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LINEAR SPIN-ZERO QUANTUM FIELDS IN EXTERNAL GRAVITATIONAL AND SCALAR FIELDS - I: A ONE-PARTICLE STRUCTURE FOR THE STATIONARY CASE

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LINEAR SPIN-ZERO QUANTUM FIELDS IN EXTERNAL GRAVITATIONAL AND SCALAR FIELDS - I: A ONE-PARTICLE STRUCTURE FOR THE STATIONARY CASE *

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ABSTRACT

We give mathematically rigorous results on the quantization of the covariant Klein-Gordon field with an external stationary scalar interaction in a stationary curved space-time.

We show how, following Segal, Weinless etc., the problem reduces to finding a "one-particle structure" for the corresponding classical system.

Our main result is an existence theorem for such a one-particle structure for a precisely specified class of stationary space-times.

Byproducts of our approach are:-

(1) A discussion of when the equal-time hypersurfaces in a given stationary space-time are Cauchy.

(2) A proof that when a one-particle structure exists it is unique - a result of general interest for the quantization of linear systems.

(3) A modification and extension of the methods of Chernoff [3] for proving the essential self-adjointness of certain partial differential operators.

§ 0 INTRODUCTION

In this series of papers, we discuss the quantization of the equation

$$\left(g^{\mu\nu}\nabla_\mu \partial_\nu + m^2 + V\right)\varphi = 0 \qquad (0.1)$$

- the covariant Klein-Gordon equation in a fixed curved space-time $(\mathcal{M}, g^{\mu\nu})$ and interacting with a fixed external scalar field V. (We shall always take $(\mathcal{M}, g)$, V to be C^∞.)

Notwithstanding much recent work on 'quantum field theory in curved space-times' [6, 12], it is perhaps fair to say that there is, as yet, no satisfactory statement of what it would mean to quantize our equation (0.1) on a generic space-time. For the case of stationary space-times, however, there is a well established procedure - at least at the heuristic level. We ought not to rest content with this procedure while the generic case remains unsolved (most space-times are not stationary, and even for stationary space-times, why should we single out the particular time coordinate for which the metric is stationary?) However, we shall show (in subsequent papers) how results on the stationary case play a role in a possible formulation of a quantization procedure suitable for generic case.

The present paper is concerned with giving rigorous mathematical results on the existence and uniqueness of quantization for the stationary case.

Sections 1 - 4 outline our general approach and state the principal results. Sections 5 - 8 contain details of proofs and further results.

§ 1 STATIONARY SPACE-TIMES

Note (1) We use Hawking and Ellis [10] (H.E.) especially chapters 1 and 6 as a reference throughout - except that we choose signature (+ - - -). (2) All space-times are assumed to be space and time orientable.

Given an (orientable) Riemannian 3-manifold $(\mathcal{C}, {}^3g)$ and given (α, β) — α a scalar field, β a vector field on $\mathcal{C}$ satisfying

$$\alpha > 0 \ , \quad \alpha^2 - {}^3g(\beta, \beta) > 0 \qquad (1.1)$$

we define $\mathrm{Stat}(\mathcal{C}, {}^3g, \alpha, \beta)$ to be the space-time $(\mathbb{R}\times\mathcal{C}, {}^4g)$ where ${}^4g$ is given (in local coordinates (t, x^i) - x^i local coordinates on $\mathcal{C}$) by

$${}^4g_{lower} = \begin{pmatrix} \alpha^2 - \beta^i\beta_i & -\beta_j \\ -\beta_i & -{}^3g_{ij} \end{pmatrix} \qquad (1.2)$$

In other words, $\mathrm{Stat}(\mathcal{C}, {}^3g, \alpha, \beta)$ is the stationary space-time having $(\mathcal{C}, {}^3g)$ as the equal-time spacelike hypersurfaces (with induced positive definite Riemannian metric ${}^3g$) and having

$$\frac{\partial}{\partial t} = \alpha N(\mathcal{C}) + \beta \qquad (1.3)$$

($N(\mathcal{C})$: unit future pointing normal vector field to $\mathcal{C}$) as the everywhere timelike Killing vector field. α and β (sometimes "N" and " $N^i\partial_i$") are usually known [17] as "lapse and shift fields".

Clearly, $\mathcal{C}$ (strictly $\{s\}\times\mathcal{C}$ for any $s\in\mathbb{R}$) is always a smooth space-like partial Cauchy surface (H.E. §6.5) in $\mathrm{Stat}(\mathcal{C}, {}^3g, \alpha, \beta)$. In the sequel, we restrict ourselves to stationary space-times in which the equal time hypersurfaces $\mathcal{C}$ are (globally) Cauchy. In other words, we restrict ourselves to those satisfying the condition:

> Along any inextendible non-spacelike curve, t takes all values in $(-\infty, \infty)$.

In §5 , we use results on the causal structure of space-times to prove some simple results explicitly characterising such space-times. (For instance, they include all cases of compact $\mathcal{C}$). The results obtained are interesting in their own right, and in any case are needed in §8.

§2 THE CLASSICAL THEORY AS A "LINEAR DYNAMICAL SYSTEM" (STATIONARY CASE)

Let us denote by $D(\mathcal{C})$ the space of real smooth Cauchy data of compact support:

$$D(\mathcal{C}) = C_0^\infty(\mathcal{C}) + C_0^\infty(\mathcal{C}) \qquad (2.1)$$

Then, the condition that $\mathcal{C}$ be Cauchy in $\mathrm{Stat}(\mathcal{C}, {}^3g, \alpha, \beta)$ guarantees that $D(\mathcal{C})$ is invariant under time evolution:

In fact, it is known (H.E. Props. 6.6.3 , 6.6.8) that the existence of a Cauchy surface is equivalent to global hyperbolicity and for globally hyperbolic space-times, we have

2.1 Leray's Theorem

Let $(\mathcal{M}, g)$ be an oriented globally hyperbolic space-time; $\mathcal{C}$ some Cauchy surface – unit future pointing normal $N(\mathcal{C})$.
Then, the Cauchy data $\phi \in D(\mathcal{C})$ given by

$$\phi = \begin{pmatrix} f \\ p \end{pmatrix} \qquad f = \varphi\Big|_{\mathcal{C}} \qquad p = N(\mathcal{C})\varphi\Big|_{\mathcal{C}}$$

define a unique solution in $C^{\infty}(\mathcal{M})$ having compact support on every other Cauchy surface. In fact, the solution has support in $J^{+}(\operatorname{supp}\phi) \cup J^{-}(\operatorname{supp}\phi)$ – the union of the causal future and the causal past of the Cauchy data.

For the proof, see Leray [15], Choquet Bruhat [4], Lichnerowicz [16]

Thus, returning to our stationary case, we may view time evolution as a one-parameter group

$$\mathcal{T}(t) : D(\mathcal{C}) \longrightarrow D(\mathcal{C})$$

Moreover, the existence of the conserved current $j_{\mu} = \varphi_1 \overleftrightarrow{\partial_{\mu}} \varphi_2$

for any pair of solutions φ_1, φ_2 of (0.1) guarantees that $\mathcal{T}(t)$ preserves the symplectic form

$$\sigma(\phi_1\ \phi_2) = \int_{\mathcal{C}} (f_1 p_2 - p_1 f_2)\, d\eta(\mathcal{C}) \tag{2.2}$$

where

$$\phi_1 = \begin{pmatrix} f_1 \\ p_1 \end{pmatrix} \in D(\mathcal{C}), \quad \phi_2 = \begin{pmatrix} f_2 \\ p_2 \end{pmatrix} \in D(\mathcal{C}),$$

$\eta(\mathcal{C})$ denotes the Riemannian volume element on $(\mathcal{C}, {}^3g)$.

