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THE STABILITY OF THE STRONG GRAVITY SOLUTION *

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ABSTRACT

We investigate the perturbation of the classical solution to a strong gravity model given by Salam and Strathdee. Using the Hamiltonian formalism it is shown that this static and spherically symmetric solution is stable under the odd parity perturbations provided some parameters in the solution are suitably restricted.

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I. INTRODUCTION

A strong gravity model has been suggested by Isham, Salam and Strathdee [1] with many interesting consequences. The model assumes that hadrons interact through the exchange of spin-two particles. The tensor field satisfies modified Einstein equations containing a strong coupling parameter and a mass term. The gauge group on which the model was constructed consists of the co-ordinate transformation of Einstein. A static spherically symmetric Schwarzschild-de Sitter solution of the model has recently been found [2] which is interpreted as an effective potential produced by a source quark. This potential is used in a Klein-Gordon equation for a scalar test particle [3] and the energy levels of the system are determined. These levels obtained follow a rising Regge trajectory with the source and the test particles confined in their mutual potential well.

The aim of this paper is to show that this static spherically symmetric solution of strong gravity, which appears to produce quark confinement, is stable. Generally, there are two methods for such an investigation. The first method is the Lagrangian formalism in which one starts from a perturbed Lagrangian, then expands it in terms of tensorial harmonics, deduces the perturbed Hamiltonian and examines the stability of the desired solution. This method is complicated, unless one works in a special gauge which simplifies the calculations. Since our Lagrangian includes a mass term, it is not gauge invariant. Thus we shall use the second method due to Arnowitt, Deser and Misner (ADM) [4] or Hamiltonian formalism of general relativity. This method has been used [5] by Moncrief for the stability analysis of the Schwarzschild black hole and the perfect fluid stellar models. We do not want to give the mathematical details of the stability analysis although, after showing that the Hamiltonian is positive definite, it will become clear that a similar analysis can be carried through. We show that the perturbed Hamiltonian is a Liapunov function [6] of the canonical variables, which is also a test for the stability of the solution.

The plan of the paper is as follows. In the next section we summarize the strong gravity model and give the action integral including the mass Lagrangian in the ADM form. In this section the initial value constraints as well as field equations are obtained and the static spherically symmetric solution, whose stability is going to be investigated for odd parity perturbation, is given. In Sec.III the perturbed action is presented for the Hamiltonian formalism. From this action, the perturbed field equations as well as the perturbed constraints are obtained. In the last section we expand all the field quantities in terms of odd parity Regge-Wheeler tensorial harmonics [7]. We show that the Hamiltonian is a positive definite function of the small perturbations and conclude that the strong gravity solution is stable for $f_{tt} = c < \frac{9}{4}$, where f_{tt} is the time-time component of the metric.

II. THE MODEL

The model is characterized by the Lagrangian [2),8)]

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_f + \mathcal{L}_{fg} , \tag{2.1}$$

$$\mathcal{L}_g = \frac{1}{k_g} \sqrt{-g}\; R(g) \qquad \mathcal{L}_f = \frac{1}{k_f^2} \sqrt{-f}\; R(f) ,$$

$$\mathcal{L}_{fg} = -\frac{M^2}{4k_f^2} \sqrt{-g}\; (f_{\mu\nu} - g_{\mu\nu})\,(f_{\kappa\lambda} - g_{\kappa\lambda})\,(g^{\kappa\mu} g^{\lambda\nu} - g^{\kappa\lambda} g^{\mu\nu}) , \tag{2.2}$$

where k_g and k_f are the gravitational constant ($k_g \sim 10^{-19}$ GeV^{-1}) and the coupling constant of the strongly interacting f meson ($k_f \sim 1$ GeV^{-1}), respectively. $\mathcal{L}_g$ is the well-known Einstein Lagrangian for the gravity field. $\mathcal{L}_f$ has the same form as $\mathcal{L}_g$ and it describes the hadronic world. All geometric quantities in $\mathcal{L}_f$ have their usual definitions in terms of $f_{\mu\nu}$ as the metric tensor, similar to those in $\mathcal{L}_g$. The mixing Lagrangian $\mathcal{L}_{fg}$ causes the interaction between the fields $f_{\mu\nu}$ and $g_{\mu\nu}$. $\mathcal{L}_{fg}$ has been chosen such that one of the rank-2 tensor fields describes a massive particle. Namely, on the basis of a linearization of the above equations, it was suggested [2)] that the equations resulting from (2.1) and (2.2) describe a massless gravitation given by the field combinations

