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Hungarian Academy of Sciences

CENTRAL
RESEARCH
INSTITUTE FOR
PHYSICS

BUDAPEST

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HU ISSN 0368 5330
ISBN 963 371 394 3

ABSTRACT

A weakly turbulent, magnetoactive plasma is considered in an inhomogeneous case, with anisotropic temperature distribution. The dispersion relation is established following a method developed by Tsytovich and Nekrasov. The correction coefficients are calculated in the three principal scaling modes: 1. the turbulent frequencies, 2. the cyclotronic velocities of the macroinstabilities, $n\omega_c/k_z$, predominate, and 3. the turbulent frequencies are lower than the other.

АННОТАЦИЯ

В работе рассматривается неоднородная, слабо турбулентная магнитоактивная плазма с анизотропным распределением температуры. Составляется дисперсионное уравнение по методу Цитовича и Некрасова. Вычисляются коррекционные коэффициенты слабой турбулентности в трех основных случаях: когда доминируют 1. частоты турбулентности, 2. циклотронная частота $/n\omega_c/k_z/$ макронеустойчивости, и, наконец, 3. когда частота турбулентности - ниже, чем остальные.

KIVONAT

Inhomogén, anizotróp hőmérsékleteloszlásu gyengén turbulens mágnesesen aktiv plazmát tekintünk. Citovics és Nyekraszov módszere szerint felépitjük a diszperziós relációt. A gyenge turbulencia korrekciós koefficienseit három fő skálaviszony esetén határoztuk meg: 1. ha a turbulens frekvenciák, 2. a makroinstabilitás ciklotron-sebessége $/n\omega_c/k_z/$ a domináns, vagy 3. ha a turbulens frekvencia a legalacsonyabb.

Introduction

It is well known that in many cases the different forms of turbulence are responsible for the anomalous diffusive properties of a plasma [1]. These anomalies may be very dangerous for the confinement of a thermonuclear device, but help to heat it [3], [6], [7].

It was Rogister [2] who first drew to the present author's attention to the importance of the problem that in a given type of turbulence the stability of a plasma may be influenced by the turbulent pulsations in a wrong direction. This problem can be tackled most simply by a consideration of the dispersion properties.

Krivoirutski, Makhankov and Tsytovich [4] studied the case of a high-frequency plasma turbulence and the stability problem against the drift waves. The calculations of ref. [4] were limited to the drift approximation, and the weak turbulence theory was used. The magnetic field was considered to be uniform.

Nekrasov [5] gave the solution of a more general problem: the frequency limits were omitted and so a non-linear dispersion relation was established. Nekrasov applied this general dispersion equation to the ionic cyclotronic turbulence spectrum.

Formulation of the dispersion relation

According to the weak turbulence theory /[6], [7] and [5]/ we rebuilt the non-linear dispersion relation for a non-uniform plasma.

Let us take an electrostatic turbulent field

$$\vec{E}^{\,turb.} = -\frac{\partial \varphi}{\partial \vec{r}}$$

of many waves in a collisionless plasma medium. In this case the Vlasov equation holds for the one-particle distribution function

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{r}} + \vec{a}\frac{\partial f}{\partial \vec{v}} = 0 \qquad (1)$$

Here $\vec{a}$ is the acceleration of a particle which stems from the external field and from the averaged field of the other particles. For the turbulent case it is usual to divide the fields and the distribution function into two parts: an averaged /regular/ and a fluctuating /turbulent/ group, viz:

$$f = f^r + f^t; \qquad \varphi = \varphi^r + \varphi^t \qquad (2)$$

From the viewpoint of electrostatic turbulence the electromagnetic component of the turbulent field is neglected,

$$\vec{B}^{t} \simeq 0 \; ; \quad \vec{E}^{t}_{\text{electromagn.}} \cong 0 \qquad (2)$$

The division of the ^{r} and ^{t} parts is defined by the averaging process

$$\langle f \rangle = f^{r} \; ; \qquad \langle \varphi \rangle = \varphi^{r}$$
$$\langle f^{t} \rangle = 0 \; ; \qquad \langle \varphi^{t} \rangle = 0 \qquad (3)$$

We suppose the averaging to be unambigous.

If the interaction between the turbulent and the regular domain is not too intensive we can say that the modification of the quasistationary states is characterised by linear deviations,

$$f^{r} = f^{r}_{0} + \varepsilon f^{r}_{1} \; ; \qquad f^{t} = f^{t}_{0} + \varepsilon f^{t}_{1}$$
$$\varphi^{r} = \varepsilon \varphi^{r}_{1} \; ; \qquad \varphi^{t} = \varphi^{t}_{0} + \varepsilon \varphi^{t}_{1} \qquad (4)$$

The coupling being weak: $\varepsilon \ll 1$, therefore we can apply the weak turbulence theory.

