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GLOBAL PROPERTIES OF SYSTEMS QUANTIZED VIA BUNDLES † *

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ABSTRACT

Take a smooth manifold M and a Lie algebra action ($\mathfrak{g}$-action) θ on M as the geometrical arena of a physical system moving on M with momenta given by θ. It is proposed to quantize the system with a Mackey-like method via the associated vector bundle ξ_ς of a principal bundle $\xi = (P,\pi,M,H)$ with model dependend structure group H and with $\mathfrak{g}$-action ϕ on P lifted from θ on M. This (quantization) bundle ξ_ς gives the Hilbert space $\mathcal{H} = L^2(\xi_\varsigma,\omega)$ of the system as the linear space of sections in ξ_ς being square integrable with respect to a volume form ω on M; the usual position operators are obtained; ϕ leads to a vector field representation $D(\phi_\varsigma,\theta)$ of $\mathfrak{g}$ in $\mathcal{H}$ and hence to momentum operators. So $\mathcal{H}$ carries the quantum kinematics. In this quantization the physically important connection between geometrical properties of the system, e.g. quasi-completeness of θ and G-maximality of ϕ_ς, and global properties of its quantized kinematics, e.g. skew-adjointness of the momenta and integrability of $D(\phi_\varsigma,\theta)$ can easily be studied. The relation to Nelson's construction of a skew-adjoint non-integrable Lie algebra representation and to Palais' local G-actions is discussed. Finally the results are applied to actions induced by coverings as examples of non-maximal ϕ_ς on E_ς lifted from maximal θ on M which lead to direct consequences for the corresponding quantum kinematics.

1. Introduction

Consider the geometrical arena on which a classical system is described. Suppose this to be a smooth manifold M or a base of a principal bundle which can be equipped with a local symmetry via a $\mathfrak{g}$-action connected with physical momenta. Then the global geometrical structure induces certain properties for the system depending on the type of theory which is constructed on the arena. Results of this type are known. We refer e.g. to classical Maxwell fields on manifolds [1,2], to gauge theories on principal bundles [3,4,5] and to spin structures on non-simply-connected manifolds and their use in many-body physics [6].

To quantize a (particle-like) system we use principal fibre bundles $\xi = (P,\pi,M,H)$ with projection $\pi: P \longrightarrow M$, structure group H, and a $\mathfrak{g}$-action (ϕ,θ) on ξ, where θ is a Lie algebra homomorphism from a finite-dimensional Lie algebra $\mathfrak{g}$ into the Lie algebra $\mathcal{V}(M)$ of smooth vector fields on M and ϕ denotes a lift of θ to P. Then any unitary finite-dimensional representation $\varsigma: H \longrightarrow \mathrm{Aut}\ V$ with associated vector bundle $\xi_\varsigma = (E_\varsigma,\pi_\varsigma,M,V)$ [7] and $\mathfrak{g}$-action (ϕ_ς,θ) induces a Mackey-type quantization via a vector field representation D of $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$ spanned by smooth sections in ξ_ς. Because the quantized kinematics is completely given by ϕ_ς on ξ_ς, we call ξ_ς quantization bundle. ϕ_ς leads to momentum operators and to any Borel set on M corresponds a position projector.

The global group theoretical properties of the $\mathfrak{g}$-action (ϕ_ς,θ) on the quantization bundle and the representation D of $\mathfrak{g}$ are directly linked: quasi-completeness (of θ) and maximality (of ϕ_ς) correspond to skew-adjointness and integrability of D. So the local $\mathfrak{g}$-actions on E_ς and M give, via their physically significant singularity [8] structure, information on the quantum mechanics on the bundle, and vice versa.

The material is organized as follows: A short description of Lie algebra actions on quantization bundles and of the corresponding quantization is given in Section 2; Lemma 1 relates the quasi-completeness of the action to the skew-adjointness of the vector field representation. Global results are presented in Section 3; using a property of skew-adjoint representations (Lemma 2) we prove that a vector field representation D is G-integrable if and only if ϕ_ς is G-maximal (Theorem 1). Relations to Nelson's construction of a skew-adjoint non-integrable representation [9] and to Palais' local G-actions [10] are given. The results are applied to actions induced by coverings in Section 4. Two types of non-integrability are distinguished: those which are induced from non-maximal θ on M and those which are induced from non-maximal ϕ_ς on E_ς, whereas the projected symmetry θ is maximal. The proofs of Lemma 1 and Theorem 1 are given in the Appendix.