$$\left[\frac{1}{k_g^2}\; g_{\mu\nu} + \frac{1}{k_f^2} f_{\mu\nu}\right] \left[\frac{1}{k_g^2} + \frac{1}{k_f^2}\right]^{-1}$$

plus a strongly interacting massive spin-2 field described by the orthogonal combination $(g_{\mu\nu} + f_{\mu\nu})$.

We want to examine here the stability of the static spherically symmetric solution of the pure strong gravity in the limit $k_g \to 0$. Thus all matter as well as ordinary gravity will be ignored. In this situation the action corresponding to the Lagrangian (2.1) reduces to the form [3)]

$$I = \int d^4x \left[\frac{1}{k_f^2} \sqrt{-f}\; R(f) + \mathcal{L}_{mass}\right] , \tag{2.3}$$

where the first term is identical in form with the Einstein action (put f instead of g in the Einstein Lagrangian). The second term, which gives mass to the tensor meson, takes the form [8)]

$$\mathcal{L}_{mass} = -\frac{M^2\sqrt{-\eta}}{4k_f^2} (f_{\kappa\lambda} - \eta_{\kappa\lambda})\,(f_{\mu\nu} - \eta_{\mu\nu})\,(\eta^{\kappa\mu} \eta^{\lambda\nu} - \eta^{\kappa\lambda} \eta^{\mu\nu}) , \tag{2.4}$$

where M is spin-2 mass and $\eta_{\mu\nu}$ denotes the flat space-time metric ($\eta = \det \eta_{\mu\nu}$). Varying $f_{\mu\nu}$ one obtains the equations of strong gravity

$$R^{\mu\nu} - \frac{1}{2} f^{\mu\nu} R = T^{\mu\nu} , \tag{2.5}$$

where the left-hand side is the usual Einstein tensor $G^{\mu\nu}$ and the right-hand side is given by

$$T^{\mu\nu} = -\frac{M^2}{2} \sqrt{\frac{\eta}{f}}\; (f_{\kappa\lambda} - \eta_{\kappa\lambda})\,(\eta^{\kappa\mu} \eta^{\lambda\nu} - \eta^{\kappa\lambda} \eta^{\mu\nu}) . \tag{2.6}$$

It is worthwhile to note that, although the left-hand side of Eq.(2.5) is a tensor, the right-hand side is not. Therefore, it is not possible to remove any components from $f_{\mu\nu}$ by way of co-ordinate conditions as one would do with a covariant system. After using the Bianchi identities and assuming spherical symmetry, the number of independent components in $f_{\mu\nu}$ is reduced from ten to four.

Since $\mathcal{L}_f$ is identical in form with the Einstein Lagrangian, the variational integral corresponding to it can be written in the Arnowitt, Deser and Misner form [4]. After discarding the surface terms, which will not contribute to the Hamilton equations, we obtain [9]

$$I_1 = \int \mathcal{L}_f \, d^4x = \frac{1}{k_f^2} \int d^4x \left[\pi^{ij} \gamma_{ij,t} - N\mathcal{H}^* - N_i \mathcal{H}^i\right] , \tag{2.7}$$

where we have defined

$$\mathcal{H}^* = \gamma^{-1/2} \left[\pi^{ij} \pi_{ij} - \frac{1}{2}(\pi^i{}_i)\right] - \gamma^{1/2} R , \tag{2.8}$$

$$\mathcal{H}^i = -2 \pi^{ij}{}_{|j} , \tag{2.9}$$

and γ_{ij}, lapse function N, shift vector N_i are related to the four-dimensional metric $f_{\mu\nu}$ through

$$\gamma_{ij} = f_{ij} , \quad N = (-f^{00})^{-1/2} , \quad N_i = f_{0i} , \quad \gamma = \det \gamma_{ij} , \quad i,j = 1,2,3 . \tag{2.10}$$

The momenta π^{ij} are related to the second fundamental form K_{ij} of spacelike surfaces $x^0 = t$ = constant through $\pi^{ij} = -\gamma^{1/2}(K^{ij} - \gamma^{ij}K^{\ell}{}_{\ell})$. Latin indices are raised and lowered with the metric γ_{ij}, a vertical bar denotes covariant differentiation with respect to this metric and R is the curvature scalar constructed from it.

