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CRITERION FOR THE NUCLEARITY OF SPACES OF FUNCTIONS OF INFINITE NUMBER OF VARIABLES *

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ABSTRACT

In this paper we shall formulate a new necessary and sufficient condition for the nuclearity of spaces of infinite number of variables, and we define new nuclear spaces which play an important role in the field of functional analysis and quantum field theory. Also we shall give the condition for nuclearity of the infinite weighted tensor product of nuclear spaces.

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I. AUXILIARY CONCEPTS AND FACTS

1. If $\{e_n\}$, $\{h_n\}$ are orthonormal bases in the Hilbert space H_1 and H_2, respectively, and $\lambda_n > 0$ are the numbers such that the series $\sum_1^\infty \lambda_n^2$ converges, then the formula $Af = \sum_{n=1}^{\infty} \lambda_n (f,e_n) h_n$ defines an operator of the Hilbert-Schmidt type, and defines a nuclear operator which maps the space H_1 into H_2 if $\sum_{n=1}^{\infty} \lambda_n < \infty$.

By a nuclear space we mean a locally convex space with the property that every linear continuous map from this space into a Banach space is nuclear [8,9,10].

2. The inductive limit $\ell\ \mathrm{ind.}_{\alpha\in A}\ \Phi_\alpha(\tau_\alpha)$ is defined as follows: We are given a family $\{\Phi_\alpha(\tau_\alpha)\}_{\alpha\in A}$ of locally compact space, a linear space Φ and a linear map $u_\alpha : \Phi_\alpha \to \Phi$ such that the family $\{u_\alpha(\Phi_\alpha)\}_{\alpha\in A}$ spans the entire space Φ . The topology τ, if Φ is defined as the finest locally convex topology such that the maps $A_\alpha : \Phi_\alpha(\tau_\alpha) \to \Phi(\tau)$, remains continuous. A basis of neighbourhoods of zero may be formed by sets $\Gamma_\alpha A_\alpha(U_\alpha)$, where U_α are neighbourhoods of zero in $\Phi_\alpha(\tau_\alpha)$. If $\Phi(\tau)$ is a Hausdorff space, then it is called the inductive limit of the spaces $\Phi_\alpha(\tau_\alpha)$, and the topology τ is called the inductive topology. We may also write

$$\Phi(\tau) = \ell.\mathrm{ind}_{\alpha\in H}\ \Phi_\alpha(\tau_a) \qquad \text{for } \Phi(\tau), \qquad \text{see [8].}$$

To define the projective limit, let Φ be a subspace of $\bigotimes_{i\in A} \Phi_i$ whose elements $\Phi = (\Phi_i)$ satisfy the relation $\Phi_i = u_{ij}(\Phi_j)$ whenever $i < j$. The space Φ is called the projective limit of a family $\{\Phi_\alpha\}_{\alpha\in A}$ of locally compact Hausdorff spaces with respect to continuous linear mappings

$$u_{ij}: \Phi_j \to \Phi_i \ (i < j). \qquad [8,9,10]$$

Wolka [4] has given an interesting criterion for the nuclearity of inductive limit and projective limit of Hilbert spaces.

Let $\Phi = \ell\ \mathrm{ind.}_{n\to\infty} H_n$ where H_n are Hilbert spaces, then it follows easily that Φ is nuclear if and only if for every m there exists $n > m$ such that the injection $u_{nm} : H_m \to H_n$ is of the Hilbert-Schmidt type. [9,10,11]

3. The infinite weighted tensor product of Hilbert spaces: The following construction of the space will be used further as shown in [5,6]. Let $(H_i)_{i=1}^{\infty}$ be a sequence of Hilbert spaces H_i, let $e = (e^{(i)})_{i=1}^{\infty}$, $(e^i \in H_i)$ be a fixed sequence of unit vectors of these spaces and let $\delta = (\delta_i)_{i=1}^{\infty}$ $(\delta_i > 0)$ be a fixed numerical sequence. The Hilbert space constructed below will be called an infinite weighted tensor product

$$H_{e,\delta} = \bigotimes_{i=1;e;\delta}^{\infty} H_i$$

of the spaces H_i with the stabilizing sequence e and the weight δ.