2. Quantization bundles with $\mathfrak{g}$-actions

1. Let F(X) be the flow of a smooth vector field $X \in \mathcal{V}(M)$

$$F(X) : (m,t) \in D(X) \subset M \times \mathbb{R} \longrightarrow M \ni F(X)(m,t) = \varphi_t^X(m)$$

with (open [11]) domain $D(X,t) = \{m \mid (m,t) \in D(X)\}$ for $t \in \mathbb{R}$. Consider a principal fibre bundle $\xi = (P,\pi,M,H)$. Then a $\mathfrak{g}$-action (ϕ,θ) on ξ consists of $\mathfrak{g}$-actions $\phi : \mathfrak{g} \longrightarrow \mathcal{V}(P)$, $\theta : \mathfrak{g} \longrightarrow \mathcal{V}(M)$ such that for $x \in \mathfrak{g}$

i) $D(\phi(x)) = (\pi \times id_{\mathbb{R}})^{-1} D(\theta(x))$;

ii) $\pi(F(\phi(x))(p,t) = F(\theta(x))(\pi(p),t)$ for $(p,t) \in D(\phi(x))$;

iii) $\phi(x)$ is H-invariant.

2. Let $\varsigma : H \longrightarrow \mathrm{Aut}\, V$ be a representation of H in a finite-dimensional complex vector space V. Then the quantization bundle $\xi_\varsigma = (E_\varsigma, \pi_\varsigma, M, V)$ is the (ς-) vector bundle associated with ξ, E_ς being the orbit space of the H-action on $P \times V$ given by $(p,v)h = (ph, \varsigma^{-1}(h)v)$; $[p,v]$ denotes the orbits and $\pi_\varsigma[p,v] = \pi(p)$ holds.
Any $\mathfrak{g}$-action (ϕ,θ) on ξ gives a $\mathfrak{g}$-action (ϕ_ς,θ) on ξ_ς via

$$F(\phi_\varsigma(x))([p,v]_\varsigma, t) = [F(\phi(x))(p,t),v]_\varsigma .$$

Here the H-invariance of $\phi(x)$ was used.

3. For a quantization consider the linear space $\mathrm{Sec}_o(\xi_\varsigma)$ of compactly supported smooth sections σ in ξ_ς. Equip V with an inner product $\langle .,. \rangle_V$ and take ς to be unitary. This induces an unitary structure on ξ_ς via

$$\langle [p,v_1]_\varsigma , [p,v_2]_\varsigma \rangle_m = \langle v_1,v_2 \rangle_V ,$$

$v_1, v_2 \in V$, $\pi(p) = m$. For a pre-Hilbert structure on $\mathrm{Sec}_o(\xi_\varsigma)$ choose a volume ω on M (M is assumed to be orientable) and define

$$\langle \sigma_1, \sigma_2 \rangle = \int_\omega \langle \sigma_1(m), \sigma_2(m) \rangle_m ,$$

$\sigma_1, \sigma_2 \in \mathrm{Sec}_o(\xi_\varsigma)$. The corresponding Hilbert space is denoted by $L^2(\xi_\varsigma, \omega)$.

4. A $\mathfrak{g}$-action (ϕ_ρ, θ) on ξ_ρ induces a representation - called vector field representation - of $\mathfrak{g}$ on $Sec_0(\xi_\rho)$ via a Lie algebra homomorphism $D(\phi_\rho, \theta) : \mathfrak{g} \longrightarrow End\ Sec_0(\xi_\rho)$ through

$$(D(\phi_\rho, \theta)(x)\sigma)(m) = \frac{d}{dt} F(\phi_\rho(x))(\sigma(F(\theta(x))(m,t)),-t).$$

To have this representation skew-adjoint, i.e. $D(\phi_\rho, \theta)(x)$ essentially skew-adjoint for $x \in \mathfrak{g}$ on $Sec_0(\xi_\rho)$, one needs some restrictions on θ. $X \in \mathcal{V}(M)$ is quasi-complete (is complete) if the set

$$E(X,t) = M \setminus D(X,t)$$

is of measure zero (is empty) for $t \in \mathbb{R}$. So θ is quasi-complete or complete if $\theta(x)$ has this property for $x \in \mathfrak{g}$. Furthermore, ω is said to be θ-invariant if $L_{\theta(x)}\omega = 0$, $x \in \mathfrak{g}$.