Equivalently, we shall sometimes write

$$\sigma(\phi_1\ \phi_2) = \langle \phi_1 \mid \mathbb{g}\, \phi_2 \rangle_{L^2(\mathcal{C},\eta)+L^2(\mathcal{C},\eta)} \tag{2.3}$$

where

$$\mathbb{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We shall use the term linear dynamical system to denote any real symplectic space (D, σ) together with a one-parameter symplectic group $\mathcal{T}(t)$.

Note that, in our case, $\mathcal{T}(t)$ is generated in the sense of classical mechanics by the Hamiltonian:

$$\begin{aligned} H(f, p) &= \frac{1}{2}\int_{\mathcal{C}} d\eta(\mathcal{C})\, \alpha\left(p^2 + {}^3g^{ij}\partial_i f\, \partial_j f + (m^2+V) f^2\right) \\ &\quad + \int_{\mathcal{C}} d\eta(\mathcal{C})\, p\, \beta^i \partial_i f \\ &= \frac{1}{2} \langle \phi \mid \mathbb{A}\, \phi \rangle_{L^2(\mathcal{C},\eta)+L^2(\mathcal{C},\eta)} \end{aligned} \tag{2.4}$$

where

$$\mathbb{A} = \begin{pmatrix} -(\partial^i\alpha)\partial_i + \alpha(m^2 - \Delta(\mathcal{C}) + V) & -(\nabla_i \beta^i + \beta^i \partial_i) \\ \beta^i \partial_i & \alpha \end{pmatrix}$$

where $\Delta(\mathcal{C})$ is the Laplace-Beltrami operator for $(\mathcal{C}, {}^3g)$;
∇_i denotes the covariant derivative for $(\mathcal{C}, {}^3g)$.

The first order form equations can be written

$$\frac{d}{dt}\mathcal{T}(t)\,\phi\Big|_{t=0} = -\mathbb{h}\,\phi \tag{2.5}$$

where $\mathbb{h} = -\mathbb{g}\,\mathbb{A}$.

§3 QUANTIZATION OF LINEAR DYNAMICAL SYSTEMS [2, 5, 14, 21, 22, 23, 25]

To quantize a linear dynamical system $(D, \sigma, \mathcal{T}(t))$, one seeks a "quantization" $(\mathcal{K}, W, \Omega, V(t))$ [25] where $\mathcal{K}$ is a (separable) Hilbert space, W a map from D into unitaries on $\mathcal{K}$, Ω a vector in $\mathcal{K}$ and $V(t)$ a strongly continuous unitary group on $\mathcal{K}$ satisfying:

$$
\begin{aligned}
&(1) \quad W(\phi + \psi) = \exp(-i\sigma(\phi\ \psi)/2)\, W(\phi+\psi) \\
&(2) \quad W(\mathcal{T}(t)\phi) = V(t)\, W(\phi)\, V(-t) \\
&(3) \quad V(t)\Omega = \Omega \qquad\qquad (3.1) \\
&(4) \quad V(t) \text{ has positive energy}^{1} \\
&(5) \quad W(s\phi) \text{ is weakly continuous in } s \in \mathbb{R}. \\
&(6) \quad \Omega \text{ is cyclic for the } W(\phi)\text{'s.} \\
&\qquad\qquad\qquad \forall\, \phi, \psi \in D
\end{aligned}
$$

Such a quantization is determined (up to equivalence) by its generating functional:

$$\langle \Omega \mid W(\phi)\, \Omega \rangle$$

We now briefly explain the reason for this definition of quantization: For the generators $R(\phi)$ of $W(t\phi)$ and H of $V(t)$, (3.1) above is a rigorous version of:

$$
\begin{aligned}
&(1) \quad [R(\phi)\ \ R(\psi)] = i\,\sigma(\phi\ \psi)\, \mathbb{1} \\
&(2) \quad [H, R(\phi)] = -i\, R(h\phi) \qquad (3.2) \\
&(3) \quad H\Omega = 0 \\
&\qquad\qquad \left(h := -\tfrac{d}{dt}\mathcal{T}(t)\Big|_{t=0} \right)
\end{aligned}
$$

Here, $R(\phi)$ may be thought of as $\sigma(\hat{\varphi}, \hat{\pi}\,;\, f, p)$ where

1 i.e. its generator is positive.

$\hat{\varphi}$, $\hat{\pi}$ are the equal-time quantum fields corresponding to f, p.
(The usual canonical momentum is $\sqrt{{}^3g}\ \hat{\pi}$)

In (3.2), (1) expresses

"$[\hat{\varphi}(x)\ \hat{\varphi}(y)] = 0$, $[\hat{\pi}(x)\ \hat{\pi}(y)] = 0$, $[\hat{\varphi}(x)\ \hat{\pi}(y)] = i\delta^3(x,y)$"

(2) are the quantum Hamiltonian equations of motion; and (3) gives the interpretation of Ω as the vacuum vector - or lowest energy state of the positive Hamiltonian H.

Roughly speaking then, seeking a quantization in our sense amounts to seeking a representation of the CCR in which a positive Hamiltonian exists generating the dynamics (with a vacuum vector Ω cyclic for the CCR)

To construct a quantization for a given linear dynamical system $(D, \sigma, \mathcal{T}(t))$, one seeks first a "<u>one particle structure</u>" $(K, \mathcal{H}, U(t))$ for $(D, \sigma, \mathcal{T}(t))$ where $\mathcal{H}$ is a separable Hilbert space, and $U(t)$ a unitary group with strictly positive energy[2] and K a real linear map from D to $\mathcal{H}$ satisfying:

$$
\begin{aligned}
&(1) \quad \operatorname{ran} K \text{ is dense in } \mathcal{H}. \qquad\qquad (3.3) \\
&(2) \quad 2\operatorname{Im}\langle K\phi \mid K\psi\rangle = \sigma(\phi\ \psi) \quad (\text{i.e. } K \text{ is symplectic}) \\
&(3) \quad K(\mathcal{T}(t)\phi) = U(t)(K(\phi)) \quad (\text{i.e. } K \text{ intertwines } \mathcal{T}(t) \text{ and } U(t))
\end{aligned}
$$

We may then quantize using the Fock representation over $\mathcal{H}$ (Segal's "free Weyl process"). E.g. we can choose for $\mathcal{K}$ Fock space over $\mathcal{H}$:

$$\mathcal{K} = \mathcal{F}(\mathcal{H}) = \mathbb{C} + \mathcal{H} + (\mathcal{H}\otimes\mathcal{H})_s + (\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H})_s + \cdots$$

whereupon we have:

$$\Omega = 1 + 0 + 0 + \cdots$$

2 i.e. it has positive self-adjoint generator h with dense range.

$$V(t) = \text{“} d\Gamma(U(t)) \text{”} = 1 + U(t) + (U(t) \otimes U(t))_s + \cdots$$

$$W(\phi) = \exp(-i R^F(K\phi))$$

where $R^F(K\phi) = 1/i \; \overline{\left(a^*(K\phi) - a^*(K\phi)^* \right)}$

($a^*(\cdot)$ the usual creation operator on $\mathcal{F}(\mathcal{H})$)

This quantization is characterized by the generating functional:

$$\langle \Omega \mid W(\phi)\Omega \rangle = \exp\left(-1/2 \, \|K\phi\|^2_{\mathcal{H}}\right)$$

The following two theorems tell us that this quantization is "almost unique" :

3.1 Theorem

Given a linear dynamical system $(D, \sigma, \mathcal{J}(t))$. Suppose there is a one-particle structure $(K, \mathcal{H}, U(t))$ for $(D, \sigma, \mathcal{J}(t))$ Then it is unique up to unitary equivalence.