The mass Lagrangian for the model, given by Eq.(2.4), can be written in terms of γ_{ij}, N and N_i. Using Eq.(2.10) and the fact that $f_{00} = -N^2 + N_i N^i$, we obtain

$$\mathcal{L}_{mass} = -\frac{M^2}{2k_f^2}\sqrt{-\eta}\left[(N^2 - N_i N^i)\chi + 2N_i\chi^i + N_i N_j \chi^{ij} + {}^{(3)}\mathcal{L}_{mass}\right] , \tag{2.11}$$

where

$$\chi = \gamma_{ij}\chi^{ij} - 3\eta^{00} \qquad \chi^{ij} = \eta^{00}\eta^{ij} - \eta^{0i}\eta^{0j}$$

$$\chi^i = \gamma_{jk}\chi^{jki} - 3\eta^{0i} \qquad \chi^{jki} = \eta^{0j}\eta^{ki} - \eta^{0i}\eta^{jk}$$

$${}^{(3)}\mathcal{L}_{mass} = \frac{1}{2}\gamma_{ij}\gamma_{k\ell}(\eta^{ik}\eta^{j\ell} - \eta^{ij}\eta^{k\ell}) - 3 \tag{2.12}$$

Thus the total action corresponding to the Lagrangian given by Eq.(2.3) can be written as follows:

$$I = \frac{1}{k_f^2}\int d^4x\left\{\pi^{ij}\gamma_{ij,t} - N\mathcal{H}^* - N_i\mathcal{H}^i - \frac{M^2}{2}\sqrt{-\eta}\left[(N^2 - N_iN^i)\chi + 2N_i\chi^i + N_iN_j\chi^{ij} + {}^{(3)}\mathcal{L}_{mass}\right]\right\} . \tag{2.13}$$

Independent variations of the γ_{ij}, π^{ij}, N and N_i give a system of equations equivalent to the Einstein equation for strong gravity, i.e. Eq.(2.5). In particular, variation of the lapse function N and the shift vector N_i leads to the equations

$$\mathcal{H}^* = -\sqrt{-\eta}\, M^2 N\chi , \tag{2.14}$$

$$\mathcal{H}^i = -\sqrt{-\eta}\, M^2 [\chi^i - N^i\chi + N_j\chi^{ij}] , \tag{2.15}$$

which are the initial value constraints. Independent variations of the γ_{ij} and π^{ij} give (for the static solution $\pi^{ij} = N_i = 0$)

$$\pi^{ij}{}_{,t} = -N\gamma^{1/2}(R^{ij} - \tfrac{1}{2}\gamma^{ij}R) + \gamma^{1/2}(N^{|ij} - \gamma^{ij}N_{|k}{}^{|k}) - \frac{M^2}{2}\sqrt{-\eta}\left[(N^2 - N_iN^i)\chi^{ij} + 2N_k\chi^{ijk} + \gamma_{k\ell}(\eta^{ik}\eta^{j\ell} - \eta^{k\ell}\eta^{ij})\right]$$

$$\gamma_{ij,t} = 0 . \tag{2.16}$$

After solving the field equations in spherical polar co-ordinates one obtains

$$f_{\mu\nu}\, dx^{\mu} dx^{\nu} = -c dt^2 - 2D\, dt\, dr + A\, dr^2 + B(d\theta^2 + \sin^2\theta\, d\varphi^2) \;, \quad (2.17)$$

where the components A, B, c and D depend only on r . It will be more convenient for the following calculation to work with a diagonal metric. Namely, from (2.17) we obtain

