-\lim_{t \to +\infty} \{\varphi(f,t) - S^{-1}\varphi_0(f,t)\, S\} = 0$ on $\mathcal{S}(\mathcal{F}) \times \mathcal{S}(\mathcal{F})$.

Proof

Let $A \in \mathcal{S}(\mathcal{F}^{(m)})$, $B \in \mathcal{S}(\mathcal{F}^{(n)})$ and $f \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^3)$, then

$$\begin{aligned}
&\langle A, \{\varphi^{(+)}(f,t) - \varphi_0(f,t)\} B\rangle = \\
&= \int f(\underline{x})\, d^3x \int_{k'}^{(m)} A^*(k') \int_{k}^{(n)} B(k) \langle k' | e^{-i\underline{x}\cdot\underline{P}} \{\varphi^{(+)}(t,\underline{0}) - \varphi_0^{(+)}(t,\underline{0})\} e^{i\underline{x}\cdot\underline{P}} | k\rangle \\
&= \int d^3x\, f(\underline{x}) \int_{k'}^{(m)} A^*(k') \int_{k}^{(n)} B(k)\, e^{i(\underline{k}-\underline{k}')\cdot\underline{x}} \langle k' | \{\varphi^{(+)}(t,\underline{0}) - \varphi_0^{(+)}(t,\underline{0})\} | k\rangle \\
&= \int_{k'}^{(m)} A^*(k') \int_{k}^{(n)} B(k)\, \tilde{f}(\underline{k}-\underline{k}') \langle k' | \{\varphi^{(+)}(t,\underline{0}) - \varphi_0^{(+)}(t,\underline{0})\} | k\rangle ,
\end{aligned} \tag{34}$$

where $\varphi^{(+)}(x) = \int_p a(p)\, e^{-ipx}$, the positive frequency component of the field. Eq.(34) takes the form:

$$\begin{aligned}
&\langle A, \{\varphi^{(+)}(f,t) - \varphi_0^{(+)}(f,t)\} B\rangle = \\
&= \int_{k'}^{(m)} A^*(k') \int_{k}^{(n)} B(k)\, \tilde{f}(\underline{k}-\underline{k}') \langle k' | U^{-1}(t,\underline{0})\, \varphi_0^{(+)}(t,\underline{0}) \{U(t,\underline{0}) - I\} | k\rangle
\end{aligned}$$

$$+\int_{k'}^{(m)} A^*(k') \int_{k}^{(n)} B(k)\, \tilde{f}(\underline{k}-\underline{k}') \langle k' | \{U^{-1}(t,\underline{0}) - I\}\, \varphi_0^{(+)}(t,\underline{0})\, | k \rangle$$

$$= \int_{k'}^{(m)} A^*(k') \int_{k}^{(n)} B(k)\, \tilde{f}(\underline{k}-\underline{k}') \int_{p} \langle k' | U^{-1}(t,\underline{0})\, a(p) \{U(t,\underline{0}) - I\} | k \rangle\, e^{-iE_p t}$$

$$+ n \int_{k'}^{(m)} A^*(k') \int_{k}^{(n)} B(k)\, \tilde{f}(\underline{k}-\underline{k}') \langle k' | \{U^{-1}(t,\underline{0}) - I\} | k \rangle'\, e^{-iE_{k_1} t} \quad , \tag{35}$$

where $|k\rangle' = |k_2, \ldots, k_n\rangle$.

Now, because $|e^{-iE_p t}| = |e^{-iE_{k_1} t}| = 1$ and there exists an $M \in \mathbb{R}$ such that $|\tilde{f}(k)| < M$ for all $k \in \mathbb{R}^3$, and because $\mathscr{S}-\lim_{t\to-\infty} \{U(t,\underline{0}) - I\} = 0$ on $\mathscr{S}(\mathcal{F})$ and $\int_p a(p)$ is $\mathscr{S}$-continuous, it follows from Eq.(35) that $w-\lim_{t\to-\infty} \{\varphi^{(+)}(f,t) - \varphi_0^{(+)}(f,t)\} = 0$ on $\mathscr{S}(\mathcal{F}) \times \mathscr{S}(\mathcal{F})$. Now, since $\varphi^{(-)}(f,t) = [\varphi^{(+)}(f^*,t)]^\dagger$, we get $w-\lim_{t\to-\infty} \{\varphi(f,t) - \varphi_0(f,t)\} = 0$ on $\mathscr{S}(\mathcal{F}) \times \mathscr{S}(\mathcal{F})$. The other limit could be proved in a similar way. ∎