Let us now perform the averaging upon the Vlasov equation (1) of the j -th component of our plasma, together with (2) and (3). We get for the zeroth and first order regular function

$$\hat{L}_j f^{r}_{0j} = \frac{e_j}{m_j} \left\langle \frac{\partial \varphi^{t}_{0}}{\partial \vec{r}} \frac{\partial f^{t}_{0j}}{\partial \vec{v}} \right\rangle \qquad (5)$$

$$\hat{L}_j f^{r}_{1j} = \frac{e_j}{m_j} \left(\frac{\partial \varphi^{r}_{1}}{\partial \vec{r}} \frac{\partial f^{r}_{0j}}{\partial \vec{v}} + \left\langle \frac{\partial \varphi^{t}_{1}}{\partial \vec{r}} \frac{\partial f^{t}_{0j}}{\partial \vec{v}} \right\rangle + \left\langle \frac{\partial \varphi^{t}_{0}}{\partial \vec{r}} \frac{\partial f^{t}_{1j}}{\partial \vec{v}} \right\rangle \right) \qquad (6)$$

Here $\hat{L}_j$ is the Vlasov operator for the one-particle case, i.e. without the averaged field

$$\hat{L}_j \doteq \frac{\partial}{\partial t} + \vec{v}\frac{\partial}{\partial \vec{r}} + \left(\vec{a}_{j\,ext} + \frac{e_j}{m_j}(\vec{v}\times\vec{B}_o)\right)\frac{\partial}{\partial \vec{v}}$$

$\vec{a}_{j\,ext}$ is the acceleration of the particle due to an external field.

Restricting ourselves to the linear expansion (4) we can find the equation for the turbulent part in zeroth and first order

$$\hat{L}_j f^t_{oj} = \frac{e_j}{m_j}\frac{\partial \varphi^t_o}{\partial \vec{r}}\frac{\partial f^r_{oj}}{\partial \vec{v}} \tag{7}$$

$$\hat{L}_j f^t_{1j} = \frac{e_j}{m_j}\left(\frac{\partial \varphi^t_1}{\partial \vec{r}}\frac{\partial f^r_{oj}}{\partial \vec{v}} + \frac{\partial \varphi^t_o}{\partial \vec{r}}\frac{\partial f^r_{1j}}{\partial \vec{v}} + \frac{\partial \varphi^r_1}{\partial \vec{r}}\frac{\partial f^t_{oj}}{\partial \vec{v}}\right) \tag{8}$$

where we meglected the $O(\varepsilon^3)$ terms.

According to the quasilinear equation (5) the zeroth order regular function exhibits a relaxation due to the right hand side of (5) , but in the quasi-stationary state the following is true:

$$\hat{L}_j f^r_{oj} = 0 \tag{9}$$

This means that up to an accuracy of $O(\varepsilon^2)$ we can use the condition (9) for calculating the turbulent corrections.

Assuming the stationarity of the f_{oj}^{r} -function, (9) asserts that it depends only upon its integrals of motion /energy, magnetic moment and velocity parallel to the magnetic field, say $\parallel z$ /,

$$f_{oj}^{r} = f_{oj}^{r}\left(w, u_{y}, v_{z}\right) \tag{10}$$

If we take into account a potential energy of constant intensity /in the direction x /, and in this direction the non-uniformity of the magnetic field,

$$w_{j} \doteq v_x^2 + v_y^2 - 2V_{pot,j} \; ; \quad V_{pot} = prop \cdot x \tag{11}$$

and let

$$B_0(x) \doteq B_0 + prop \cdot x = B_0 + \mathcal{O}(prop\, x \cdot \varepsilon) \tag{12}$$

we can write

$$u_y \doteq v_y + \frac{e_j}{m_j c} B_0 x + \mathcal{O}(\varepsilon, x^2) \tag{13}$$

The Poisson equation relates the electric fields and the distribution functions,

$$\Delta\varphi = -4\pi \sum_j e_j \int f_j \, d\vec{v} \tag{14}$$

According to the scaling (4) and (5), (6), (7), (8), the regular parts depend upon the averages of the turbulent functions too.

Should we wish to obtain a relation for the stability in the presence of the turbulent fluctuations, we must calculate the regular part of the potential in the first order

$$\varphi_1^r\left(\varphi_0^t,\ \varphi_1^t\right) \tag{15}$$

We can use the method of integration along the unperturbed trajectories /see e.g. [1]/. Let us introduce a Fourier representation $(\vec{k},\omega),(\vec{k}',\omega')$, etc. in space and time, we find the solution of the equation (7) in a series of Bessel functions, J_n -s. Using the notation of [5],

$$f^t_{0j,k'\omega'} = -\frac{e_j}{m_j}\varphi^t_{0k'\omega'}\sum_{n,p} e^{i(p-n)(\theta-\alpha_{\vec{k}'})}\hat{L}^{np}_{j\vec{k}'\omega'} f^r_{0j}\ ; \tag{16}$$

It is mentioned here that f^r_{0j} denotes the quasi-stationary distribution function, i.e. it satisfies (9) and (10). The operator $\hat{L}^{np}_{j\vec{k},\omega}$ is the following

$$\hat{L}^{np}_{j\vec{k},\omega} \doteq \frac{J_n(k_\perp v_\perp/\omega_{cj})\, J_p(k_\perp v_\perp/\omega_{cj})}{\omega - n\omega_{cj} - k_y v_{Dj} - k_z v_z} \times \tag{17}$$

$$\times\left[2(n\omega_{cj}+k_y v_{Dj})\frac{\partial}{\partial w} + k_y\frac{\partial}{\partial u_y} + k_z\frac{\partial}{\partial v_z}\right]$$

ω_{cj} is the j-th particle cyclotron frequency, v_{0j} is the drift velocity induced by the inhomogeneity and the external potential V_{pot}. The angles θ and $\alpha_{\vec{k}}$-s are defined by the cylindrical coordinate system,

$$\left[k_\perp \cos\alpha_{\vec{k}},\ k_\perp \sin\alpha_{\vec{k}},\ k_z\right];\quad \left[v_\perp \cos\theta,\ v_\perp \sin\theta + v_{0j},\ v_z\right] \tag{18}$$