$$f_{\mu\nu}\, dx^{\mu} dx^{\nu} = -c dt^2 + \frac{\Delta}{c} dr^2 + B(d\theta^2 + \sin^2\theta\, d\varphi^2) \;, \qquad (2.18)$$

where the convenient combination

$$\Delta = Ac + D^2 \qquad (2.19)$$

is used. In this co-ordinate, $\eta_{\mu\nu}$ is given by

$$\eta_{\mu\nu}\, dx^{\mu} dx^{\nu} = -dt^2 - \frac{2D}{c} dr\, dt + \left(1 - \frac{D^2}{c^2}\right) dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \;. \qquad (2.20)$$

This is the static spherically symmetric solution of strong gravity and we want to examine its stability. The functions c, B and Δ are given as follows:

$$B = \frac{3}{2} r^2 \;, \quad \frac{c}{\Delta} = \frac{2}{3}\left\{1 + \frac{1}{6}\sqrt{\Lambda}\, M^2 r^2\right\} \;, \quad \Delta = \text{constant} \;,$$

$$A + c = \frac{3}{2} + \frac{2}{3}\Delta \;, \qquad \Delta < \frac{9}{4} \;. \qquad (2.21)$$

III. PERTURBATIONS

Generally, there are two different methods for obtaining the Euler equations governing small perturbations. One of them is the Jacobi method [5),6),10)] of second variation and the other, which will be employed here, was suggested in Ref.9 for the derivation of the basic equations of the short-wave approximation. We do not review this method here but give the appropriate formulae for the Hamilton formalism.

We define the perturbed functions by

$$\gamma'_{ij} = \gamma_{ij} + \phi_{ij} \;; \quad \gamma'^{ij} = \gamma^{ij} - \phi^{ij} + \phi^{ik}\phi_k^{\ j} + \cdots \;,$$
$$\pi'^{ij} = \pi^{ij} + p^{ij} \;; \quad \mathcal{N} = N + N' \;, \quad \mathcal{N}_i = N_i + N'_i \;, \qquad (3.1)$$

where γ_{ij}, π^{ij}, N, N_i are background quantities, which are defined in Sec.II, and ϕ_{ij}, p^{ij}, N', N'_i are small perturbations, which are much less than the background quantities. The perturbed curvature scalar is taken as follows:

$$R'^{i}_{\ jk\ell} = \Gamma'^{i}_{\ j\ell.k} - \Gamma'^{i}_{\ jk.\ell} + \Gamma'^{i}_{\ mk}\Gamma'^{m}_{\ j\ell} - \Gamma'^{i}_{\ m\ell}\Gamma'^{m}_{\ jk} \;. \qquad (3.2)$$

We raise and lower the indices of ϕ_{ij} with γ_{ij} and signify the covariant derivatives relative to γ_{ij} and γ'_{ij} by a vertical bar "|" and by a dot ".", respectively.

We write $\Gamma'^{i}_{\ jk}$, which are the Christoffel symbols with respect to the metric γ'_{ij}, in terms of γ_{ij} and use Eq.(3.1); up to the second order we obtain

$$\begin{aligned}\sqrt{\gamma'}\, R' = {} & \sqrt{\gamma}\, R + \sqrt{\gamma}\left[\tfrac{1}{2}\phi R - \phi^{ij}R_{ij} + W^{i}_{\ j}{}^{j}{}_{|i} - W^{i}_{\ ji}{}^{|j}\right] \\ & + \sqrt{\gamma}\Big[\tfrac{1}{8}(\phi^2 - 2\phi^{ij}\phi_{ji})R - \tfrac{1}{2}\phi\phi^{ij}R_{ij} + \phi^{ik}\phi_k^{\ j}R_{ij} + \\ & + \tfrac{1}{2}\phi(W^{i}_{\ j}{}^{j}{}_{|i} - W^{i}_{\ ji}{}^{|j}) - \phi^{ij}(W^{k}_{\ ij|k} - W^{k}_{\ ik|j}) + \\ & + W^{i}_{\ ji}W^{j}_{\ k}{}^{k} - W^{i}_{\ j}{}^{k}W^{j}_{\ ki} + (\phi^{i}_{\ j}W^{j}_{\ ki})^{|k} - \\ & - (\phi^{i}_{\ j}W^{j}_{\ k}{}^{k})_{|i}\Big] \;,\end{aligned} \qquad (3.3)$$