It is well known that the free field expansion $\varphi_0(x) = \int_p \{a(p)\, e^{-ipx} + a^\dagger(p)\, e^{ipx}\}$ is invertible. $a(p) = i \int_{x_0=t} e^{ipx} \overleftrightarrow{\frac{\partial}{\partial x_0}} \varphi_0(x)\, d^3x$ and $a^\dagger(p) = -i \int_{x_0=t} e^{-ipx} \overleftrightarrow{\frac{\partial}{\partial x_0}} \varphi_0(x)\, d^3x$. In general, one defines the time t annihilation and creation operators as follows:

$$a(p,t) = i \int_{x_0=t} e^{ipx} \overleftrightarrow{\frac{\partial}{\partial x_0}} \varphi(x)\, d^3x \quad \text{and} \quad a^\dagger(p,t) = -i \int_{x_0=t} e^{-ipx} \overleftrightarrow{\frac{\partial}{\partial x_0}} \varphi(x)\, d^3x$$

Theorem 5

The vacuum and the one-particle states are stable.

Proof

First we notice that if $|0\rangle$ is the vacuum state then $\varphi(x)|0\rangle$, for, by Theorem 3 , $U(x)|0\rangle = |0\rangle$. Hence, $\varphi(x)|0\rangle = U^{-1}(x)\, \varphi_0(x)|0\rangle = U^{-1}(x) \int_p |p\rangle\, e^{ipx} = \int_p |p\rangle\, e^{ipx} = \varphi_0(x)|0\rangle$.

Now $a(p,t)|0\rangle = i \int_{x_0=t} e^{ipx} \overleftrightarrow{\frac{\partial}{\partial x_0}} \varphi(x)|0\rangle = i \int_{x_0=t} e^{ipx} \overleftrightarrow{\frac{\partial}{\partial x_0}} \varphi_0(x)|0\rangle =$ $a(p)|0\rangle = 0$. Hence, for any $t \in \mathbb{R}$, $a(p,t)$ annihilates the vacuum and the vacuum is stable. Also $a^\dagger(p,t)|0\rangle = a^\dagger(p)|0\rangle = |p\rangle$, and the creation of a particle from the vacuum state is time independent. ∎

3. Crossing and PCT

We discuss how one can impose crossing relations on a theory of a spinless neutral particle. Assume that $\tilde{\Delta}_n$ is a symmetric function of n $\mathbb{R}^4$ variables with the properties:

i) $\tilde{\Delta}_n^*(k_1, \ldots, k_n) = \tilde{\Delta}_n(-k_1, \ldots, -k_n)$;

ii) $\tilde{\Delta}_n(\Lambda k_1, \ldots, \Lambda k_n) = \tilde{\Delta}_n(k_1, \ldots, k_n)$; $\Lambda \in SO(1,3)$.

Then the function $\Delta_n(x; y_1, \ldots, y_n) = \int d^4k_1 \cdots \int d^4k_n\, e^{ik_1(y_1-x)} \cdots e^{ik_n(y_n-x)}$ $\tilde{\Delta}_n(k_1, \ldots, k_n)$ is symmetric in the y variables, with the properties:

i) $\Delta_n^*(x; y_1, \ldots, y_n) = \Delta_n(x; y_1, \ldots, y_n)$;

ii) $\Delta_n(\Lambda x + a; \Lambda y_1 + a, \ldots, \Lambda y_n + a) = \Delta_n(x; y_1, \ldots, y_n)$; $(\Lambda, a) \in E(1,3)$.

Consider the quadratic form

$$Q_n(x) = \int d^4y_1 \cdots \int d^4y_n\, \Delta_n(x; y_1, \ldots, y_n) : \varphi_0(y_1) \cdots \varphi_0(y_n) : \tag{36}$$

We can write

$$\varphi_0(x) = \int d^4p\, \theta(p_0)\, \delta(p^2 - m^2) \{ a(p)\, e^{-ipx} + a^\dagger(p)\, e^{ipx} \}$$
$$= \int d^4p\, \delta(p^2 - m^2) \{ \theta(p^0) a(p) + \theta(-p^0) a^\dagger(-p) \} e^{-ipx} . \tag{37}$$

Substituting (37) in (36) we get

$$
\begin{aligned}
Q_n(x) = {} & \int d^4y_1 \cdots \int d^4y_n \int d^4k_1 \cdots \int d^4k_n \; e^{ik_1(y_1-x)} \cdots e^{ik_n(y_n-x)} \, \tilde{\Delta}_n(k_1,\ldots,k_n) \cdot \\
& \cdot \int d^4p_1 \, \delta(p_1^2-m^2) \cdots \int d^4p_n \, \delta(p_n^2-m^2) :\{\theta(p_1^0)\, a(p_1) + \theta(-p_1^0)\, a^+(-p_1)\} \cdots \{\theta(p_n^0)\, a(p_n) + \theta(-p_n^0)\, a^+(-p_n)\}: \\
& \cdot e^{-ip_1y_1} \cdots e^{-ip_ny_n} \\
= {} & \sum_{j=0}^{n} \int_{k'}^{(j)} \int_{k}^{(n-j)} C_n^j \, \tilde{\Delta}_n(-k_1',\ldots,-k_j', k_1,\ldots,k_{n-j}) \, e^{i(k_1'+\cdots+k_j'-k_1-\cdots-k_{n-j})x} \cdot \\
& \cdot a^+(k_1') \cdots a^+(k_j')\, a(k_1) \cdots a(k_{n-j}) \, .
\end{aligned}
\tag{38}
$$