3.2 Theorem

In the case where a one-particle structure exists, the above quantization is the only one for which the map $K\phi \mapsto W(\phi)$ is strongly continuous on $\mathcal{H}$ in its norm topology.

These theorems are of general interest for the quantization of any linear Bose systems. Theorem 3.2 is proved in Segal [21]. Surprisingly, a proof of Theorem 3.1 does not appear to have been given before - though the result is implicit in Weinless' paper [25]. In §6, we give a proof based on Weinless' techniques.

Note: We used the phrase 'almost unique' since in the definition of a quantization, one demands only that $W(\cdot)$ be strongly continuous on finite dimensional subspaces of D (c.(5) of 3.1 above) which is a weaker condition than that of continuity on $\mathcal{H}$ required for Theorem 3.2. For theorems on the existence and uniqueness of a quantization as such, see again Weinless [25] - where again the concept of a 'one particle structure' plays an important role.

§4 EXISTENCE OF A ONE-PARTICLE STRUCTURE FOR EQUATION (0.1) (IN THE STATIONARY CASE)

The main result of this paper is on the existence of a one-particle structure for the linear dynamical system we constructed in §2 for our equation:

4.1 Theorem: Given equation (0.1) on Stat($\mathcal{C}$ ^3g α β).

Suppose (i) V is stationary and satisfies $V(x) > -m^2 + \epsilon$ everywhere for some $\epsilon > 0$.

(ii) $\mathcal{C}$ is Cauchy in Stat($\mathcal{C}$ ^3g α β)

(iii) α is bounded below away from zero:

$\alpha > \epsilon_1 > 0$ for some ϵ_1

(iv) $\alpha - \beta^i\beta_i/\alpha$ is bounded below away form zero:

$\alpha - \beta^i\beta_i/\alpha > \epsilon_2 > 0$ for some ϵ_2.

Then, on the corresponding linear dynamical system (as in §2) there exists a (unique) one particle structure .

Moreover, letting h be the self-adjoint generator of $U(t)$: $U(t) = e^{-ith}$

(1) h has bounded inverse (i.e. there is a "mass gap")

(2) $KD + i\,KD$ is an invariant domain of essential self-adjointness for h.

Note (a) that condition (i) may be interpreted as avoiding "Klein paradox" situations; condition (iii) prevents the Killing vector from becoming arbitrarily small; condition (iv) prevents it from approaching a light-like vector.

(b) In the case of compact $\mathcal{C}$, (ii), (iii), (iv) are automatically satisfied.

In §7, we prove this theorem, and in §8, we give a more detailed discussion of the static case when it is possible to construct $(K, \mathcal{H}, U(t))$ in a more concrete way.

§5 WHEN IS $\mathcal{C}$ CAUCHY IN STAT($\mathcal{C}$ ${}^3g$ α β)?

We use the

5.1 Lemma: Let $\mathcal{S}$ be a partial Cauchy surface in some space-time $(\mathcal{M}, g)$; then $\mathcal{S}$ is a global Cauchy surface if and only if every inextendible null geodesic in $\mathcal{M}$ intersects $\mathcal{S}$.

Note: With slightly different definitions for the technical terms, this would be a special case of 'Property 6' of Geroch [9]. A proof in our case (i.e. following the definitions of H. E.) can easily be constructed by combining the Corollary to H.E. Prop. 6.5.3 with the easily proved fact that a partial Cauchy surface is a global Cauchy surface if and only if its Horizon is empty. For details see [13].

Firstly, in the "special static case of lapse 1 and shift 0", we have:

5.2 Proposition: $\mathcal{C}$ is Cauchy in Stat($\mathcal{C}$ ${}^3g$ 1 0) if and only if $(\mathcal{C}\ {}^3g)$ is complete.

(Recall: [11] p. 56: for Riemannian manifolds, geodesic completeness is equivalent to metric space completeness in the metric of geodesic length)

Proof: It is easy to see that the null geodesics in

$$\begin{pmatrix} 1 & 0 \\ 0 & -{}^3g \end{pmatrix}$$ can all be parametrized:

$$t \longmapsto (t, \gamma(t)) \qquad \text{(A)}$$

where $t \longmapsto \gamma(t)$ is a geodesic in $(\mathcal{C}\ {}^3g)$ parametrized by its Riemannian length. We have, (Lemma 5.1) $\mathcal{C}$ is Cauchy $\Longleftrightarrow$ the image of every inextendible null geodesic cuts every $\{t\}\times\mathcal{C}$ $\Longleftrightarrow$ (by (A) above) every inextendible geodesic

$t \mapsto \gamma(t)$ is defined for all $t \in (-\infty, \alpha) \Leftrightarrow (\mathcal{C}\ {}^3g)$ is complete.

$\square$

For more general lapse and shift, we have partial results:

5.3 Proposition: A sufficient condition for $\mathcal{C}$ Cauchy in Stat($\mathcal{C}\ {}^3g\ \alpha\ \beta$) is if $(\mathcal{C}\ {}^3g)$ complete and α bounded on $\mathcal{C}$.

Proof: We first prove the

Lemma: Suppose $\mathcal{C}$ is complete in some Riemannian metric ${}^3h$ (perhaps, but not necessarily ${}^3g$) and $\exists\ \epsilon > 0$ s.t.

$${}^3h_{ij}\, n^i n^j / n^{o2} < \epsilon$$

for every null-vector $(n^o\ n^i)$ in the tangent bundle. Then, $\mathcal{C}$ is Cauchy in Stat($\mathcal{C}\ {}^3g\ \alpha\ \beta$)

Proof: Parametrize null geodesics in Stat($\mathcal{C}\ {}^3g\ \alpha\ \beta$) by t (thus $t \mapsto (t, \gamma(t))$). Suppose an inextendible such geodesic has l.u.b. $\{t\} = T$. Let $t_i \to T$, then $\{\gamma(t_i)\}$ is Cauchy since

$$d(\gamma(t_i), \gamma(t_j)) \leq \int_{t_i}^{t_j} \left({}^3h_{ij} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\right)^{1/2} dt < \epsilon^{1/2}(t_j - t_i)$$

$\therefore \gamma(t_i) \to \gamma(T)$ (say) (since $t \mapsto \gamma(t)$ continuous and $(\mathcal{C}\ {}^3h)$ complete). Also, (using ${}^3h(\dot\gamma(t_i)\dot\gamma(t_j)) < \epsilon$ again) we easily have $(1, \dot\gamma(t_i)) \to (1, \dot\gamma(T))$ ($\dot\gamma(T)$ "left tangent at T ") in the tangent bundle to our space-time whereupon, we can extend the geodesic beyond $(T, \gamma(T))$ contra. hypoth. – so the geodesic is defined for all $t > T$. Similarly, it is extendible into the past.

$\therefore$ every inextendible null geodesic takes on all values of t.

$\therefore$ $\mathcal{C}$ is Cauchy.