Lemma 1 : Let (ϕ_ρ, θ) be a $\mathfrak{g}$-action on ξ_ρ such that θ is quasi-complete. Take an θ-invariant ω. Then the induced vector field representation $D(\phi_\rho, \theta)$ is skew-adjoint on $Sec_0(\xi_\rho) \subset L^2(\xi_\rho, \omega)$.

For the proof see the Appendix.

5. $D(\phi_\rho, \theta)$ can be constructed also from (ϕ, θ) on ξ using the space $\mathcal{E}_0(P, \rho)$ of smooth and compactly supported equivariant functions $f : P \longrightarrow V$, $f(ph) = \rho^{-1}(h)f(p)$, since the representation $D_\rho(\phi, \theta) : \mathfrak{g} \longrightarrow End\ \mathcal{E}_0(P, \rho)$ given by

$$(D_\rho(\phi,\theta)(x)f)(p) = \frac{d}{dt} f(F(\phi(x))(p,t))$$

is unitarily equivalent to $D(\phi_\rho, \theta)$ via the isomorphism $\psi_\rho : \mathcal{E}_0(P,\rho) \longrightarrow Sec_0(\xi_\rho)$, $(\psi_\rho(f))(m) = [p, f(p)]_\rho$ for $p \in \pi^{-1}(m)$.

6. In the special case where (ϕ, θ) is the differential of a G-action on ξ which is free and transitive on P, i.e. if ξ is isomorphic to $(G, \pi, G/H, H)$, the quantization gives an induced representation in the sense of Mackey ([12], [13]).

3. Loop criteria

1. A skew-adjoint representation $D : \mathfrak{g} \longrightarrow \mathcal{A}(\mathcal{D})$ is called G-integrable (G being a connected Lie group with Lie algebra $\mathfrak{g}$), if there exists a unitary representation $U : G \longrightarrow \mathcal{U}(\mathcal{H})$ with differential $D_U = D$, i.e. if the diagram

$$\begin{array}{ccc} G & \xrightarrow{\;U\;} & \mathcal{U}(\mathcal{H}) \\ \exp \uparrow & & \uparrow Exp \\ \mathfrak{g} & \xrightarrow{\;D\;} & \mathcal{A}(\mathcal{D}) \end{array}$$

is commutative. Here $\mathcal{A}(\mathcal{D})$ is the subset of essentially skew-adjoint operators in the Lie algebra $S(\mathcal{D})$ of symmetric operators on a common dense invariant domain $\mathcal{D} \subset \mathcal{H}$.
As a simple geometric tool to decide on integrability we use loops in G and in $\mathcal{U}(\mathcal{H})$. Take the set $l(G)$ of all loops in G starting at e and consider

$$\hat{l}(G) = \{(x_1, \dots, x_k) \mid k \in \mathbb{N},\ x_i \in \mathfrak{g}\ ,\ \exp x_1 \dots \exp x_k = e\},$$

which can be regarded as a subset of $l(G)$ via the mapping $j : \hat{l}(G) \longrightarrow l(G)$,

$$(j(x_1, \dots, x_k))(t) = \exp x_1 \dots \exp x_{n-1} \exp(kt-n+1)x_n$$

for $t \in \Delta_n/k$, $\Delta_n = [n-1,n]$, $n = 1,\dots,k$. Denote by $C(\mathcal{U}(\mathcal{H}))$ and $l(\mathcal{U}(\mathcal{H}))$ the set of curves and loops, respectively, in $\mathcal{U}(\mathcal{H})$ starting at 1. Then there exists a natural map $\delta(D,G) : \hat{l}(G) \longrightarrow C(\mathcal{U}(\mathcal{H}))$ with

$$\delta(D,G)(x_1,\dots,x_k)(t) = Exp\ D(x_1) \dots Exp\ D(x_{n-1})\ Exp(kt-n+1)D(x_n)$$

for $t \in \Delta_n/k$, $\Delta_n = [n-1,n]$, $n = 1,\dots,k$. This map controls the integrability of D. Any $g \in G$ (G is connected) can be written as $g = \exp x_1 \dots \exp x_k$ for suitable $x_i \in \mathfrak{g}$; hence the defining diagram gives

Lemma 2 : A skew-adjoint representation D of $\mathfrak{g}$ in $\mathcal{H}$ is G-integrable if and only if $Im\ \delta(D,G) \subset l(\mathcal{U}(\mathcal{H}))$.