From (36) one gets the important formula

$$
F^{(j,k)}(p_1',\ldots,p_j'; p_1,\ldots,p_k) = \frac{(j+k)!}{j!\,k!} \, \tilde{\Delta}_{j+k}(-p_1',\ldots,-p_j', p_1,\ldots,p_k) \, . \tag{39}
$$

Eq.(39) connects different physical transitions. Although, upon integrating $Q_n(x)$ over R^4, the contribution from the terms with less than two creation or annihilation operators vanishes, the form (36) is only locally integrable. This is a consequence of Theorem 3.

We define the "R product" $R\{:\varphi(y_1)\cdots\varphi(y_n):\}$ as the initial product, but with the terms of less than two creation or annihilation operators dropped. Hence we now define

$$
G_n(x) = \int d^4y_1 \cdots \int d^4y_n \, \Delta_n(x; y_1,\ldots,y_n)\, R\{:\varphi(y_1)\cdots\varphi(y_n):\} \tag{40}
$$

and the effective potential density takes the form $G(x) = \sum_{n=4}^{\infty} G_n(x)$.

To prove PCT invariance, all that is necessary then is $\tilde{\Delta}_n(-p_1,-p_2,\ldots,-p_n) = \tilde{\Delta}_n(p_1,p_2,\ldots,p_n)$. But this is a consequence of Poincaré invariance, because $\tilde{\Delta}_n$ should be formed by the scalar product of 4 momenta.

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APPENDIX TO PART I

The asymptotic limit of non-bounded operators

If A is a non-bounded operator and $\mathcal{D}_A$ is the set of all vectors $f \in \mathcal{H}$ such that $\|A_0(t)f\|$, $\|A_0(t)Sf\|$ and $\|A_0^\dagger(t)\,U(t)f\|$ are bounded for all $t \in \mathbb{R}$, then for every $f,g \in \mathcal{D}_A$,

$$w\text{-}\lim_{t\to-\infty} \langle f, \{\hat{A}(t) - A_0(t)\} g\rangle = w\text{-}\lim_{t\to+\infty} \langle f, \{\hat{A}(t) - S^{-1}A_0(t)S\} g\rangle = 0 .$$

The proof follows the same steps, with the exception of the following: set $M_f = \sup_{t\in\mathbb{R}} \|A_0^\dagger(t)\,U(t)f\|$ and $M_g = \sup_{t\in\mathbb{R}} \|A_0(t)\,g\|$, then

$$|\langle f, \{\hat{A}(t) - A_0(t)\} g\rangle| \leqslant M_f \|\{U(t)-I\}g\| + M_g \|\{U(t)-I\}f\| .$$

The integrability of the effective potential

Let $f \in \mathcal{H}$, then $\|U(t)f - f\|^2 = \langle U(t)f-f,\, U(t)f-f\rangle = 2\langle f,f\rangle - 2\,\mathrm{Re}\,\langle f, U(t)f\rangle$. Now $U(t) = e^{itH_0/\hbar}\, W_-^\dagger\, e^{-itH_0/\hbar} = \int_0^{2\pi} e^{i\lambda}\, dE_\lambda(t)$, where $E_\lambda(t) = e^{itH_0/\hbar} E_\lambda\, e^{-itH_0/\hbar}$. Since $s-\lim_{t\to-\infty} U(t) = 0$, it follows that $\lim_{t\to-\infty} \mathrm{Re}\,\langle f, U(t)f\rangle = \langle f,f\rangle$. But $\int_0^{2\pi} d\,\langle f, E_\lambda(t)f\rangle = \langle f, E_{2\pi}(t)f\rangle - \langle f, E_0(t)f\rangle = \langle f,f\rangle$. Hence

$$\lim_{t\to-\infty} \mathrm{Re} \int_0^{2\pi} (e^{i\lambda}-1)\, d\langle f, E_\lambda(t) f\rangle = 0$$

or

$$\lim_{t\to-\infty} \int_0^{2\pi} \sin^2\frac{\lambda}{2}\, d\langle f, E_\lambda(t) f\rangle = 0 .$$

a) Assume that there exists a $\delta > 0$ such that $E_{2\pi-\delta} = I$, then

$$0 \leqslant \int_{2\pi-\delta}^{2\pi} \sin^2\frac{\lambda}{2}\, d\langle f, E_\lambda(t) f\rangle \leqslant \int_{2\pi-\delta}^{2\pi} d\langle f, E_\lambda(t) f\rangle = \langle f, E_{2\pi}(t) f\rangle - \langle f, E_{2\pi-\delta}(t) f\rangle = 0 .$$