Proof of Theorem: It remains to verify the conditions of our Lemma. Now, given a null vector $(n^o\ n^i)$, we have (using eq. 1.2)

$$(\alpha^2 - \beta^i\beta_i)n^{o2} - 2n^o\beta^i n_i - {}^3g_{ij}n^i n^j = 0$$

i.e. $$g_{ij}\left(\frac{n^i}{n^o} + \beta^i\right)\left(\frac{n^j}{n^o} + \beta^j\right) = \alpha^2$$

In other words: calling n^i/n^o "N" and writing $\|\ \ \|$ for the Euclidean norm in our tangent space:

$$\| N + \beta \| = \alpha$$

also, $\alpha^2 - \beta^i\beta_i > 0 \Rightarrow \|\beta\| < \alpha$

so $\|N\| \leq \|N + \beta\| + \|\beta\| \leq 2\alpha$

i.e. ${}^3g_{ij}\, n^i n^j / n^{o2} \leq 4\alpha^2$

$\square$

5.4 Corollary $\mathcal{C}$ compact implies $\mathcal{C}$ Cauchy in Stat($\mathcal{C}\ {}^3g\ \alpha\ \beta$) for any ${}^3g\ \alpha\ \beta$ on $\mathcal{C}$.

Proof: $(\mathcal{C}\ {}^3g)$ complete, since in metric spaces compact $\Rightarrow$ complete

Finally, $\mathcal{C}$ compact and α continuous $\Rightarrow$ α bounded. $\square$

5.5 Examples

Intuitively, $\mathcal{C}$ fails to be Cauchy in Stat($\mathcal{C}\ {}^3g\ \alpha\ \beta$) if signals (null or timelike curves) can hit the "edge" in a finite time. We give a series of examples for 2 dimensional space-times.

(1) Choose $\mathcal{C} = (0, 1)$ with usual metric, $\alpha = 1$, $\beta = 0$
$\mathcal{C}$ is not Cauchy. (see figure 1)

(2) We could also envisage the lapse function growing too fast at "infinity " on a complete manifold:–

Take the previous example under a conformal transformation which sends $(0, 1)$ (usual metric) to $(-\infty, \infty)$ (usual metric) i.e. stretches it out lengthways thus making $(\mathcal{C}, g)$ complete but preserving the causal structure. (Note: conformal transformations on a space-time always preserve the causal structure)

See figure 2

(3) Finally, we show that if the lapse function <u>shrinks</u> fast enough, $(\mathcal{C}, g)$ need not be geodesically complete for $\mathcal{C}$ to be Cauchy. Take the "Rindler Wedge" i.e. one of the connected pieces of 2 dimensional Minkowski space-time which is spacelike separated from some origin O. Define α and β by taking for $\frac{\partial}{\partial t}$ the Killing vector tangent to Lorentz boosts.

$\mathcal{C} = (O, \infty)$ is clearly Cauchy.

See figure 3

§6 THE UNIQUENESS OF ONE PARTICLE STRUCTURES IN GENERAL

In this section, we prove Theorem 3.1

<u>Proof</u>: Suppose there are two such one particle structures $(K_1\ \mathcal{H}_1\ U_1(t)), (K_2\ \mathcal{H}_2\ U_2(t))$: Then $T := K_2 \circ K_1^{-1}$ is a real, linear invertible, symplectic from the real linear dense domain $K_1 D_1$ in $\mathcal{H}_1$ to the real linear dense domain $K_2 D$ in $\mathcal{H}_2$. Also (3) of equation 3.3 implies $K_1 D$ is invariant for $U_1(t)$, and $K_2 D$ is invariant for $U_2(t)$.

We shall show that T extends to a unitary.

We have (a) $T U_1(t) = U_2(t) T$ on $K_1 D$

(b) $T^+ \supset -i T^{-1} i$ on domain $i K_2 D$ where

T^+ denotes the real adjoint of T — $\mathcal{H}_{1,2}$ considered as real Hilbert spaces (i.e. with inner product $\mathrm{Re}\langle \cdot | \cdot \rangle$)

((b) easily follows since T symplectic)

We now show that T has a closed complex linear extension:-

Consider

$$f_{x,y} : t \mapsto \left(\langle U_1(t) x | T^+ y\rangle_{\mathcal{H}_1} - \langle U_2(t) T x | y\rangle_{\mathcal{H}_2}\right)$$
$$\forall x \in K_1 D \;;\; y \in D(T^+)$$

By a well known argument (see e.g. [25]) the strictly positive energy of $U_1(t), U_2(t)$ guarantees that $f_{x,y}$ extends to a function which is bounded and holomorphic in the lower half t-plane, vanishes as Im t $\to -\infty$ and is continuous and bounded on the real axis. We also have $\mathrm{Re}\, f_{x,y}(t)$ vanishes on the real axis by definition of the real adjoint. The Schwartz reflection principle then implies that $f_{x,y}(t)$ is bounded and holomorphic in the plane (and vanishes at ∞). So by Liouville's theorem it vanishes everywhere.

We then have, setting $t = 0$

$$\langle x | T^+ y\rangle_{\mathcal{H}_1} = \langle T x | y \rangle_{\mathcal{H}_2} \quad \forall x \in K_1 D,\ y \in D(T^+)$$

whereupon $\langle Tx | iy\rangle = i\langle Tx|y\rangle = i\langle x|T^+y\rangle = \langle x | iT^+ y\rangle$

showing $iy \in D(T^+)$ and $T^+(iy) = iT^+y$

$\therefore$ T^+ is complex linear, whereupon $\bar{T} = T^{++}$ is complex linear. Taking the closure of both sides of (b), we recover

$$\bar{T}^* \supset -i\bar{T}^{-1} i = \bar{T}^{-1}$$

since, for the complex linear $\bar{T}$, the real adjoint $\bar{T}^+$ and complex adjoint $\bar{T}^*$ coincide. Whereupon, $\bar{T}$ is a closed norm preserving map with dense range - hence unitary. ∎

§7 PROOF OF THE EXISTENCE THEOREM

In this section, we prove Theorem 4.1. At a heuristic level, the strategy is well known (introduction of "complex structure" etc. [1, 2, 6, 12, 22, 23]). The main technical steps involve proving the essential self-adjointness of certain Hilbert space operators related to our differential equation. For these essential self-adjointness proofs, it turns out that Leray's theorem combines nicely with a well known Lemma of Nelson [18]. In fact, in §8, we shall need to use a generalization of Nelson's Lemma due to Chernoff which we now state:

7.1 Chernoff's Lemma [3]: Let T be a symmetric operator with dense domain $\mathcal{D} \subset \mathcal{H}$ a (complex) Hilbert space. Suppose T maps $\mathcal{D}$ into itself. Suppose in addition that there is a one-parameter group $V(t)$ of unitary operators on $\mathcal{H}$ such that $V(t)\mathcal{D} \subset \mathcal{D}$, $V(t)T = TV(t)$ on $\mathcal{D}$ and

$$\frac{d}{dt} V(t)u = iTV(t)u \quad ; u \in \mathcal{D}$$

Then T^n is essentially self-adjoint for all n.