where $\gamma' = \det \gamma'_{ij}$, $\phi = \phi_i^{\ i}$ and

$$W^{i}_{\ jk} = \frac{1}{2}\gamma^{i\ell}\left\{\phi_{\ell j|k} + \phi_{\ell k|j} - \phi_{jk|\ell}\right\} \;. \qquad (3.4)$$

When we insert $\sqrt{\gamma'}\, R'$ in the perturbed action the divergence terms of the second order in ϕ will not give any contribution. Therefore they can be discarded. However, in the action (2.13) the term $\sqrt{\gamma'}\, R'$ will be multiplied by the perturbed lapse function N', thus the divergence terms which are first order in ϕ (since our equations are correct up to the second order) in Eq.(3.3) cannot be discarded. Indeed they will contribute to the perturbed Hamilton equations as will be seen below.

It is easier to obtain the perturbed action corresponding to the mass Lagrangian. Namely we use Eq.(3.1) and obtain the following total perturbed action:

$$
\begin{aligned}
I_{pert} = \frac{1}{k_f^2} \int d^4x \Big\{ & p^{ij} \phi_{ij,t} - N \gamma^{-1/2} (p^{ij} p_{ij} - \tfrac{1}{2} p^2) - \tfrac{1}{4} N \gamma^{1/2} \big[\phi^{ij} \phi_{ij} R - \\
& - 4 \phi^i{}_k \phi^{kj} R_{ij} + 2 \phi \phi^{ij} R_{ij} - \tfrac{1}{2} \phi^2 R + \phi^{ij|k} (\phi_{ij|k} - 2 \phi_{ik|j}) - \\
& - \phi_{|i} (\phi^{|i} - 2 \phi^{ij}{}_{|j}) \big] - \gamma^{1/2} N' \big[\phi_{|i}{}^{|i} - \phi_{ij}{}^{|ij} + \\
& + \phi_{ij} (R^{ij} - \tfrac{1}{2} \gamma^{ij} R) \big] + 2 N'_i p^{ij}{}_{|j} - \\
& - \frac{M^2 \sqrt{-\eta}}{2} \big[N'^2 \chi + 2 N N' \chi' - N'_i N'^i \chi + 2 N'_i \chi'^i \\
& + N'_i N'_j \chi^{ij} + {}^{(3)}\mathcal{L}^*_{mass} \big] \Big\} .
\end{aligned}
\qquad (3.5)
$$

where

$$\eta = \det \eta_{\mu\nu}$$

$$\chi' = \phi_{ij} \chi^{ij} \quad , \quad \chi'^i = \phi_{jk} \chi^{jki}$$

$${}^{(3)}\mathcal{L}^* = \tfrac{1}{2} \phi_{ij} \phi_{kl} (\eta^{ik} \eta^{jl} - \eta^{ij} \eta^{kl}) \; ; \; i,j,k,l = 1,2,3 . \qquad (3.6)$$

In Eq.(3.5) all indices are raised or lowered with the unperturbed metric γ_{ij} and its inverse γ^{jk}, the covariant derivatives are taken with respect to this metric and R is the curvature scalar constructed from it.

Varying I_{pert} with respect to the perturbed lapse function N' and perturbed shift vector N'_i we get the equation of constraints

$$\phi_{|i}{}^{|i} - \phi_{ij}{}^{|ij} + \phi_{ij} (R^{ij} - \tfrac{1}{2} \gamma^{ij} R) = -M^2 \gamma^{-1/2} \sqrt{-\eta} (N \chi' + N' \chi), \quad (3.7)$$

$$2p^{ij}{}_{|j} = M^2 \sqrt{-\eta} [\chi'^i + N'_j \chi^{ij} - N'^i \chi] . \qquad (3.8)$$