The form (16) of the zeroth order turbulent quantities expresses the proportionality in the linearized case, and it is a generalization of the well known formula of the linear theory

$$E_{\vec{k}} \quad \text{prop} \quad f^{1}_{\vec{k}} \tag{19}$$

The first order term of the turbulent distribution function is given by equation (8). If we restrict ourselves to the first order terms it is sufficient to take into account only the right hand side of (8), i.e. to neglect the turbulent terms in (6),

$$\hat{L}_j f^{r\,0}_{1j} = \frac{e_j}{m_j} \frac{\partial \varphi^{r}_{1}}{\partial \vec{r}} \frac{\partial f^{r}_{0j}}{\partial \vec{v}} \tag{6*}$$

That is, we have considered only the developed turbulence and neglected the evolution of the spectrum. The solution of (6*) is then similar to (16), so

$$f^{r\,0}_{1j\,\vec{k},\omega} = -\frac{e_j}{m_j} \varphi^{r}_{1\vec{k},\omega} \sum_{n,p} e^{i(p-n)(\theta - \omega t)} \hat{L}^{np}_{j\,\vec{k},\omega} f^{r}_{0j} \tag{16*}$$

The turbulent part of the distribution function is, in the first order,

$$f^{t}_{1\vec{k}'\omega'} = \Big\{ -\frac{e_j}{m_j}\, \varphi^{t}_{1\vec{k}'\omega'} \sum_{n,p} e^{-i(n-p)(\theta-\alpha_{\vec{k}'})}\, \hat{L}^{np}_{j\,\vec{k}'\omega'} +$$

$$+ \frac{e_j}{m_j} \int d\vec{k}''\, d\omega'' \Big(\varphi^{r}_{1\vec{k}''\omega''}\, \varphi^{t}_{0\,\vec{k}'-\vec{k}'',\,\omega'-\omega''} + \varphi^{r}_{1\,\vec{k}'-\vec{k}'',\,\omega'-\omega''}\, \varphi^{t}_{0\vec{k}'',\omega''} \Big) \times \tag{20}$$

$$\times \sum_{npqr} \exp\Big[-i(n-p+q-r)\theta + i(q-r)\alpha_{k'} + i(n-p)\alpha_{\vec{k}'-\vec{k}''}\Big] \times$$

$$\times\, \hat{L}^{npqr}_{j\,\vec{k}',\vec{k}'',\omega'}\, \hat{L}^{np}_{j\,\vec{k}'-\vec{k}'',\,\omega'-\omega''} \Big\} f^{r}_{0j}$$

The operator $\hat{L}^{npqr}$ is given by the following expression,

$$\hat{L}^{npqr}_{j\,\vec{k},\vec{k}',\omega} \doteq \frac{1}{\omega-(n-p+q)\omega_{cj}-k_y v_{Dj}-k_z v_z} \times$$

$$\times \Big[k'_{\perp}/2\; J_{q+1}\, J_r\, e^{i(\alpha_{\vec{k}}-\alpha_{\vec{k}'})} \Big(2 v_{\perp} \frac{\partial^{*}}{\partial w} + \frac{n-p}{v_{\perp}} \Big) +$$

$$+ k'_{\perp}/2\; J_{q-1}\, J_r\, e^{-i(\alpha_{\vec{k}}-\alpha_{\vec{k}'})} \Big(2 v_{\perp} \frac{\partial^{*}}{\partial w} - \frac{n-p}{v_{\perp}} \Big) + \tag{21}$$

$$+ J_q\, J_r \Big[2 k'_y\, v_{Dj} \frac{\partial}{\partial w} + k'_y \frac{\partial}{\partial u_y} + k'_z \frac{\partial}{\partial v_z} \Big]$$

In $\hat{L}^{npqr}$ the operator $v_{Dj}\frac{\partial}{\partial w}$ acts upon the distribution function only, $\partial^{*}/\partial w$ denotes $\partial/\partial v_{\perp}^{2}$ if it acts upon the Bessel functions and $\partial/\partial w$ otherwise.

The regular part of the first-order distribution function is given by (6), here the expressions (20) are to be used, viz:

$$
\begin{aligned}
f^{r}_{1 j \varkappa} = \Big\{ & - \frac{e_j}{m_j} \varphi^{r}_{1\varkappa} \sum_{n,p} e^{i(p-n)(\theta - \alpha_{\vec{k}})} \hat{L}^{np}_{j\varkappa} + \\
& + \frac{e_j^2}{m_j^2} \int d\varkappa' \left[\langle \varphi^{t}_{1\varkappa'} \varphi^{t}_{0\,\varkappa-\varkappa'} \rangle + \langle \varphi^{t}_{1\,\varkappa-\varkappa'} \varphi^{t}_{0\varkappa'} \rangle \right] \times \\
& \times \sum_{npqr} e^{-i(n-p+q-r)\theta + i(q-r)\alpha_{\vec{k}} + i(n-p)\alpha_{\vec{k}'}} \hat{L}^{npqr}_{j\,\vec{k},\vec{k}',\omega} \hat{L}^{np}_{j\,\varkappa-\varkappa'} - \\
& - \frac{e_j^3}{m_j^3} \int d\varkappa' d\varkappa'' \Big[\varphi^{r}_{1\varkappa''} \langle \varphi^{t}_{0\varkappa'} \varphi^{t}_{0\,\varkappa-\varkappa'-\varkappa''} \rangle + \\
& + \varphi^{r}_{1\,\varkappa-\varkappa'-\varkappa''} \langle \varphi^{t}_{0\varkappa'} \varphi^{t}_{0\varkappa''} \rangle + \varphi^{r}_{1\varkappa'} \langle \varphi^{t}_{0\varkappa''} \varphi^{t}_{0\,\varkappa-\varkappa'-\varkappa''} \rangle \Big] \times \\
& \times \sum_{npqrst} \left[e^{-i(n-p+q-r+s-t)\theta + i(s-t)\alpha_{\vec{k}} + i(q-r)\alpha_{\vec{k}-\vec{k}'} + i(n-p)\alpha_{\vec{k}-\vec{k}'-\vec{k}''}} \right] \\
& \times \hat{L}^{npqrst}_{j\,\vec{k},\vec{k}',\omega} \hat{L}^{npqr}_{j\,\vec{k}-\vec{k}',\vec{k}'',\omega-\omega'} \hat{L}^{np}_{j\,\varkappa-\varkappa'-\varkappa''} \Big\} f_{0j} \; ;
\end{aligned}
\tag{22}
$$