(We shall refer to "Nelson's Lemma" when we only use the case $n = 1$)

Proof of Theorem 4.1: We proceed in 5 stages:-

(1) $\langle \phi | \mathbb{A} \phi \rangle_{L^2+L^2} > \epsilon \| \phi \|^2_{L^2+L^2}$ for some $\epsilon > 0$

Proof: by equation 2.4:

$$\langle \phi | \mathbb{A} \phi \rangle_{L^2+L^2} = \int_\Sigma d\eta \, \alpha \left(p^2 + {}^3g^{ij} \partial_i f \partial_j f + (m^2+V) f^2\right) + 2 \int_\Sigma d\eta \, p \beta^i \partial_i f$$

$$= \int_\Sigma d\eta \, \alpha \left\{ \left(1 - \frac{\beta_i \beta^i}{\alpha^2}\right) p^2 + {}^3g^{ij} \left(\partial_i f + \frac{\beta_i}{\alpha} p\right)\left(\partial_j f + \frac{\beta_j}{\alpha} p\right) + (m^2 + V) f^2 \right\}$$

$$\geq \int_\Sigma d\eta \left\{ \epsilon_2 p^2 + \epsilon_1 \epsilon f^2 \right\}$$

($\epsilon, \epsilon_1, \epsilon_2$ as in §4)

$$\geq \min(\epsilon_2, \epsilon_1 \epsilon) \int d\eta \, (p^2 + f^2)$$

(2) Define $\mathcal{A}$: the completion of $D(\Sigma)$ in $\mathbb{A}$ norm ($\mathbb{A}$ norm: $\|\phi\|^2_{\mathcal{A}} := \langle \phi | \mathbb{A} \phi \rangle_{L^2+L^2}$)

then

(a) $\mathbb{h} : D \to D$ and is skew symmetric on $D \subset \mathcal{A}$

(b) σ is continuous in $\mathbb{A}$ norm i.e.

$$\sigma(\phi, \psi) \leq C \|\phi\|_{\mathcal{A}} \|\psi\|_{\mathcal{A}}$$

(c) $\mathcal{J}(t) : D \to D$ extends to a strongly continuous unitary group $\mathcal{J}'(t)$ with $\frac{d\mathcal{J}'(t)}{dt}\Big|_{t=0} \overset{s}{=} -\mathbb{h}$ on D and $\mathcal{J}'(t)\mathbb{h} = \mathbb{h}\mathcal{J}'(t)$ on D.

Proof

(a) Clearly, $\mathbb{h} : D \to D$ since $\mathbb{h} = -g\mathbb{A}$ and both g and $\mathbb{A}$ are matrices of C^∞ differential operators.

Let $\phi, \psi \in D$ then $\langle \phi | \mathbb{h} \psi \rangle_{\mathcal{A}} = \langle \phi | \mathbb{A}(-g\mathbb{A}) \psi \rangle_{L^2+L^2}$
$= \langle g\mathbb{A}\phi | \mathbb{A}\psi \rangle_{L^2+L^2}$ (since $g^\dagger = -g$ on L^2+L^2)
$= -\langle \mathbb{h}\phi | \psi \rangle_{\mathcal{A}}$

(b) $\sigma(\phi, \psi) = \langle \phi | g \psi \rangle_{L^2+L^2} \leq \|\phi\|_{L^2+L^2} \|\psi\|_{L^2+L^2}$ (since g bounded in L^2+L^2) $\leq \frac{1}{\epsilon^2} \|\phi\|_{\mathcal{A}} \|\psi\|_{\mathcal{A}}$ by (1) above.

(c) follows easily after checking that $\mathcal{J}(t)$ is:

(i) isometric on D

(ii) strongly continuous on D

(iii) $\frac{d}{dt}\mathcal{J}(t)\Big|_{t=0} \overset{s}{=} -\mathbb{h}$ on D

(i) follows from

$$\frac{d}{dt} \langle \mathcal{J}(t)\phi | \mathcal{J}(t)\phi \rangle_{\mathcal{A}} = \frac{d}{dt} \int \widetilde{\mathcal{J}(t)\phi} \, \mathbb{A} \, \mathcal{J}(t)\phi \, d\eta = 0$$

(ii) from $\lim\limits_{t\to 0} \|(\mathcal{J}(t)-1)\phi\|^2 = \lim\limits_{t\to 0} \int \overline{(\mathcal{J}(t)-1)\phi}\; \mathbb{A}\,(\mathcal{J}(t)-1)\phi\, d\eta = 0$

and (iii) from

$$\lim_{t\to 0} \left\| \left(\frac{\mathcal{J}(t)-1}{t} + \mathbb{h}\right)\phi \right\| = \lim_{t\to 0} \int \overline{\left(\frac{\mathcal{J}(t)-1}{t} + \mathbb{h}\right)\phi}\; \mathbb{A}\left(\frac{\mathcal{J}(t)-1}{t} + \mathbb{h}\right)\phi\, d\eta = 0$$

In each case, the last equality following by interchanging the derivative in (i) or limit in (ii) and (iii) with the integral sign and applying the pointwise results ($\lim (\mathcal{J}(t)-1) = 0$,

$$\left.\frac{d\,\mathcal{J}(t)}{dt}\right|_{t=0} = -\mathbb{h}\,\mathcal{J}(t)$$)

– In each case the conditions justifying the interchange being that the integrand (and its t derivative for (i)) is a jointly continuous function of t and x and that the integral over x is over a compact set with finite measure (cf Dieudonné [7] I p. 176 § VII.11) These conditions holding thanks to Leray's theorem.

(3)

(a) $\mathbb{h}$ is essentially skew-adjoint on D

(b) $\overline{\mathbb{h}}^{-1}$ exists and is bounded

(c) σ extends uniquely to σ' on $\mathcal{A}$. Moreover

$$\sigma'(\phi\ \psi) = \langle \phi \,|\, \overline{\mathbb{h}}^{-1}\psi\rangle_{\mathcal{A}}$$

Proof

3 (a) follows by Nelson's Lemma[3] using 2 (a) and 2(c). 2(b) and the Riesz lemma imply $\exists$ a bounded skew adjoint operator T s.t.

$$\sigma(\phi\ \psi) = \langle \phi \,|\, T\psi\rangle_{\mathcal{A}} \qquad \phi, \psi \in D$$

---

[3] applied to the "natural complexification" $\mathcal{A}_{\mathbb{C}}$ of $\mathcal{A}$:
i.e. $\mathcal{A} + \mathcal{A}$ with $i\,(x, y) = (-y, x)$,

$$\langle (x_1\, y_1) \,|\, (x_2\, y_2)\rangle_{\mathcal{A}_{\mathbb{C}}} = \langle x_1|x_2\rangle_{\mathcal{A}} + \langle y_1|y_2\rangle_{\mathcal{A}} + i\left(\langle x_1|y_2\rangle_{\mathcal{A}} - \langle y_1|x_2\rangle_{\mathcal{A}}\right)$$

This last equation then extends σ uniquely by continuity to $\mathcal{A}$ – call the extension σ'. Now note: $\phi, \psi \in D \Rightarrow$

$$\langle \phi \,|\, T\mathbb{h}\psi\rangle_{\mathcal{A}} = \sigma(\phi\ \ -\mathfrak{g}\mathbb{A}\psi) = \langle \phi \,|\, \mathfrak{g}(-\mathfrak{g}\mathbb{A})\psi\rangle_{L^2+L^2}$$
$= \langle \phi, \psi\rangle_{\mathcal{A}}$ i.e. $T\mathbb{h} = 1$ on D (*)

$\therefore$ ran T dense and T^{-1} is an unbounded skew-adjoint operator with $D(T^{-1}) = \operatorname{Ran} T$ (see e.g. Rudin [20] thm. 13.11 (b))