2. To apply this loop criterion to vector field representations, we introduce the sets $C(P,p)$ and $l(P,p)$ of curves and loops in P starting at p. Consider $\phi : \mathfrak{g} \longrightarrow \mathcal{W}(P)$, take the set

$$l(\phi,G,p) = \left\{(x_1,\dots,x_k) \in \hat{l}(G) \,\middle|\, \varphi_1^{\phi(x_k)} \cdots \varphi_1^{\phi(x_1)}(p) \text{ exists}\right\}$$

and construct, as above, $\delta(\phi,G,p) : l(\phi,G,p) \longrightarrow C(P,p)$ with

$$\delta(\phi,G,p)(x_1,\dots,x_k)(t) = \varphi_{kt-n+1}^{\phi(x_n)} \varphi_1^{\phi(x_{n-1})} \cdots \varphi_1^{\phi(x_1)}(p)$$

for $t \in \Delta_n/k$, $\Delta_n = [n-1,n]$, $n = 1,\dots,k$. Again, this δ-map completely controls the integrability of skew-adjoint vector-field representations.

Theorem 1 : A skew-adjoint vector field representation $D(\phi_\xi,\theta)$ induced from a $\mathfrak{g}$-action (ϕ_ξ,θ) is G-integrable if and only if

$$\operatorname{Im} \delta(\phi_\xi,G,q) \subset l(E_\xi,q) \qquad (1)$$

for $q \in E_\xi$.

For the proof see the Appendix. Integrability criteria for symmetric or arbitrary skew-adjoint D of $\mathfrak{g}$ can be found in [14], [15].

3. This result relates the integrability of $D(\phi_\xi,\theta)$ to a global property of the $\mathfrak{g}$-action. Condition (1) can be identified with an infinitesimal version of Palais' maximal local G-action [10]. Hence we call ϕ_ξ on E_ξ G-maximal if condition (1) is fulfilled and the results in [10] imply that θ is G-maximal if and only if it is the differential of a 'restriction' of a global G-action $\bar{\phi}$ on a manifold $\bar{M}$ which contains M as an open submanifold. Furthermore, any complete θ is $\tilde{G}$-maximal; $\tilde{G}$ is the unicover of G.

4. The construction of a skew-adjoint non $\mathbb{R}^2$-integrable representation by Nelson [9] takes for $\theta : \mathbb{R}^2 \rightarrow \mathcal{W}(M)$ a non-$\mathbb{R}^2$-maximal action, which is generated (up to one point) from a covering (see Section 4) of the natural $\mathbb{R}^2$-action on the torus minus one point; in this case any lift of θ to an action on the total space of a bundle over M is non-$\mathbb{R}^2$-maximal too. In [16], [17] a non-maximal θ given by a double covering of $\mathbb{R}^2-(0,0)$ is used.

4. An application: Coverings

1. G-maximality of ϕ implies G-maximality of θ ; however, the converse is not true. To construct an example, take as geometrical arena a physically distinguished, connected, but not simply connected manifold M (e.g. $\mathbb{R}^2-(0,0)$, $\mathbb{R}^3-\mathbb{R}$) and a regular covering $p : M^N \longrightarrow M$ characterized by a normal subgroup N of the non-trivial fundamental group $\pi_1(M,m_o)$. Such a covering is a simple case of a non-trivial principal fibre bundle, given here as

$$\xi^{M,N} = (M^N,p,M,\pi_1(M,m_o)/N);$$

the structure group $\pi_1(M,m_o)/N$ is discrete and the coverings are classified by normal subgroups $N \subset \pi_1(M,m_o)$.