$$
\varkappa \doteq [\vec{k}, \omega] \; ; \qquad d\varkappa \doteq d\vec{k}\, d\omega \tag{22a}
$$

The calculations are similar for the f^{r}_{1} part. The operator $\hat{L}^{npqrst}$ is the same as $\hat{L}^{npqr}$, the numbers $n-p+q-r$ and $n-p$ are to be substituted by $n-p+q-r+s-t$ and $n-p+q-r$, and the Bessel functions J_r and J_q by J_t and J_s. In the last bracket the contribution of the quasilinear relaxation is to be subtracted /cf. equation (5)/.

We now need an expression for φ_1^t , in order to get an equation for φ_1^r , by using (22).

The linear part $\hat{L}^{np}$ connects the linear dielectric response $\varepsilon(\vec{k},\omega)$ and the stationary distribution function f_{oj}^r as is usual,

$$\varepsilon(\vec{k},\omega) = 1 + \sum_j \sum_{np} \frac{4\pi e_j^2}{m_j} \frac{1}{k^2} \int d\vec{v}\, \hat{L}_{j\kappa}^{np} f_{oj}^r ; \quad (23)$$

The field φ_1^t is induced by the density f_1^t , so

$$\varepsilon(x')\varphi_{1x'}^t = \frac{4\pi e_j^2}{m_j^2 k'^2} \int dx'' \left[\varphi_{1x''}^r \varphi_{0x'-x''}^t + \right.$$

$$\left. + \varphi_{1x'-x''}^r \varphi_{0x''}^t \right] \sum_{j,npqr} e^{-i(n-p+q-r)\theta + i(q-r)\alpha_{\vec{k}'} + i(n-p)\alpha_{\vec{k}'-\vec{k}''}} \times$$

$$\times \int d\vec{v}\, \hat{L}_{j\,\vec{k}',\vec{k}''\omega'}^{npqr} \hat{L}_{j\,\vec{k}'-\vec{k}'',\omega'-\omega''}^{np} f_{oj}^r \qquad (24)$$

In equations (23) and (24) the variable θ reflects the anisotropy in the $\vec{v}$ - more accurately in the $v_\perp$ - space. If the drift velocity and the inhomogeneity are not too large the phase value in the exponential varies more quickly than $v_\perp \left\{ \begin{matrix} \sin\theta \\ \cos\theta \end{matrix} \right\}$ in the multiplicative terms. Averaging is such a manner, the integration gives selection rules for (23) and (24),

$$n-p=0 \; ; \qquad n-p+q-r=0$$

and naturally $d\vec{v} \rightarrow 2\pi \, dv_\perp \, v_\perp \, dv_z$

The final form of the dispersion equation is the following /for the regular part of the potential disturbance φ_1^r /

$$0 = \varepsilon(\varkappa) - \int d\varkappa' \, |\varphi^t_{0\varkappa'}|^2 \Big\{ \frac{1}{\varepsilon(\varkappa-\varkappa')} \big[A(\varkappa-\varkappa',\varkappa) + A(\varkappa-\varkappa',-\varkappa') \big] \times$$
$$\times \big[A(\varkappa,\varkappa') - A(\varkappa,\varkappa-\varkappa') \big] + B(\varkappa,\varkappa',\varkappa) + B(\varkappa,\varkappa',-\varkappa') + \qquad (25)$$
$$+ B^*(\varkappa,\varkappa-2i0,\varkappa'+i0) \Big\}$$

Here $\varkappa \doteq [\vec{k},\omega]$, the symbols A and B are abbreviations from (22) and (24), respectively:

$$A(\varkappa_1,\varkappa_2) \doteq \sum_j \frac{8\pi^2 e_j^3}{m_j^2 \, k_1^2} \sum_{\substack{npqr \\ n-p+q-r=0}} \exp\big(-i(n-p)(\alpha_{\vec{k}_1} - \alpha_{\vec{k}_1-\vec{k}_2})\big) \times \qquad (26)$$
$$\times \int_0^\infty v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z \, \hat{L}^{npqr}_{j\,\vec{k}_1,\vec{k}_2,\omega_1} \, \hat{L}^{np}_{j\,\vec{k}_1-\vec{k}_2,\omega_1-\omega_2} \, f^r_{j0} \, ;$$