(*) $\Rightarrow T^{-1} = \mathbb{h}$ on D

$\therefore\ T^{-1} = \overline{\mathbb{h}}$ since $\mathbb{h}$ essentially skew-adjoint on D

We have thus proved 3(b), 3(c)

Note: By this stage, we have extended dynamics to the linear dynamical system $(\mathcal{A}, \sigma', \mathcal{J}'(t))$ – $(\mathcal{A}, \sigma')$ represents physically the phase space of all finite-energy classical solutions. Mathematically, the fact that our enlarged symplectic space is a (real) Hilbert space (and not just a prehilbert space) allows us to complete the construction of K and $\mathcal{H}$:–

(4) Construction of $(K, \mathcal{H}, U(t))$

Firstly, we seek a "complex structure" J on $\mathcal{A}$ i.e. a real unitary operator satisfying

(i) $J^2 = -1$

(ii) $J\overline{\mathbb{h}} = \overline{\mathbb{h}}J$

(iii) $\sigma'(\phi\ J\psi) > 0 \quad (\Leftrightarrow \overline{\mathbb{h}}^{-1}J \geq 0)$

The following constructions for J are easily seen to be equivalent and to satisfy (i),(ii),(iii) above:–

Either define J via the polar decomposition

$$J = |\overline{\mathbb{h}}|^{-1}\,\overline{\mathbb{h}} \quad (= \overline{\mathbb{h}}\,|\overline{\mathbb{h}}|^{-1})$$

Or, equivalently, note that $i(\bar{h} + \bar{h})$ on $\mathcal{A}_c$ (see footnote 3) is self-adjoint. Defining P^+, P^- the projections onto the positive and negative parts of its spectrum, we have

$$J = i(P^+ - P^-)$$

– restricted to $\mathcal{A} \oplus 0$

(This is the appropriate generalization of the familiar "positive/negative frequency decomposition")

Finally, we define $(K, \mathcal{H}, U(t))$, $K : \mathcal{A} \to \mathcal{H}$ using J :-

(a) $(a + ib)K(\phi) := K(a\phi) + K(bJ\phi) \quad a, b \in \mathbb{R}$

(b) $\langle K(\phi) | K(\psi) \rangle_{\mathcal{H}} := \frac{1}{2}\sigma'(\phi\ J\psi) + \frac{1}{2} i \sigma(\phi\ \psi)$
$\qquad \equiv \frac{1}{2} \langle \phi | |\bar{h}|^{-1} \psi \rangle + \frac{1}{2} i \langle \phi | \bar{h}^{-1} \psi \rangle_{\mathcal{A}}$

(c) $\mathcal{H}$ is the completion of $K(\mathcal{A})$ in $\mathcal{H}$ norm given by (b) (it being easily checked that $K(\mathcal{A})$ defined by (a),(b) is a complex prehilbert space)

(d) $U(t)K(\phi) := K(e^{-\bar{h}t}\phi)$

We need to check that KD is dense in $\mathcal{H}$:-

we have $\|K(\phi)\|^2_{\mathcal{H}} = \frac{1}{2}\sigma'(\phi\ J\phi) = \frac{1}{2}\langle \phi | |\bar{h}|^{-1} \phi \rangle_{\mathcal{A}}$
$\qquad \leq \| |\bar{h}|^{-1} \|_{\mathcal{A}} \|\phi\|^2_{\mathcal{A}}$

verifying K continuous from $D \subset \mathcal{A}$ to $\mathcal{H}$

The result follows since in topological spaces, D dense in S_1 and $K : D \to S_2$ continuous $\Rightarrow KD$ dense in S_2.

That K is symplectic follows from (b)

$U(t)$ as defined in (d) is clearly an isometric group on $K(\mathcal{A})$ in $\mathcal{H}$; in fact it is easy to check it preserves both real and imaginary parts of the inner product since $\bar{h}, |\bar{h}|, e^{-\bar{h}t}$ all commute. We extend it by continuity to $\mathcal{H}$ calling the extension $U(t)$ also.

The strictly positive energy is verified in :-

(5) Verification of last paragraph of theorem

Using $K : \mathcal{A} \supset D \to \mathcal{H}$ continuous (proved in (4) above) and $e^{-\bar{h}t} : D \to D$ ($y'(t) = e^{-\bar{h}t}$ by 2(c), 3(a) and Stone's theorem) we have by 4(d) that $U(t)$ is strongly differentiable on KD. Writing $U(t) = e^{-iht}$ we have from 4(d)

$$-i\,h\,K(\phi) = K(-\bar{h}\phi) = -K(\bar{h}\phi)$$

Clearly, all the hypotheses of Nelson's lemma then hold for $U(t)$, h by taking as our invariant (complex linear) domain $K(D) + iK(D)$ so h is essentially self-adjoint on $K(D) + iK(D)$ ($= K(D + JD)$)

To show that h has bounded inverse we proceed as in the proof of 3(b) above. Firstly, we show that $F := i K \bar{h}^{-1} K^{-1} \equiv K|\bar{h}|^{-1}K^{-1}$ is bounded and positive on $K(\mathcal{A}) \subset \mathcal{H}$. (Note; $K(\mathcal{A})$ is a complex linear domain containing $K(D) + iK(D)$ and F is easily seen to be complex linear:

F bounded: $\|FK\phi\|^2_{\mathcal{H}} = \|K|\bar{h}|^{-1}\phi\|^2_{\mathcal{H}}$

$= \frac{1}{2}\langle |\bar{h}|^{-1}\phi \,|\, |\bar{h}|^{-1}\bar{h}^{-1}\phi \rangle_{\mathcal{A}} = \frac{1}{2}\langle (|\bar{h}|^{-1})^2 (|\bar{h}|^{-1/2}\phi \,|\, |\bar{h}|^{-1/2}\phi \rangle_{\mathcal{A}}$

$\leq \frac{1}{2}\|\bar{h}^{-1}\|^2_{\mathcal{A}} \langle \phi | |\bar{h}|^{-1}\phi \rangle_{\mathcal{A}} = \frac{1}{2}\|\bar{h}^{-1}\|^2_{\mathcal{A}} \|\phi\|^2_{\mathcal{H}} \quad \phi \in D$

F positive on KD :

$\langle K\phi | FK\phi \rangle_{\mathcal{H}} = \langle K\phi | K|\bar{h}|^{-1}\phi \rangle_{\mathcal{H}} = \frac{1}{2}\langle \phi | |\bar{h}|^{-2}\phi \rangle_{\mathcal{A}}$

$= \frac{1}{2}\langle |\bar{h}|^{-1}\phi \,|\, |\bar{h}|^{-1}\phi \rangle_{\mathcal{A}} > 0$

Calling the continuous extension of F, F also – which is then a bounded positive operator; we easily have $Fh = 1$ on $KD + iKD$ whereupon $F^{-1} \supset (h \text{ on } KD + iKD)$ and hence (since h is essentially self-adjoint on $KD + iKD$)

$$F^{-1} = \bar{h}$$

$\square$

§8 THE SPECIAL STATIC CASES

When our space-time is not just stationary but static, i.e. when we have lapse but no shift; the matrix $\mathbb{A}$ (eq 2.4) becomes diagonal:

$$\mathbb{A} = \begin{pmatrix} -(\partial^i\alpha)\partial_i + \alpha(m^2 - \Delta(\mathcal{C}) + V) & 0 \\ 0 & \alpha \end{pmatrix} \quad (8.1)$$

and we can construct K and $\mathcal{H}$ in a more concrete way. The simplest case to deal with is when the lapse function is actually 1 i.e. when the unit normals coincide with the time-like Killing vectors:-

$$\mathbb{A} = \begin{pmatrix} m^2 - \Delta(\mathcal{C}) + V & 0 \\ 0 & 1 \end{pmatrix} \quad (8.2)$$

8.1 Theorem (cf Chernoff [3] last paragraph)

Let V satisfy (i) of Theorem 4.1 and $(\mathcal{C}\ {}^3g)$ be complete. Then $m^2 - \Delta(\mathcal{C}) + V$ is essentially self-adjoint on $C_0^\infty(\mathcal{C}) \subset L^2(\mathcal{C}\ d\eta)$ - its closure being positive, invertible.