Hence

$$\lim_{t\to-\infty} \int_0^{2\pi-\delta} \sin^2\frac{\lambda}{2}\, d\,\langle f, E_\lambda(t) f\rangle = 0 .$$

But, if $0 \leqslant \lambda \leqslant 2\pi-\delta$, then $0 \leqslant \frac{\lambda}{2\pi-\delta} \sin\frac{\delta}{2} \leqslant \sin\frac{\lambda}{2}$, hence

$$\lim_{t\to-\infty} \int_0^{2\pi-\delta} \lambda^2 d\langle f, E_\lambda(t) f\rangle \leqslant \lim_{t\to-\infty} \int_0^{2\pi-\delta} \frac{(2\pi-\delta)^2}{\sin^2\frac{\delta}{2}} \sin^2\frac{\lambda}{2}\, d\langle f, E_\lambda(t) f\rangle = 0 .$$

But $\int_0^{2\pi-\delta} \lambda^2\, d\,\langle f, E_\lambda(t) f\rangle = \|A(t)f\|^2$, hence $s-\lim_{t\to-\infty} A(t) = 0$ and $U(t) = \exp\left\{ i \int_{-\infty}^{t} F(t')\, dt' \right\}$.

b) (Weaker than (a)). Assume that for every $f \in \mathcal{H}$ and $\varepsilon > 0$ there exists $\delta > 0$ and $L > 0$ such that for every $t < -L$, $\langle f, E_{2\pi-\delta}(t) f\rangle > 1-\varepsilon$. Then, given $f \in \mathcal{H}$ and $\varepsilon > 0$, one can find two numbers $\delta > 0$ and $L_1 > 0$ such that for any $t < -L_1$ we have $\langle f, E_{2\pi-\delta}(t) f\rangle > 1 - \frac{\varepsilon}{8\pi^2}$. Also, one can find a number $L_2 > 0$ such that for any $t < -L_2$ we have

$$\int_0^{2\pi} \cos^2\frac{\lambda}{2}\, d\langle f, E_\lambda(t) f\rangle \left(\frac{2\pi-\delta}{\sin\delta/2}\right)^2 \frac{\varepsilon}{2}$$

Take $L = \max(L_1, L_2)$. Hence, for every $t < -L$ we have

$$\begin{aligned}\int_0^{2\pi} \lambda^2 d\langle f, E_\lambda(t) f\rangle &= \int_0^{2\pi-\delta} \lambda^2 d\langle f, E_\lambda(t) f\rangle + \int_{2\pi-\delta}^{2\pi} \lambda^2 d\langle f, E_\lambda(t) f\rangle \\ &\leqslant \int_0^{2\pi-\delta} \frac{(2\pi-\delta)^2}{\sin^2\delta/2} \sin^2\frac{\lambda}{2}\, d\langle f, E_\lambda(t) f\rangle + \int_{2\pi-\delta}^{2\pi} 4\pi^2\, d\langle f, E_\lambda(t) f\rangle \\ &\leqslant \frac{\varepsilon}{2} + 4\pi^2\left(1 - \langle f, E_{2\pi-\delta}(t) f\rangle\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .\end{aligned}$$

Hence, $\lim_{t\to-\infty} \int_0^{2\pi} \lambda^2\, d\,\langle f, E_\lambda(t) f\rangle = 0$ or $s-\lim_{t\to-\infty} A(t) = 0$.

APPENDIX TO PART II

Proof of Theorem 3

Since $G(x)$ is strongly integrable, it follows that $G(x)\,G(y)$ is weakly integrable. Now, $\{(x,y) : x \in \mathbb{R}^3$ and $y \in \Gamma(x)\} \subseteq \mathbb{R}^4 \times \mathbb{R}^4$. Hence $\int_{x\in\mathbb{R}^4} d^4x \int_{y\in\Gamma_-(x)} d^4y\; G(x)\,G(y)$ exists. But $H(x) = G(x) \int_{y\in\Gamma_-(x)} d^4y\, G(y)$ is a covariant function of $x \in \mathbb{R}^4$. Hence the integral $\int_{x\in\mathbb{R}^4} H(x)\, d^4x$ diverges if $G(x)$ does not annihilate $\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$, as can be seen explicitly. ∎

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