4. Recall that an operator C acting from a separable Hilbert space H_1 to a Hilbert space H_2 is of the Hilbert-Schmidt (HS) type if for an orthonormal basis $e_1, e_2, \ldots,$ of the spaces H_1 the series

$$|C| = \sum_{j=1}^{\infty} ||C\, e_j||^2_{H_2}$$

converges. If H_1 is contained in H_2, we say that the inclusion is quasi-nuclear if the inclusion operator is of the Hilbert-Schmidt type. The norm $|C|$ is called the Hilbert norm of the operator C, see [1]. For the chain of the Hilbert space, see [1,12].

In order to formulate our main problem we need to do the following:

a) Let $(e_k)_{k=0}^{\infty}$ be an orthonormal basis in a separable Hilbert space H_0 and let

$$P = \{P^{\tau} = (P_k^{\tau})_{k=0}^{\infty} : P^{\tau} \in R^{\infty} = R^1 x R^1 x \ldots;\ P_0^{\tau} = 1;\ P_k^{\tau} \geq 1 \text{ for every } k, \tau \in T\}$$

be a system of weights. Then by every weight we define the corresponding Hilbert space in the form:

$$H_{\tau} = \{u = \sum_{k=0}^{\infty} C_k e_k \mid ||u||^2_+ = \sum_{k=0}^{\infty} |C_k|^2 P_k^{\tau} < \infty; (C_k)_{k=0}^{\infty} \in C^{\infty} = C^1 x C^1 x \ldots\}$$

Define the space Φ_P as the projective limit of the Hilbert spaces H_{τ} in the form [2,3,4,10]

$$\Phi_P = \bigcap_{\tau \in T} H_{\tau}.$$

Definition

We shall say that the system of weights P has the property (N) if:

$$\text{for every } \tau' \in T \text{ there exists } \tau'' \in T: \sum_{k=0}^{\infty} P_k^{\tau'}/P_k^{\tau''} < \infty \qquad (1)$$

and then we can give the condition for the nuclearity of Φ_P by:

Theorem 1

The space Φ_P is nuclear if and only if P has the property (N).

Proof

Let $P^{\tau'}$ be an arbitrary weight from P and $H_{\tau'}$ be the corresponding Hilbert space. If P has the property N, then according to (1) there exists a weight $P^{\tau''} \in P$ such that

$$\sum_{k=0}^{\infty} P_k^{\tau'}/P_k^{\tau''} < \infty .$$

Let $\{(P_k^{\tau''})^{-1/2} e_k\}_{k=0}^{\infty}$ be an orthonormal basis in the corresponding space $H_{\tau''}$ for the weight $P^{\tau''}$. If $O_{\tau'}^{\tau''} : H_{\tau''} \to H_{\tau'}$ is the inclusion operator then

$$|O_{\tau'}^{\tau''}|^2 = \sum_{k=0}^{\infty} ||(P_k^{\tau''})^{-1/2} e_k||^2_{+\tau'} = \sum_{k=0}^{\infty} P_k^{\tau'}/P_k^{\tau''} < \infty \qquad (2)$$

and, conversely, from the last equality it can easily be shown that if Φ_P is nuclear, then P satisfies the property (N).

b) Now we consider $H_{-\tau} = H'_{\tau}$ to be the corresponding adjoint space of an antilinear functional on H_0, $(e_k)_{k=0}^{\infty}$ to be an orthonormal basis on it and Φ'_P to denote the adjoint of Φ_P. Now Φ'_P is defined as the inductive limit of the Hilbert spaces $H_{-\tau}$ where $H_{-\tau}$ and Φ'_P have the form:

$$H_{-\tau} = \{w = \Sigma w_k e_k,\ (w_k)_{k=0}^{\infty} \varepsilon C^{\infty} \mid \|w\|^2_{-\tau} = \sum_{k=0}^{\infty} |w_k|^2 (P_k^{\tau})^{-1} < \infty\},$$

and

$$\Phi_P' = \bigcup_{\tau\varepsilon T} H_{-\tau} = \{w = \sum_{k=0}^{\infty} w_k e_k,\ (w_k)_{k=0}^{\infty} \varepsilon C^{\infty} \mid \|w\|^2_{-\tau} < \infty \text{ for every } \tau\varepsilon T\}. \qquad (3)$$