Proof

Completeness of $(\mathcal{C}\ {}^3g)$ gives $\mathcal{C}$ Cauchy in Stat($\mathcal{C}\ {}^3g\ \ 1\ \ 0$) by Proposition 5.2. Calling $m^2 - \Delta(\mathcal{C}) + V$ "A", we have

$$\langle x | A x \rangle_{L^2} \geq \epsilon \langle x | x \rangle_{L^2} \quad (8.3)$$
$$x \in C_0^\infty(\mathcal{C})$$

- ϵ as in (i) of theorem 4.1

Now, 2(a) and 2(c) of the proof of theorem 4.1 allow us to conclude by Chernoff's lemma that $\bar{h}^2$ is essentially self-adjoint on $C_0^\infty(\mathcal{C}) + C_0^\infty(\mathcal{C})$ in $\mathcal{A}$. ($h = -g\mathbb{A}$)

Here: $\langle f_1\ p_1 | f_2\ p_2 \rangle_{\mathcal{A}} = \langle f_1 | A f_2 \rangle_{L^2} + \langle p_1 | p_2 \rangle_{L^2}$

i.e. $\mathcal{A} = \mathcal{A}_1 + L^2$

where $\mathcal{A}_1$ is the completion of C_0^∞ in $\langle \cdot | A \cdot \rangle_{L^2}$

and L^2 is $L^2(\mathcal{C}\ \eta)$

Also

$$h = -g\mathbb{A} = \begin{pmatrix} 0 & -1 \\ A & 0 \end{pmatrix} \quad \therefore \quad h^2 = \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}$$

whereupon (by projection) A essentially self adjoint in $C_0^\infty(\mathcal{C}) \subset L^2(\mathcal{C}\ \eta)$

Furthermore, $\bar{A}$ being a unique s.a. extension must coincide with the Friedrichs extension (see Reed and Simon II [19] §X.3) which, having the same lower bound as A, has (by eq 8.3) lower bound to its spectrum ϵ. $\square$

We then obtain a concrete form for $(K, \mathcal{H}, U(t))$ (easily checked to satisfy the necessary properties and therefore equivalent to the unique one particle structure constructed in Theorem 4.1)

Let $\mathcal{H} = L^2_{\mathbb{C}}(\mathcal{C}, \eta)$ and define

k (replacing K): $C_0^\infty(\mathcal{C}) + C_0^\infty(\mathcal{C}) \longrightarrow L^2_{\mathbb{C}}(\mathcal{C}\ \eta)$

$$(f, p) \mapsto \frac{\bar{A}^{1/4} f + i \bar{A}^{-1/4} p}{\sqrt{2}}$$

$k(f, p)$ generalizes the familiar Newton-Wigner wave-function

– <u>note</u> we have a preferred complex conjugation on $\mathcal{H}$. (The natural one on $L^2_c(\mathcal{C}\ \eta_i\ !)$ representing time-reversal.

Finally, $U(t)$ is represented by $\exp(-i\,\bar{A}^{1/2}t)$

i.e. h becomes $\bar{A}^{1/2}$.

We now deal with the slightly more complicated general case (eq 8.1)

Abbreviating $-(\partial^i\alpha)\partial_i + \alpha(m^2 - \Delta(\mathcal{C}) + V)$ to A also, we have

$$\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}$$

<u>8.2 Theorem: Let</u> V <u>satisfy (i) of Theorem 4.1;</u> $(\mathcal{C}\ {}^3g)$ <u>be complete and</u> α <u>be bounded above and below away from zero.</u>

$$0 < \epsilon_1 < \alpha < C$$

<u>Then</u> $A(= -(\partial^i\alpha)\partial_i + \alpha(m^2 - \Delta\mathcal{C}) + V)$ <u>and</u> α <u>are essentially self-adjoint on</u> $C_0^\infty(\mathcal{C}) \subset L^2(\mathcal{C}\,\eta)$ – <u>their closure being positive, invertible.</u>

<u>Proof</u>

Completeness of $(\mathcal{C}\ {}^3g)$ together with α bounded above gives by proposition 5.3 that $\mathcal{C}$ is Cauchy in Stat($\mathcal{C}\ {}^3g\ \alpha\ 0$)

V satisfying (i) of th. 4.1 and $\alpha > \epsilon_1 > 0$ gives

$$\langle x \mid A x\rangle_{L^2} \geqslant \epsilon' \langle x \mid x\rangle_{L^2} \qquad (8.4)$$
$$\text{for some } \epsilon' > 0$$

Now, note α bounded above and below away from zero gives immediately that – as a multiplication operator on $L^2(\mathcal{C}\ \eta)$,

$\bar{\alpha}$ is bounded, invertible, positive self-adjoint. We can deal with A by an extension of Chernoff's method (see previous theorem) exploiting the "very nice properties of α":–

Write $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$

where $\mathcal{A}_1$ is the completion of C_0^∞ in $\langle\cdot \mid A\,\cdot\rangle$ norm

and $\mathcal{A}_2$ is the completion of C_0^∞ in $\langle\cdot \mid \alpha\,\cdot\rangle$ norm

Also $\mathbb{h} = -g\mathbb{A} = \begin{pmatrix} 0 & -\alpha \\ A & 0 \end{pmatrix}$

therefore $\mathbb{h}^2 = \begin{pmatrix} -\alpha A & 0 \\ 0 & -A\alpha \end{pmatrix}$

By restriction, $A\alpha$ is ess. s.a. on $C_0^\infty \subset \mathcal{A}_2$.

Now, $U : \mathcal{A}_2 \longrightarrow L^2(\mathcal{C}\ \eta)$

defined by $C_0^\infty(\mathcal{C}\ \eta) \ni p \longmapsto \alpha^{1/2}p$

is unitary. Moreover $U : C_0^\infty \rightarrow C_0^\infty$

Hence, $\alpha^{\frac{1}{2}}A\alpha^{\frac{1}{2}}$ is ess. s.a. on $C_0^\infty \subset L^2$

(since it is unitarily equivalent to A on $C_0^\infty \subset \mathcal{A}_2$)

Now, using eq (8.4) and the fact that is bounded below away from zero: we get (Friedrichs extension = s.a. extension since unique)

$\overline{\alpha^{\frac{1}{2}}A\alpha^{\frac{1}{2}}}$ positive s.a. invertible.