2. Let $\theta : \mathfrak{g} \longrightarrow \mathcal{W}(M)$ be G-maximal. Since p is a local diffeomorphism, θ induces a $\mathfrak{g}$-action

$$\theta^N : \mathfrak{g} \longrightarrow \mathcal{W}(M^N)$$

on M^N such that (θ^N,θ) is a $\mathfrak{g}$-action on $\xi^{M,N}$. Consider

$$\varepsilon(\theta,G,m) = \nu(m) \circ \delta(\theta,G,m)$$

with $\nu(m) : C(M,m) \longrightarrow \Omega(M,m)$ being the natural surjection onto the set of homotopy classes of curves in $C(M,m)$. Since a covering has unique path lifting, the following conditions are equivalent:

i) θ^N is G-maximal;
ii) $\operatorname{Im} \varepsilon(\theta,G,p(m')) \subset p_*\pi_1(M^N,m')$ for $m' \in M^N$.

For transitive G-maximal θ this criterion can be simplified: The covering lift θ^N of θ is G-maximal iff there exists a $m_o' \in M^N$ such that

$$\operatorname{Im} \varepsilon(\theta,G,p(m_o')) \subset p_*\pi_1(M^N,m_o').$$

For $m_o' \in p^{-1}(m_o)$ we have $p_*\pi_1(M^N,m_o') = N$.

3. For the construction of an associated vector bundle we use a unitary faithful finite-dimensional representation $\varrho : \pi_1(M,m_0)/N \longrightarrow \mathrm{Aut}\, V$. By Lemma 1 the corresponding vector field representation $D(\theta_\varrho^N,\theta)$ of $\mathfrak{g}$ on $\mathrm{Sec}_0(\xi_\varrho^{M,N})$ is skew-adjoint if θ is quasi-complete and ω is θ-invariant. For the integrability, Theorem 1 yields:

Lemma 3 : A skew-adjoint vector field representation $D(\theta_\varrho^N,\theta)$ induced from a transitive and G-maximal $\mathfrak{g}$-action θ on M via a unitary and faithful ϱ is G-integrable if and only if

$$\mathrm{Im}\ \varepsilon(\theta,G,m_0) \subset N.$$

4. As an example take $M = \mathbb{R}^2-(0,0)$, $\omega = dx_1 \wedge dx_2$, $\mathfrak{g} = \mathbb{R}^2$ (two-dimensional abelian Lie algebra) and define $\theta : \mathfrak{g} \longrightarrow \mathcal{V}(M)$ by

$$\theta(a,b) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}.$$

θ is quasi-complete and transitive, ω is θ-invariant. Furthermore, θ is G-maximal for $G = \mathbb{R}^2$. Because

$$\pi_1(\mathbb{R}^2-(0,0),*) = \mathrm{Im}\ \varepsilon(\theta,\mathbb{R}^2,*) \cong \mathbb{Z},$$

Lemma 3 implies that $D(\theta_\varrho^N,\theta)$ is $\mathbb{R}^2$-integrable iff $\mathbb{Z} \subset N$, i.e. iff $N = \mathbb{Z}$. Hence for $N = 2\mathbb{Z} \subset \mathbb{Z}$ (which corresponds to a double covering of $\mathbb{R}^2-(0,0)$), and

$$\varrho : \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 \longrightarrow \mathrm{Aut}\ \mathbb{C}$$

given by $\varrho(0) = 1$, $\varrho(1) = e^{i\pi}$, the skew-adjoint representation

$$D(\theta_\varrho^{2\mathbb{Z}},\theta) : \mathbb{R}^2 \longrightarrow \mathcal{A}(\mathrm{Sec}_0\xi^{M,2\mathbb{Z}})$$

is non-G-integrable for any G with $\mathfrak{g} \cong \mathbb{R}^2$.

5. Closing remarks

1. A system with a manifold M as configuration space and a $\mathfrak{g}$-action θ on M connected with physical momenta can be quantized through a quantization bundle $\xi_\varrho = (E_\varrho,\pi_\varrho,M,V)$. This is an associated vector bundle of a principal fibre bundle $\xi = (P,\pi,M,H)$ with structure group H and $\mathfrak{g}$-action (ϕ,θ), where ϕ is a lift of θ. The physical interpretation of H and its representation ϱ is model dependent. In special cases (as for quantum mechanics on homogeneous spaces [18]) the Lie algebra of H may be chosen as a subalgebra of $\mathfrak{g}$.