Since for the spaces $(H_\tau)_{\tau\in T}$ we find $e_0 \in H_\tau\, \forall\, \tau\in T$, $\|e_0\|_{+\tau} = 1$ and $P_0^{\tau} = 1$ for every $\tau \in T$. Then for every $\tau = (\tau_i)_{i=1}^{\infty}$, where $\tau = Tx\,Tx\,\ldots$ we can define the infinite tensor product $\mathcal{H}_\tau = \bigotimes_{i=1}^{\infty} H_i$ constructed by the stabilizing sequence $e = (e_0, e_0, \ldots)$ as in [5, 6].

Let $\mathcal{H}_0$ be the infinite tensor product of H_0 constructed by the stabilizing sequence $e = (e_0, e_0, \ldots)$, and $e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots$ be considered as an orthonormal basis in $\mathcal{H}_0$, and $e_\alpha^{\tau} = \{e_\alpha^{\tau}\}_\alpha = (P_{\alpha_1}^{\tau_1})^{-1/2} \ldots (P_{\alpha_\nu}^{\tau_\nu})^{-1/2}\, e_\alpha$ be an orthonormal basis on $\mathcal{H}_\tau$. We define the projective limit of the Hilbert spaces $\mathcal{H}_\tau$ by the form

$$\Phi_\rho = \bigcap_{\tau\varepsilon T} \mathcal{H}_\tau .$$

Then $\mathcal{H}_\tau$ and Φ_P have the following form:

$$\mathcal{H}_\tau = \{u = \sum_\alpha u_\alpha e_\alpha;\ (u_\alpha)_\alpha\ \varepsilon C^{\infty} \mid \|u\|^2_{+\tau} = \Sigma |u_\alpha|^2 (P_{\alpha_1}^{\tau_1} \ldots P_{\alpha_\nu}^{\tau_\nu}) < \infty\};$$

and

$$\Phi_\rho = \{u = \sum_\alpha u_\alpha e_\alpha;\ (u_\alpha)_\alpha \varepsilon C^{\infty} \mid \|u\|_{+\tau} < \infty\} .$$

Since $\|e_0\|_{-\tau} = 1$ for every $\tau\in T$ we can define $\mathcal{H}_{-\tau} = \bigotimes_{i=1}^{\infty} H_{-\tau_i}$ for every $\tau\in T$ constructed by the stabilizing sequence $e = (e_0, e_0, \ldots)$ with basis $(e_\alpha^{-\tau})$; $e_\alpha^{-\tau} = (P_{\alpha_1}^{\tau_1})^{1/2} \ldots (P_{\alpha_\nu}^{\tau_\nu})^{1/2}\, e_\alpha$. Define the space Φ_P' as inductive limit for

the spaces $\mathcal{H}_{-\tau}$, where $\mathcal{H}_{-\tau} = \{\omega = \sum \omega_\alpha e_\alpha \mid \|\omega\|^2_{-\tau} = \sum |\omega_\alpha|^2 (P_{\alpha_1}^{\tau_1} \ldots P_{\alpha_\nu}^{\tau_\nu})^{-1} < \infty\}$ and $\Phi_P' = \{\omega = \sum \omega_\alpha e_\alpha \mid \|\omega\|_{-\tau} < \infty \text{ for every } \tau\in T\}$.

From the above we can prove the following.