Now, $\bar{\alpha}^{\frac{1}{2}}$ is bounded invertible and maps C_0^∞ onto C_0^∞

We therefore have $\overline{\alpha^{\frac{1}{2}}A\alpha^{\frac{1}{2}}} = \overline{\bar{\alpha}^{\frac{1}{2}}A\bar{\alpha}^{\frac{1}{2}}} = \bar{\alpha}^{\frac{1}{2}}\bar{A}\bar{\alpha}^{\frac{1}{2}}$

(proof that $\overline{\bar{\alpha}^{\frac{1}{2}}A\bar{\alpha}^{\frac{1}{2}}} = \bar{\alpha}^{\frac{1}{2}}\bar{A}\bar{\alpha}^{\frac{1}{2}}$:–

Note that $\bar{\alpha}^{\frac{1}{2}} : L^2 \rightarrow L^2$ homeorphism in norm topology

$\therefore\ \bar{\alpha}^{-\frac{1}{2}} \oplus \bar{\alpha}^{\frac{1}{2}} : L^2 + L^2 \longrightarrow L^2 + L^2$

$$\{p, Ap\} \longmapsto \{\bar{\alpha}^{-\frac{1}{2}}p,\ (\bar{\alpha}^{\frac{1}{2}}A\bar{\alpha}^{\frac{1}{2}})\bar{\alpha}^{\frac{1}{2}}p\}$$

is a homeorphism sending the graph of A to the graph of $\bar{\alpha}^{\frac{1}{2}}A\bar{\alpha}^{\frac{1}{2}}$

$\therefore$ it sends the graph of $\bar{A}$ to the graph of $\overline{\bar{\alpha}^{\frac{1}{2}}A\bar{\alpha}^{\frac{1}{2}}}$ which must therefore equal $\bar{\alpha}^{\frac{1}{2}}\bar{A}\bar{\alpha}^{\frac{1}{2}}$.)

Finally, then, $\bar{A}$ must be self-adjoint (and positive with bounded inverse) since $\bar{A}^{-1} = \bar{\alpha}^{\frac{1}{2}}\bar{\alpha}^{-\frac{1}{2}}\bar{A}^{-1}\bar{\alpha}^{-\frac{1}{2}}\bar{\alpha}^{\frac{1}{2}}$

$$= \bar{\alpha}^{\frac{1}{2}}(\bar{\alpha}^{\frac{1}{2}}\bar{A}\bar{\alpha}^{\frac{1}{2}})^{-1}\bar{\alpha}^{\frac{1}{2}} = \bar{\alpha}^{\frac{1}{2}}(\overline{\bar{\alpha}^{\frac{1}{2}}A\bar{\alpha}^{\frac{1}{2}}})^{-1}\bar{\alpha}^{\frac{1}{2}}$$

which is bounded, self-adjoint, positive. $\square$

We sketch a "concrete" description of $(\mathcal{K}, \mathcal{H}, U(t))$

(1) $\bar{A}$, $\bar{\alpha}$ positive, self-adjoint, invertible allow us to identify $\mathcal{A}$ as $L^2(\mathcal{C}\,\eta) + L^2(\mathcal{C}\,\eta)$

($\mathcal{A}_1 \supset C_0(\mathcal{C}) \ni f \longmapsto \bar{A}^{\frac{1}{2}} f \in L^2(\mathcal{C}\,\eta)$, and

$\mathcal{A}_2 \supset C_0^\infty(\mathcal{C}) \ni p \longmapsto \bar{\alpha}^{\frac{1}{2}} p \in L^2(\mathcal{C}\,\eta)$ define unitaries)

With this identification, it is easy to follow through the construction in §7 to get

$$J = \begin{pmatrix} 0 & -j \\ j^+ & 0 \end{pmatrix} \tag{8.5}$$

where the real-unitary operator $j = \bar{A}^{\frac{1}{2}} \bar{\alpha}^{\frac{1}{2}} (\bar{\alpha}^{\frac{1}{2}} \bar{A} \bar{\alpha}^{\frac{1}{2}})^{-\frac{1}{2}}$

Ashtekar and Magnon [1] have previously given an equivalent (non-rigorous) formula.

(2) From J above, there are several (unitarily equivalent) analogues to Newton-Wigner fields in which $\mathcal{H}$ appears as $L^2_{\mathbb{C}}(\mathcal{C}\,\eta)$

Take

$$\begin{aligned} k:\ C_0^\infty(\mathcal{C}) + C_0^\infty(\mathcal{C}) &\longrightarrow L^2_{\mathbb{C}}(\mathcal{C}\,\eta) \\ (f, p) &\longmapsto \frac{Xf + iX^{+-1}p}{\sqrt{2}} \end{aligned} \tag{8.6}$$

where X is closed and satisfies:

$$X^+ X = \bar{\alpha}^{-\frac{1}{2}} j^+ \bar{A}^{\frac{1}{2}} = \bar{\alpha}^{-\frac{1}{2}} (\bar{\alpha}^{\frac{1}{2}} \bar{A} \bar{\alpha}^{\frac{1}{2}})^{\frac{1}{2}} \bar{\alpha}^{-\frac{1}{2}}$$

E.g. let $X = (\bar{\alpha}^{\frac{1}{2}} \bar{A} \bar{\alpha}^{\frac{1}{2}})^{1/4} \bar{\alpha}^{-\frac{1}{2}}$

We then have the one particle Hamiltonian

$$h = (\bar{\alpha}^{\frac{1}{2}} \bar{A} \bar{\alpha}^{\frac{1}{2}})^{1/2}$$

(cf Klein [14] theorem 2 : which gives a unitarily equivalent construction in a special case – see [13])

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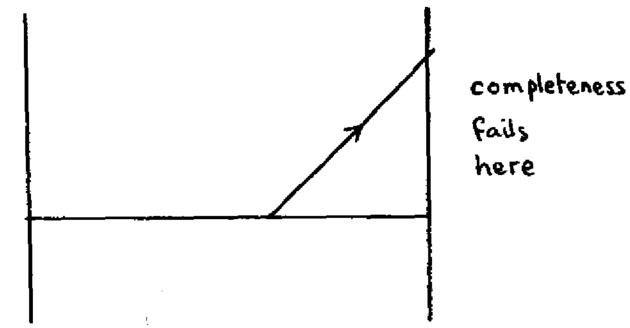

Figure 1

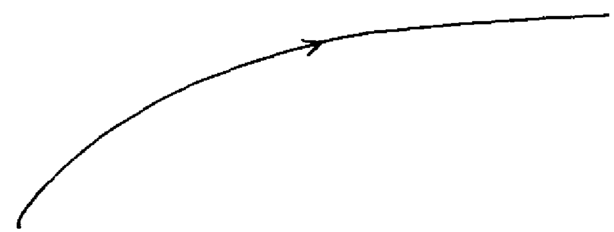

Figure 2

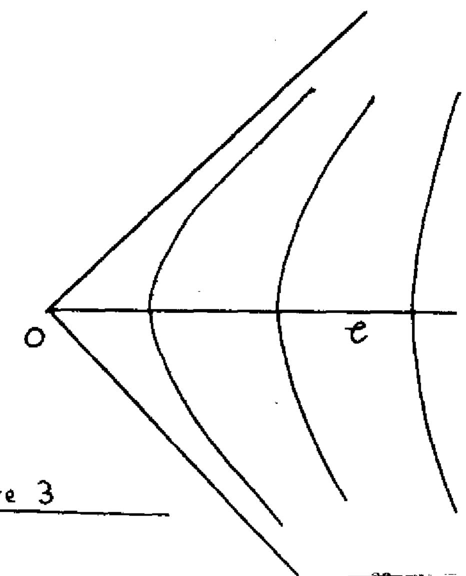

Figure 3