Variation of I_{pert} with respect to p^{ij} and ϕ_{ij} gives the Hamilton equations

$$\phi_{ij,t} = \frac{\delta \mathcal{H}_T}{\delta p_{ij}} \qquad p^{ij}{}_{,t} = - \frac{\delta \mathcal{H}_T}{\delta \phi_{ij}} . \qquad (3.9)$$

where $\delta(\ )/\delta(\ )$ signifies functional differentiation and $\mathcal{H}_T$ is defined as follows:

$$I_{pert} = \frac{1}{k_f^2} \int d^4x \left[p^{ij} \phi_{ij,t} - \mathcal{H}_T \right] . \qquad (3.10)$$

IV. ODD PARITY PERTURBATIONS AND THE POSITIVITY OF THE HAMILTONIAN

We use the Regge and Wheeler form [7] for odd parity perturbations. Since we have spherical symmetry, we use the perturbations of order $(\ell,0)$, which are taken to be

$$\phi_{ij} = h_1(r,t) (\hat{e}_1)_{ij} + h_2(r,t) (\hat{e}_2)_{ij} , \qquad (4.1)$$

where

$$\hat{e}_1 = - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} (1-z^2) \frac{dY_\ell}{dz} \quad ; \quad \hat{e}_2 = -\frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} (1-z^2)^{3/2} \frac{d^2 Y_\ell}{dz^2} . \qquad (4.2)$$

In these formulae Y_ℓ is the standard spherical harmonic function of order $(\ell,0)$ and $z = \cos\theta$. The odd parity perturbation N' of the lapse function vanishes and the perturbed shift vector is expressed as

$$[N_i'] = h_0(r,t)\ [0,0,-1]\ (1-z^2)\ \frac{dY_\ell}{dz} \quad . \tag{4.3}$$

The labels $(\ell,0)$ on h_0, h_1, h_2, $\hat{e}_1$, $\hat{e}_2$ and the label (0) on Y_ℓ have been suppressed in the above equations. Here we shall examine the modes with $\ell \geqslant 2$. The mode with $\ell = 1$ is non-derivative and requires special treatment [5),11)].

Since the perturbed momenta p^{ij} transform as components of a tensor density, $\gamma^{-1/2}\, p_{ij}$ is expanded in odd parity tensor harmonics as follows:

$$\gamma^{-1/2}\, p_{ij} = p_1(r,t)\ (\hat{e}_1)_{ij} + p_2(r,t)\ (\hat{e}_2)_{ij} \quad . \tag{4.4}$$

Thus using Eqs.(4.1)-(4.4) we obtain

$$\phi = N' = p = 0 \ , \tag{4.5}$$

and for odd parity perturbations (where $p = p^i{}_i$) we have

$$I_{\rm pert}^{\rm (odd)} = \frac{1}{k_f^2} \int d^4x\ [p^{ij}\ \phi_{ij,t} - \mathcal{H}_T] \quad , \tag{4.6}$$

where

$$\mathcal{H}_T = N\gamma^{-1/2} p^{ij} p_{ij} - 2N_i' p^{ij}{}_{|j} + \frac{1}{4} N\gamma^{1/2} \left[\phi^{ij}\phi_{ij} R - 4\phi^i{}_k \phi^{kj} R_{ij} + \phi^{ij|k}(\phi_{ij|k} - 2\phi_{ik|j}) \right] + \frac{1}{2} M^2 \sqrt{-\eta} \left[N'_i N'^i \chi + 2 N'_i \chi'^i + N'_i N'_j \chi^{ij} + {}^{(3)}\mathcal{L}^*_{\rm mass} \right] . \tag{4.7}$$

Now each term of the perturbed odd parity action will be expanded in appropriate spherical harmonics. Then, after some re-arrangement, the angular integration can be performed.