<u>Lemma 1</u>

The adjoint spaces for $\mathcal{H}_\tau$, Φ_P are $\mathcal{H}_{-\tau}$, Φ_P', respectively. Thus we have the following chains of spaces

$$\mathcal{H}_\tau \subseteq \mathcal{H}_0 \subseteq \mathcal{H}_{-\tau} \qquad \Phi_\rho \subseteq \mathcal{H}_0 \subseteq \Phi_\rho' .$$

The proof of this lemma can easily be shown as in Ref.[1](Chapt.1). Since

$$\|\cdot\|_{-\tau} \le \|\cdot\|_0 \le \|\cdot\|_\tau \quad \|\cdot\|_{-P} \le \|\cdot\|_0 \le \|\cdot\|_P .$$

At the end of this article we shall obtain the criterion for nuclearity of the infinite weighted tensor product of nuclear spaces.[7]

We have the following definition.

<u>Definition</u>

The system of weights P satisfies the property $N^{\oplus}$ if for every $\tau' \in T$, $\varepsilon > 0$ there exists $\tau'' \in T$ such that

$$\sum_{k=0}^{\infty} P_k^{\tau'} / P_k^{\tau''} \le 1 + \varepsilon .$$

Then we have the following theorem.

<u>Theorem 2</u>

The space Φ_P is nuclear if and only if P has the property $N^{\oplus}$.

<u>Proof</u>

Let P, which has the property $N^{\oplus}$ and $\tau' = (\tau_i')_{i=1}^{\infty}$, be an arbitrary element, we can choose $\tau'' = (\tau_i'')_{i=1}^{\infty}$ such that

$$\sum_{k=0}^{\infty} P_k^{\tau'_i} / P_k^{\tau''_i} \leq 1 + \frac{1}{2i} \qquad (i = 1,2,\ldots) .$$

The Hilbert norm for the inclusion operator

$$O_{\tau'_i}^{\tau''_i} ; \quad H_{\tau''_i} \to H_{\tau'_i}$$

has the form

$$|O_{\tau'_i}^{\tau''_i}|^2 = \sum_{k=0}^{\infty} P_k^{\tau'_i} / P_k^{\tau''_i} .$$

Then we have $|O_{\tau_i}^{\tau''}|^2 < \infty$, which gives the condition for nuclearity of Φ_P, where $O_{\tau'}^{\tau''} : \mathcal{H}_{\tau''} \to \mathcal{H}_{\tau'}$ is the inclusion operator.

Conversely, suppose that Φ_P is nuclear, and if $\tau' \in T$, $\varepsilon > 0$, then we can find $\tau'' = (\tau''_i)_{i=1}^{\infty}$ such that the Hilbert norm of the inclusion operator is convergent, i.e.

$$|O_{\tau'_i}^{\tau''_i}|^2 < \infty \ , \text{ then } \quad |O_{\tau'_i}^{\tau''_i}|^2 - 1 \to 0 \text{ as } i \to \infty ,$$

and we can choose i_0 such that $i > i_0$, $|O_{\tau'_i}^{\tau''_i}|^2 - 1 \leqslant \varepsilon$, which is the condition of nuclearity when $i > i_0$.

II. *NUCLEAR SPACES OF FUNCTIONS OF INFINITE NUMBER OF VARIABLES*

In this section we shall construct two illustrative examples for these spaces.

Let $L(S^1, dS)$ be a Hilbert space with complex-valued functions which are summable with respect to normed Lebesgue measure $dS = \frac{1}{2\pi} dt$ $(t \in R^1)$ and $(e^{ikt})_{-\infty}^{\infty}$ be its orthonormal basis. Let $T^{\infty} = S^1 \times S^1 \times \ldots$

a) <u>The space $K(T^{\infty})$</u>

Consider the Sobolev space $W_2^m(S^1, dS(t))$ as the completion of trigonometric polynomials $T_n(t) \sum_{|k| < n} C_k e^{ikT}, n = 0,1,\ldots,$ by the norm

$$||T_n||^2_{W_2^m(S^1)} = \sum_{i=0}^{m} ||T_n^{(i)}||^2_{L_2(S^1)} = \sum_k |C_k|^2 (1 + k^2 + \ldots + k^{2m}),$$

and define the locally convex linear topological space $K(S^1) = \bigcap_{m=0}^{\infty} W_2^m(S^1)$ with topology to be considered as the inductive topology of the Sobolev space [7]. It is clear that $K(S^1)$ is considered as a space of infinite differentiable functions on S^1.