$$B(\varkappa_1,\varkappa_2,\varkappa_3) \doteq -\sum_j \frac{8\pi^2 e_j^4}{m_j^3 \, k_1^2} \sum_{\substack{npqrst \\ n-p+q-r+s-t=0}} \exp\big(-i(n-p)(\alpha_{\vec{k}_1} - \alpha_{\vec{k}_1-\vec{k}_2-\vec{k}_3}) -$$
$$- i(q-r)(\alpha_{\vec{k}_1} - \alpha_{\vec{k}_1-\vec{k}_2})\big) \times \int_0^\infty v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z \; \times$$
$$\times \hat{L}^{npqrst}_{j\,\vec{k}_1,\vec{k}_2,\omega_1} \, \hat{L}^{npqr}_{\vec{k}_1-\vec{k}_2,\vec{k}_3,\omega_1-\omega_2} \, \hat{L}^{np}_{\vec{k}_1-\vec{k}_2-\vec{k}_3,\omega_1-\omega_2-\omega_3} \times \qquad (27)$$
$$\times f^r_{j0}$$

Evaluation of the formulae, discussion

To evaluate the formula (25) with (26) and (27) it is necessary to know the stationary distribution function f_{oj}^{r} satisfying (10). Taking into account the influence of a strong magnetic field the temperature in the parallel and vertical direction to the magnetic field lines may exhibit different values, $T_{\parallel}$ and $T_{\perp}$. If the drift velocities v_{D_j} are smaller than the wave velocities and if the density, magnetic field and potential energy vary smoothly compared with the wave phenomena, the following simple form holds for the ground state distribution function

$$f_{oj}^{r} = f_{oj}^{r}\left(w_j, u_y, v_z\right) =$$

$$= n_{oj}\left(\frac{m_j}{2\pi T_{eff\,j}}\right)^{3/2}\left(1+\frac{\kappa_j}{\omega_{cj}}u_y\right)\exp\left[-\frac{m_j}{2}\left(\frac{w_j}{T_{\perp j}}+\frac{v_z^2}{T_{\parallel j}}\right)\right] \qquad (28)$$

Here the w_j contain a smooth variation in x, defined by (11). The effective temperature T_{eff} and the coefficient κ_j are results of normalization. If the inhomogeneity length scale is considerably larger than the turbulence and instability length scales, i.e. if the plasma in the x - direction is limited to L /thereafter it may be periodic/, in the y and z directions to $A\,[cm^2]$, then

$$\int_0^L dx\left[n_{oj}\left(\frac{T_{\perp j}^2 T_{\parallel j}}{T_{eff\,j}}\right)^{1/2} A\left(1+\frac{\kappa_j}{\omega_{cj}}\int\omega_{cj}(x)\,dx\right)e^{m_j g_j x/T_{\perp j}}\right] = n_o V \qquad (29)$$

If a linear dependence of B_0 is supposed,

$$B_0(x) = B_0(1 + \delta x) \tag{30}$$

it is simple to get an equation for $T_{eff,j}$, viz.

$$T_{eff,j} \doteq \frac{(T_{\parallel j} T_{\perp j})^{1/3}}{L^{2/3}} \, \frac{1}{a^2} \Big\{ a^2 (e^{aL} - 1) + \alpha_j a \big[(aL - 1) e^{aL} - 1 \big] + \alpha_j \delta \Big[\Big(1 - aL + \frac{a^2 L^2}{2}\Big) e^{aL} - 1 \Big] \Big\}^{2/3} \tag{31}$$

Here a stands as an abbreviation,

$$a \doteq m_j g_j / T_{\perp j} \tag{32}$$

It is easy to see that if $a \to \alpha_j \to \delta \to 0$, $T_{eff,j}$ has the plausible value

$$T_{eff,j} \to \left(T_{\perp j}^2 T_{\parallel j} \right)^{1/3}$$

The constant α_j is to be calculated from the diffusion,

$$\frac{1}{n_j} \frac{d n_j}{d x} = j$$

in the vicinity of $x \approx 0$

$$j_{x \approx 0} = \left. \frac{\frac{d}{dx} \int f_{0j} \, d\vec{v}}{\int f_{0j} \, d\vec{v}} \right|_{x=0} = \alpha_j + \frac{m_j g_j}{T_{\perp j}} \tag{33}$$

The drift velocity v_{Dj} is given by the non-uniformity of the magnetic field and by the conservative force,

$$v_{Dj} = \frac{1}{2}\,\delta\,\frac{v_\perp^2}{\omega_{cj}} - \frac{g_i}{\omega_{cj}}$$

The drift velocity is a constant of the motion in the first approximation.

It is now possible to evaluate expressions (25), (26) and (27) with the distribution function (28). The operator $\hat{L}^{np}$ acting upon f_{oj}^{r} is the following

$$\hat{L}^{np}_{j\kappa}\, f_{oj}^{r} = \Gamma^{*}_{nk}\, f_{oj}^{r}$$

$$\Gamma^{*}_{nk} \doteq \tag{34}$$

$$= \frac{J_{nk} J_{pk}}{\omega - \Omega_{nk}}\left[\left(-\frac{m_j}{T_{\perp j}}\right)\left(n\omega_{cj} + k_y\left(v_{Dj} - \frac{\kappa^{*} T_{\perp j}}{\omega_{cj} m_j}\right) + \gamma_j k_z v_z\right)\right]$$

Here

$$\gamma_j \doteq T_{\perp j}/T_{\parallel j}\;; \qquad J_{nk} \doteq J_n\left(k_\perp v_\perp/\omega_{cj}\right)$$

$$\Omega_{ak} = a\,\omega_{cj} + k_y v_{Dj} + k_z v_z; \quad a = \begin{cases} n & \Rightarrow \Omega_{nk} \\ n-p+q & \Rightarrow \Omega_{qk} \\ n-p+q-r+s & \Rightarrow \Omega_{sk} \end{cases} \tag{35}$$

The derivations $\partial/\partial v_\perp$ upon J_{nk} give vanishing terms.