First of all we want to define our canonical variables. Using expansions given by Eqs.(4.1)-(4.4), after performing the angular integration, we obtain

$$\ominus = \frac{1}{k_f^2} \int d^3x\ p^{ij}\ \phi_{ij,t} = \frac{1}{k_f^2} \int dr \left[\hat{p}_1(r,t) \frac{\partial}{\partial t} h_1(r,t) + \hat{p}_2(r,t) \frac{\partial}{\partial t} h_2(r,t) \right] , \tag{4.8}$$

where

$$\hat{p}_1 = 2\ell(\ell+1)\left(\frac{c}{\Delta}\right)^{1/2} p_1 \quad , \quad \hat{p}_2 = \frac{1}{2B}(\ell-1)\ell(\ell+1)(\ell+2)\left(\frac{\Delta}{c}\right)^{1/2} p_2 \ . \tag{4.9}$$

Thus $\hat{p}_1(r,t)$ and $\hat{p}_2(r,t)$ are the canonical momenta conjugate to $h_1(r,t)$ and $h_2(r,t)$.

Before writing the final expression for H_T we want to note the following. It is not necessary to calculate the term $\int d^3x\ N_1'\ p^{ij}{}_{|j}$ separately. Namely, we use the constraint given by Eq.(3.8) and write the covariant derivatives of the perturbed momenta $p^{ij}{}_{|j}$ in the other terms of the Hamiltonian, i.e. Eq.(4.7). Consequently, after angular integration, the variation with respect to h_0 will not give any constraint. The value of $h_0(r,t)$ can be obtained from the perturbed constraint Eq.(3.8). Thus after straightforward but lengthy calculations, we found

$$H_T = \frac{1}{\ell(\ell+1)} \int_0^\infty dr \left[\frac{1}{2}\Delta^{1/2} \hat{p}_1^2 + \frac{2Bc}{(\ell-1)(\ell+2)} \hat{p}_2^2 \right] + \frac{\ell(\ell-1)(\ell+1)(\ell+2)}{2\sqrt{\Delta}} \int_0^\infty dr\ \frac{c}{B} \left[h_1 + \frac{1}{2}\left(\frac{\partial h_2}{\partial r} - \frac{B'}{B} h_2 \right) \right]^2 + M^2 \ell(\ell+1) \int_0^\infty dr \left[\left(\frac{1}{2} - \frac{2}{9}c\right) h_1^2 + \frac{(\ell-1)(\ell+2)}{4B}\left(\frac{3}{4} - \frac{2}{9}\Delta\right) h_2^2 + \frac{M^2 r^2 \Delta^{3/2}}{12c^2} h_0^2 \right] , \tag{4.10}$$

where we have suppressed the labels $(\ell,0)$ on H_T to simplify the notation. Since $\Delta < \frac{9}{4}$ (Eq.(2.20)), H_T is a positive definite function of small perturbations for $c \leq \frac{9}{4}$, which gives a restriction on the tensor mass M and the range of the potential r. Namely, from $c \leq \frac{9}{4}$ using Eq.(2.20) we obtain

$$M^2 r^2 \leq \frac{3(27 - 8\Delta)}{4\,\Delta^{3/2}} \quad . \tag{4.11}$$

If we eliminate h_0 from H_T by using the perturbed constraint Eq.(3.8), we obtain

$$\begin{aligned} H_T = \frac{1}{\ell(\ell+1)} \int_0^\infty dr \left[\frac{1}{2}\Delta^{1/2} \hat{\varphi}_1^2 + \frac{2Bc}{(\ell-1)(\ell+2)} \hat{\varphi}_2^2 \right] + \frac{\ell(\ell-1)(\ell+2)(\ell+1)}{2\sqrt{\Delta}} \int_c^\infty dr\, \frac{c}{B} \left[h_1 + \frac{1}{2}\left(\frac{\partial h_2}{\partial r} - \frac{B'}{B} h_2 \right) \right]^2 \\ + M^2 \ell(\ell+1) \int_0^\infty dr \left\{ \frac{(\ell-1)(\ell+2)}{4B} \left(\frac{3}{4} - \frac{2}{9}\Delta \right) h_2^2 + \frac{6c(9-4c)}{\ell(\ell+1) M^4 r^2 \Delta^{3/2}} \left(\hat{\varphi}_2 + \frac{B'}{2B}\hat{\varphi}_1 + \frac{1}{2}\frac{\partial \hat{\varphi}_1}{\partial r} \right)^2 \right. \\ \left. + \frac{12c^2 \Delta^{-3/4}}{\ell(\ell+1) M^4 r^2} \left[\sqrt{3\left(1 - \frac{3}{2c}\right)} \left(\hat{\varphi}_2 + \frac{B'}{2B}\hat{\varphi}_1 + \frac{1}{2}\frac{\partial \hat{\varphi}_1}{\partial r} \right) + \frac{\ell(\ell+1) M^3 r \Delta^{3/4}}{6\sqrt{3}\,c} h_1 \right]^2 \right\} . \end{aligned} \tag{4.12}$$