The quantization method leads to compactly supported sections in ξ_ϱ as states and, with a volume form ω on M, to the Hilbert space $L^2(\xi_\varrho,\omega)$; the Borel sets on M correspond to projection operators and the $\mathfrak{g}$-action leads to a vector field representation $D(\phi_\varrho,\theta)$ of $\mathfrak{g}$. The generators of $D(\phi_\varrho,\theta)$ can be interpreted as observables (momentum operators) if they are essentially skew-adjoint. This is the case if θ on M is quasi-complete and if the form ω is θ-invariant.

As the $\mathfrak{g}$-action on E_ϱ does not need to be G-maximal, also $D(\phi_\varrho,\theta)$ on $\mathrm{Sec}_0(\xi_\varrho) \subset L^2(\xi_\varrho,\omega)$ is not necessarily the differential of a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$, i.e. $D(\phi_\varrho,\theta)$ may be non-G-integrable. However, its integrability depends on the geometry of (ϕ_ϱ,θ) on ξ_ϱ only. The key is the loop criterion (see Lemma 2,3) or, as an equivalent condition, the G-maximality of (ϕ_ϱ,θ). So the integrability for vector field representations is reduced to a pure geometrical problem. Skew-adjoint representations of $\mathfrak{g}$ not being vector field representations will not appear as a result of this geometrical quantization procedure.

2. For the M given, some physical information is needed to choose a $\mathfrak{g}$-action θ. If θ is taken to be G-maximal, it can be considered as a 'restriction' of a complete $\mathfrak{g}$-action on some manifold $\overline{M} \supset M$ (see Section 3.3) and it is justified to call points in $\overline{M} - M = M'$ 'singularities' of θ; they cannot be reached by the system moving on M. The skew-adjointness of $D(\phi_\varsigma, \theta)$ is directly connected with M' and is assured if the integral curves ending in a singularity build a set of measure zero; under some restrictions (c.f. the examples given above) this is equivalent to

$$\dim \overline{M} - \dim M' \geq 2.$$

This condition is related to the construction of non-integrable representations via coverings (Section 4), because only for

$$\dim \overline{M} - \dim M' = 2$$

the homomorphism

$$i_* : \pi_1(\overline{M} - M', m) \longrightarrow \pi_1(\overline{M}, m)$$

is not injective in general [19], which gives the possibility to use the non-simply-connectedness of $\overline{M} - M'$ (Lemma 3).

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6. Appendix

1. Proof of Lemma 1: Put $(x \in \mathfrak{g})$

$$\Gamma(\theta(x), t) = \{\sigma \in \mathrm{Sec}_o(\xi_\varsigma) \mid \operatorname{supp}\sigma \subset D(\theta(x), t)\}.$$

Define $U_t(\phi_\varsigma, \theta)(x) : \Gamma(\theta(x), -t) \longrightarrow \Gamma(\theta(x), t)$ by $(\sigma \in \Gamma(\theta(x), -t))$

$$(U_t(\phi_\varsigma, \theta)(x)\sigma)(m) = \begin{cases} F(\phi_\varsigma(x))(\sigma(F(\theta(x))(m,t)), -t) & \text{if } m \in D(\theta(x), t); \\ 0_m & \text{otherwise} \end{cases}$$

which is bijective and, moreover, isometric since

i) $M - D(\theta(x), t)$ is of measure zero;

ii) $L_{\theta(x)}\omega = 0$;

iii) for $m \in D(\theta(x), t)$ the association

$$q \in \pi_\varsigma^{-1}(m) \longrightarrow F(\phi_\varsigma(x))(q, t) \in \pi_\varsigma^{-1}(F(\theta(x))(m, t))$$

is unitary.