The evaluation of expression [25] is lengthy therefore we introduce some approximations. The phase functions $\exp[i(n\varphi \ldots)]$, taking into account the randomness of the turbulent pulsations in cylindrical symmetry, give a vanishing value for the coefficients of $J_r J_{q\pm1}$ and $J_s J_{s\pm1}$, and selection rules

$n-p=0$ for the A -s and

$n-p=0;\ q-r=0$ for the B -s (35a)

The selection rule (35a) helps the integration in $v_\perp$, so it is possible to put

$$\int_0^\infty v_\perp e^{-\frac{m_j}{2T_\perp} v_\perp^2} J_{nk}^2 \, dv_\perp \doteq \Lambda_n\left((r_L k_\perp)^2\right) \tag{36}$$

here

$$\Lambda_n(x) = I_n(x)\, e^{-x} \tag{37}$$

/see, for example, Ichimaru [1]/.

It is useful to describe an intermediate step here. Say for the B -s,

$$\frac{1}{\omega_2 - \Omega_{sk_2}}\left(-\frac{m_j}{T_{\perp j}} k_{y4} v_{0j} + \frac{\alpha_j^*}{\omega_{cj}} k_{y4} + k_{z4}\frac{\partial}{\partial v_z}\right) \times$$

$$\times\left[\frac{1}{\omega_1 - \Omega_{qk_1}}\left(\left(-\frac{}{T_{\perp j}} k_{y2} v_{0j} + \frac{\alpha_j^*}{\omega_{cj}} k_{y2} + k_{z2}\frac{\partial}{\partial v_z}\right)\times\right.\right. \tag{38}$$

$$\left.\times\left(\frac{\Gamma_{nk}^*}{\omega - \Omega_{nk}}\right)\right) f_{0jn}^r\Big]$$

The asterisk in Γ^{*}_{nk} and α^{*}_{j} means that thereafter the dependence upon u_{y} in f^{r}_{oj} is vanishing. The similar expression for the A-s is given by the bracket $[\ ]$. The selection rule is used, therefore in the expression f^{r}_{jn} only $\mathfrak{z}^{2}_{nk}$ takes part, and

$$f^{r}_{ojn} = \mathrm{const}\cdot(\mathfrak{z}_{nk}\,\mathfrak{z}_{nk})\ \exp\left[-\frac{v_2^2}{2T_{oj}}m_j + V_{pot}/T_{1j}\right] \qquad (39)$$

The summation upon Ω_q and Ω_s is possible, say for s

$$\sum_s \left(\omega_1 - s\omega_{cj} - k_{y1}v_{oj} - k_{z1}v_z\right)^{-1} = \sum_s \frac{1}{\alpha - s\omega_{cj}} = \frac{1}{\alpha} +$$

$$+\sum_{1}^{\infty}\left(\frac{1}{\alpha - s\omega_{cj}} + \frac{1}{\alpha + s\omega_{cj}}\right) = \frac{1}{\alpha} - \frac{2\alpha}{\omega_{cj}^2}\sum\frac{1}{s^2 - \alpha^2/\omega_{cj}^2} \qquad (40)$$

This last expression is given by the formula [8]

$$\cos ax = \frac{2a\sin a\pi}{\pi}\left\{\frac{1}{2a^2} + \frac{\cos x}{1^2 - a^2} - \frac{\cos 2x}{2^2 - a^2} + - \dots\right\} \qquad (41)$$

So by a ctg function,

$$\sum_q \frac{1}{\omega_1 - \Omega_{qk_1}} = \frac{\pi}{\omega_{cj}}\,\mathrm{ctan}\left[\frac{\pi}{\omega_{cj}}(\omega_1 - k_{y1}v_{0j} - k_{z1}v_z)\right] \qquad (42)$$

Therefore the integration has singularities of the type $1/x$ and $1/x^2$. For the coefficients A and B we have

$$\int_{-\infty}^{\infty}\frac{\pi^2}{2\alpha_1}\,\mathrm{ctan}\left(\frac{\alpha_1\pi}{\omega_{cj}}\right)\left(K(2) + k_{z2}\frac{\partial}{\partial v_z}\right)\frac{\Gamma^{*}_{nk}f^{r}_{ojn}}{\omega - \Omega_{nk}}\,dv_z \qquad (43)$$

$$\int_{-\infty}^{\infty} \frac{\bar{\kappa}^2}{2\alpha_2^2} \operatorname{ctan}\left(\frac{\kappa_2 \bar{\kappa}}{\omega_{cj}}\right)\left(K(4) + k_{z4}\frac{\partial}{\partial v_z}\right) \times$$

$$\times \frac{\bar{\kappa}^2}{2\alpha_1^2} \operatorname{ctan}\left(\frac{\kappa_1 \bar{\kappa}}{\omega_{cj}}\right)\left(K(2) + k_{z2}\frac{\partial}{\partial v_z}\right) \times$$

$$\times \left[\left(\frac{\Gamma^{\alpha}_{sk}}{\omega - \Omega_{nk}}\right) f^{r}_{j o n}\right] d v_z \qquad (44)$$