Again H_T is a positive definite function of small perturbations for $c \leq \frac{9}{4}$ [i.e. (4.11)]. Since Δ is an arbitrary constant with values $0 < \Delta < \frac{9}{4}$, the Hamiltonian is a positive definite function for all values of r provided Δ is suitably restricted. Thus the energy density is bounded from below and the $m \to 0$ limit exists [12].

We have given a form of perturbation theory suitable for strong gravity and have shown that the Hamiltonian is a positive definite function of odd parity perturbations. A similar method has been used by Moncrief to investigate the stability of the Schwarzschild black hole and the perfect fluid stellar models. We can use the same procedure as followed by Moncrief [5),11)] to show the stability of the strong gravity solutions. Namely, new canonical variables,

$$k_1 = h_1 + \frac{1}{2}\left(\frac{\partial h_2}{\partial r} - \frac{B'}{B} h_2 \right) , \qquad k_2 = h_2 ,$$

$$\pi_1 = \hat{\varphi}_1 , \qquad \pi_2 = \hat{\varphi}_2 + \frac{1}{2}\left(\frac{\partial \hat{\varphi}_1}{\partial r} + \frac{B'}{B}\hat{\varphi}_1 \right) , \tag{4.13}$$

can be defined. It can be shown that under the transformation (4.13), Eq.(4.8) takes the form

$$\theta = \int \left(\pi_1 \frac{\partial k_1}{\partial r} + \pi_2 \frac{\partial k_2}{\partial t} \right) dr \quad . \tag{4.14}$$

Thus π_1 and π_2 are new momenta conjugate to k_1 and k_2. While the new canonical variable k_2 is gauge dependent, it can be shown that k_1 is gauge independent. Writing the Hamiltonian in terms of new variables one can obtain an equation for k_1 similar to the Regge-Wheeler equation [7)] with some additional terms coming from $\mathcal{L}_{mass}$. We do not repeat these mathematical details here, but it is clear that a similar analysis can be carried through.

It is not difficult to show that the perturbed Hamiltonian (H) given by Eq.(4.10) is a Liapunov function. That is:

a) H is continuous together with its first partial derivatives in a certain open region Ω ;
b) $H(0) = 0$;
c) H is non-negative and vanishes only at the origin;
d) Using the Hamilton equations given by Eq.(3.9) we obtain $dH/dt = 0$.

Thus, after showing that the Hamiltonian is a Liapunov function, we conclude that our static spherically symmetric solution of strong gravity is stable under odd parity perturbations for $c \leq \frac{9}{4}$. Even parity perturbations are under investigation.

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$$\mathcal{L}_{mass} = - \frac{M^2\sqrt{-\eta}}{4k_f^2} \left(f^{\kappa\lambda} - \eta^{\kappa\lambda}\right) \left(f^{\mu\nu} - \eta^{\mu\nu}\right) \left(g_{\kappa\mu}g_{\lambda\nu} - g_{\mu\nu}g_{\kappa\lambda}\right) .$$

We prefer to use the form given by Eq.(2.2) (its linearization gives 2-4) which is more convenient for the expansion of $\mathcal{L}_{mass}$ in terms of the lapse function N and shift vector N_i defined in Sec.III. The corresponding solutions differ from one another by a constant multiple (compare the values given by (2.11) with the values given in Refs.2 and 3). Therefore the two forms of $\mathcal{L}_{mass}$ have the same physical consequences. I am grateful to Professor Strathdee for pointing out to me this possibility.

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