$\Gamma(\theta(x), t)$ is dense in $\mathrm{Sec}_o(\xi_\varsigma)$, so $U_t(\phi_\varsigma, \theta)(x)$ extends to a unitary operator on $L^2(\xi_\varsigma, \omega)$ and

$$t \in \mathbb{R} \longrightarrow U_t(\phi_\varsigma, \theta)(x) \in \mathcal{U}(\mathcal{H})$$

is a strongly continuous one-parameter unitary group. By Stone's theorem, there is a (unique) skew-adjoint operator $A(x)$ on $\mathcal{D}(A(x)) \subset L^2(\xi_\varsigma, \omega)$, $\mathcal{D}(A(x)) \supset \mathrm{Sec}_o(\xi_\varsigma)$, such that

$$A(x) \big| \mathrm{Sec}_o(\xi_\varsigma) = D(\phi_\varsigma, \theta)(x).$$

Thus $D(\phi_\varsigma, \theta)(x)$ is skew-symmetric on $\mathrm{Sec}_o(\xi_\varsigma)$. It is even essentially skew-adjoint on $\mathrm{Sec}_o(\xi_\varsigma) \subset L^2(\xi_\varsigma, \omega)$, which can be shown directly, using the method indicated in [9]. By definition, $\mathrm{Sec}_o(\xi_\varsigma)$ is a common invariant domain for all $D(\phi_\varsigma, \theta)(x)$, $x \in \mathfrak{g}$.

2. Proof of Theorem 1: Consider the sets ($x_i \in \mathfrak{g}$)

$$D(\theta;x_1,\ldots,x_k) = \left\{ m\in M \,\middle|\, \varphi_1^{\theta(x_k)} \cdots \varphi_1^{\theta(x_1)}(m) \text{ exists} \right\}$$

and

$$\Gamma(\theta;x_1,\ldots,x_k) = \left\{ \sigma \in \mathrm{Sec}_o(\xi_\varsigma) \,\middle|\, \mathrm{supp}\,\sigma \subset D(\theta;x_1,\ldots,x_k) \right\}.$$

Since $D(\theta(x),t)$ is open in M, the same holds for $D(\theta;x_1,\ldots,x_k)$. By the quasi-completeness of θ ,

$$E(\theta;x_1,\ldots,x_k) := M - D(\theta;x_1,\ldots,x_k)$$

is a set of measure zero. So $\Gamma(\theta;x_1,\ldots,x_k)$ is dense in $\mathrm{Sec}_o(\xi_\varsigma)$. Because of Lemma 2 we have to prove that

i) $\mathrm{Im}\,\delta(D(\phi_\varsigma,\theta),G) \subset 1(\mathcal{U}(L^2(\xi_\varsigma,\omega)))$

and

ii) $\mathrm{Im}\,\delta(\phi_\varsigma,G,q) \subset 1(E_\varsigma,q)$ for $q\in E_\varsigma$

are equivalent. Now condition i) is equivalent to

$$(*) \qquad \mathrm{Exp}\, D(\phi_\varsigma,\theta)(x_1) \ldots \mathrm{Exp}\, D(\phi_\varsigma,\theta)(x_k)\,\sigma = \sigma$$

for $(x_1,\ldots,x_k) \in \hat{1}(G)$, $\sigma\in\Gamma(\theta;x_1,\ldots,x_k)$. Stone's theorem gives (compare the discussion in the proof of Lemma 1)

$$(\mathrm{Exp}\, D(\phi_\varsigma,\theta)(x)\sigma)(m) = (U_1(\phi_\varsigma,\theta)(x)\sigma)(m)$$
$$= F(\phi_\varsigma(x))(\sigma(F(\theta(x))(m,1)),-1)$$

for $x\in\mathfrak{g}$, $\sigma\in\Gamma(\theta;x)$, $m\in D(\theta;x)$. Hence (*) is equivalent to

$$\delta(\phi_\varsigma,G,\sigma(\delta(\theta,G,\pi_\varsigma(q))(x_1,\ldots,x_k)(1)))(-x_k,\ldots,-x_1)(1) = \sigma(\pi_\varsigma(q))$$

for $q\in D(\phi_\varsigma;x_1,\ldots,x_k)$, $\sigma\in\Gamma(\theta;x_1,\ldots,x_k)$, $(x_1,\ldots,x_k) \in \hat{1}(G)$. Because $D(\phi_\varsigma;x_1,\ldots,x_k)$ is open in E_ς, (*) is equivalent to

$$\delta(\phi_\varsigma,G,q)(x_1,\ldots,x_k)(1) = q$$

for $(x_1,\ldots,x_k) \in \hat{1}(G)$, $q \in D(\phi_\varsigma;x_1,\ldots,x_k)$. But this is equivalent to condition ii).

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