Here

$$\alpha_1 \doteq \omega_1 - k_{y1} v_{Dj} - k_{z1} v_z \,; \qquad K(i) \doteq \left(-\frac{m_j}{T_{\perp j}} v_{Dj} + \frac{\kappa^{*}}{\omega_{cj}}\right) k_{yi}$$

$$\alpha_2 \doteq \omega_2 - k_{y2} v_{Dj} - k_{z2} v_z \,;$$

$$f^{r}_{o j n} \doteq \qquad (45)$$

$$\doteq 2\pi\, n_o \left(\frac{m_j}{2\pi T_{eff j}}\right)^{3/2} \left(\frac{T_{\perp j}}{m_j}\right) \Lambda_n\left(r_L^2 k_\perp^2\right) e^{-\frac{m_j v_z^2}{2 T_{\parallel j}} + V_{pot}/T_{\perp j}}$$

To evaluate (43) and (44) we use the approximation

$$\frac{1}{x} \operatorname{ctan} x \rightarrow \frac{1}{x^2} + \text{const} \qquad (46)$$

supposing the quick vanishing of the exponential in (45). So we approximate (43) by

$$\text{prop} \int \left(\frac{1}{(b_1 - x)^2} - \frac{\beta_1^2}{3}\right)\left(\varkappa(2) - \frac{\partial}{\partial x}\right)\left[\frac{a_2 + x}{a_2 - x} e^{-x^2/2}\right] dx \qquad (47)$$

v_z being normalized to the parallel thermal velocity, $\sqrt{T_{\parallel j}/m_j}$.

We substitute the expression of the type

$$\frac{1}{\sqrt{\pi}} \int \frac{e^{-x^2}}{x - z}\, dx \qquad (48)$$

by the well known plasma dispersion function $Z(z)$ tabulated by Fried and Conte [9] and by Fadeyeva and Terentiev [10]. The other combinations of the rational functions and the exponential are reducible too in derivatives of Z and other well known functions.

Formula (39) is determined in the same manner.

According to the nature of the turbulent spectrum we distinguish three characteristic cases.

1. The turbulent frequencies are very high compared with all other frequencies,

$$\omega'/k'_z \gg n\omega_{cj}/k_z \gg m\omega_{cj}/k'_z \gg \omega/k_z\,,\ \frac{v_{Dj}k_y}{k_z} \qquad (49)$$

2. The magnetic field predominates

$$n\omega_{cj}/k_z \gg \omega'/k'_z \gg n\omega_{cj}/k'_z \qquad (50)$$

3. In the case of an extremely high magnetic field

$$n\omega_{cj}/k_z \gg n\omega_{cj}/k'_z \gg \omega'/k'_z \qquad (51)$$

The results of calculation for the coefficients are given in Table 1.

For the evaluation we used the known formulae for the $Z(z)$-function /see for example ref.[9]/.

In the nearly resonant case the coefficients A are considerably greater than the B values because $\varepsilon \approx 0$.

Using approximating values for the $\perp_n$ -functions, in the first, $(n=1)$ case we get the situation shown in Fig. 1. We can thus appreciate $\perp_n(x)\perp_n(x')$ for practical cases $(r_L k_\perp \sim 0.1 - 0.01,\ r_L k'_\perp \sim 10 - 100)$ to $\sim 10^{-3}$

It is to noteworths that the thermal isotropy, $\gamma_j = 1$ gives vanishing correction - in the first order - for the A coefficients. It is possible to explain it in the following way. In the high magnetic field the effects of magnetic drift motions predominate. The charge separation is influenced by the fact that in the parallel direction the particles flow relatively freely: in the perpendicular direction they are connected to the field lines. In the isotropic case the charge separation due to the fluctuating field is compensated by these freely flowing charges and in first approximation it vanishes. In the anisotropic case there is a survival part which influences the dispersion relation.

In a given turbulent spectrum the corrections to the linear dispersion relation, according to (25), are given by quadratures.

$$\sum A \sum A = \sum_j \sum_n{}' \omega_{pj}^2 \frac{e_j^2}{m_j^2} \frac{\pi}{2} \left(\frac{m_j}{T_{eff\,j}}\right)^3 \frac{T_{\perp j}^2}{m_j^2} \left(\frac{m_j}{T_{\parallel j}}\right)^3 \times \frac{\Lambda_n(r_i^2 k_\perp^2)\,\Lambda_n(r_i^2 k_\perp'^2)}{\varepsilon(\vec{k}-\vec{k}', \omega-\omega')} \cdot \frac{1}{k^2 k'^2} \frac{\omega_{cj}}{n} \times$$

$$\times \begin{cases} \dfrac{k_z'}{k_z} \dfrac{1}{\omega' - k_y' v_{Dj}} \quad \cdots\cdots & 1. \\[2ex] \dfrac{k_z'}{k_z} \dfrac{1}{\omega' - k_y' v_{Dj}} \quad \cdots\cdots & 2. \\[2ex] -\dfrac{(2+(1-1/\gamma_j)^2)(1-1/\gamma_j)}{n\,\omega_{cj}} \quad \cdots & 3. \end{cases}$$

$$\sum B = -\sum_j 8\,\omega_{pj}^2 \left(\frac{\pi}{2}\right)^{3/2} \frac{e_j^2}{m_j} \gamma_j \left(\frac{m_j}{T_{\parallel j}}\right)^{1/2} \left(\frac{m_j}{T_{eff\,j}}\right)^{3/2} \frac{1}{k^2 k_z} \begin{cases} \dfrac{2 k_z'^2}{k_z \gamma_j (\omega' - k_y' v_{Dj})^2} \quad \cdots\cdots & 1. \\[2ex] \dfrac{8 k_z}{(\omega' - k_y' v_{Dj})^2} \quad \cdots\cdots & 2. \\[2ex] \dfrac{\sqrt{2}\; k_z'^2 k_z}{(\omega' - k_y' v_{Dj})^3 (\omega - k_y v_{Dj})} \quad \cdots\cdots & 3. \end{cases}$$

Table 1.