<u>Definition</u>

Consider the space $K(T^{\infty}) = \bigotimes_{i=1;e}^{\infty} K(S^1)$ to be **regarded as the** infinite direct product of spaces constructed by the stabilizing sequence $e = \{e^K\}_{K=1}^{\infty}$ [7].

On the other hand, let $H_0 = L_2(S^1)$, $e_k = e^{ikt}$, $k = 0,1,2,\ldots,$ and

$$P = \Sigma = \{P^m = (P_k^m)_{k=0}^{\infty} , \ P_k^m = (1 + |k|)^m\}_{m=0}^{\infty}$$

be a set of countable weights. It is easy to see that $P_0^m = 1$ for every $m = 0,1,\ldots$ and $P_k^m > 1$ for every m, k, and Σ satisfies the condition of nuclearity $N^{\oplus}$. For every weight P^m we can write the corresponding space in the form:

$$K^m = \{u(t) = \sum_{k=0}^{\infty} u_k e^{ikt} \mid ||u||^2_{+m} = \sum_{k=0}^{\infty} |u_k|^2 (1 + |k|)^m < \infty\}$$

and the corresponding spaces for every index $m = (m_i)_{i=1}^{\infty}$ can be written in the form:

$$K^m(T^\infty) = \bigotimes_{i=1;e}^{\infty} K^{m_i}(S^1)$$

$$= \{u(x) = \sum_\alpha u_\alpha e_\alpha^{i(\alpha_1 x_1 + \ldots + \alpha_\nu x_\nu)} \mid ||u||^2_{+m} = \Sigma|u_\alpha|^2(1+|\alpha_1|)^{m_1} \ldots$$

$$\ldots (1 + |\alpha_\nu|)^{m_\nu} < \infty;\ x = (x_i)_{i=1}^{\infty} \in T^\infty\}$$

Then we can define $\hat{K}(T^\infty)$ as the projective limit of the spaces $K^m(T^\infty)$, i.e.

$$\hat{K}(T^\infty) = \bigcap_{m=(m_i)_{i=1}^{\infty}} K^m(T^\infty)$$

and we can prove that the space $K(T^\infty)$ coincides with $\hat{K}(T^\infty)$ in the topological sense.

Theorem 3

Using the above-mentioned it can easily be shown that the space $K(T^\infty)$ is nuclear, since the system of weights Σ satisfies the condition $N^{\oplus}$.

b) The space $\mathcal{A}(T^\infty)$

Let $H_0 = L_2(S^1)$, $P = W$ be a system of weights such that

$$W = \{P^m = (P_k^m)_{k=0}^{\infty} \mid P_k^m = m^{|k|}\}_{m=1}^{\infty} .$$

It is clear that this system of weights satisfies the condition of nuclearity $N^{\oplus}$ and the corresponding space for each weight has the form

$$A^m(S^1) = \{u(t) = \sum_{k=0}^{\infty} u_k e^{ikt} \mid ||\ ||^2_{+m} = \sum_{k=0}^{\infty} |u_k|^2 m^{|k|} < \infty\} ,$$

and we can define $A(S^1)$ as the inductive limit of the spaces $A^m(S^1)$.

Definition

$$A(T^\infty) = \mathop{\times}_{i=1;e}^{\infty} A(S^1)$$

defines the infinite tensor product of spaces with weights ([7]).

Theorem 4

The space $\mathcal{A}(T^\infty)$ is nuclear.

This fact can easily be proved since the system of weights W satisfies the condition of nuclearity $N^{\oplus}$.

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