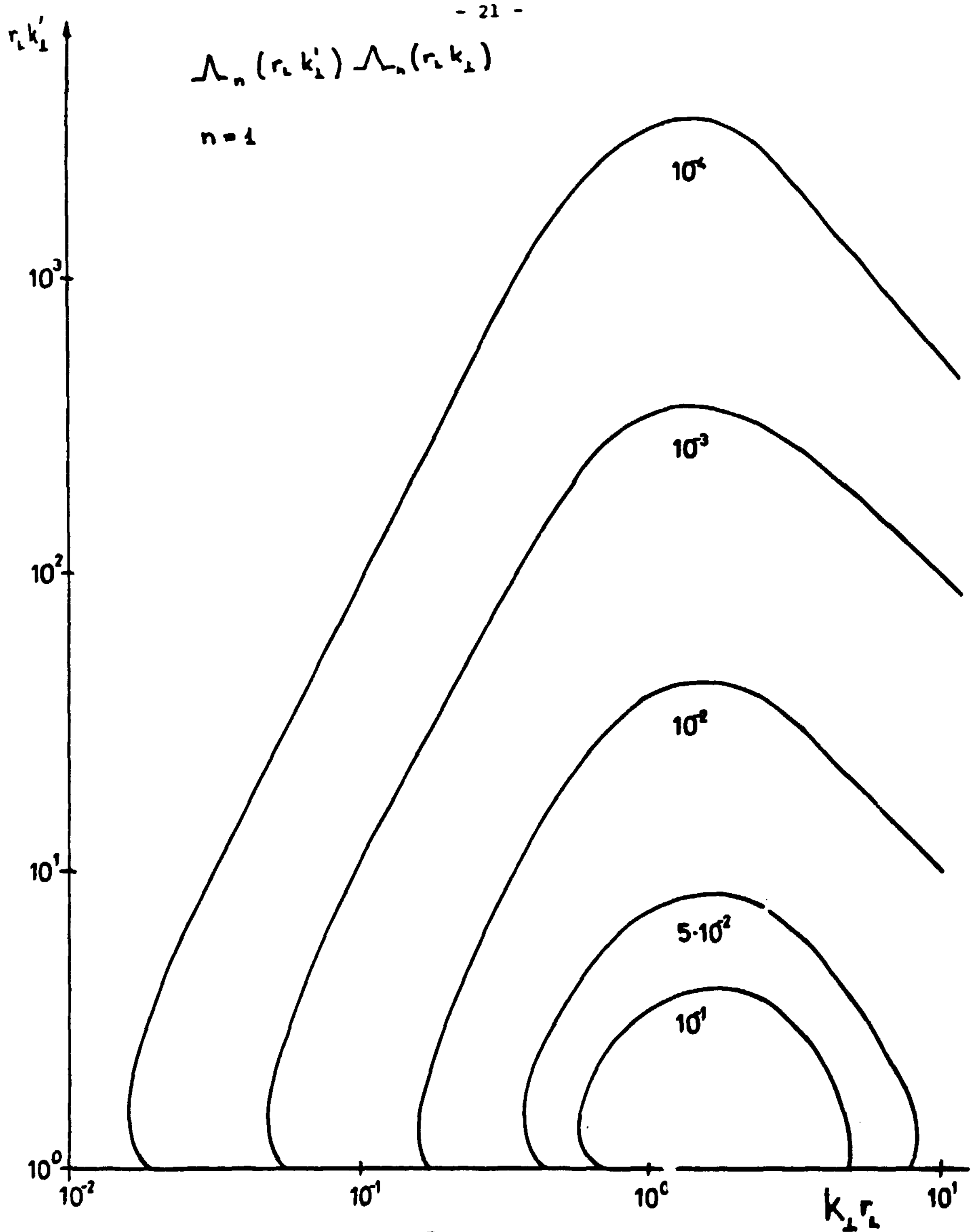
$r_L k'_\perp$
$\Lambda_n(r_L k'_\perp)\,\Lambda_n(r_L k_\perp)$
n = 1
10^{-4}
10^{-3}
10^{-2}
$5\cdot 10^{-2}$
10^{-1}
10^3
10^2
10^1
10^0
10^{-2}
10^{-1}
10^0
10^1
$k_\perp r_L$

Fig. 1.

Acknowledgement

The author is grateful to Gy. Páris for his support and frequent discussions.

References

[1] See for example S. Ichimaru: Basic principles of plasma physics, W.A. Benjamin Inc. Reading, Mass. /1973/

[2] A. Rogister, H. Hasselberg: On the parametric heating of plasmas, IPP of the KFA Jülich, manuscript

[3] See for example Á. Ág, K.H. Finken, R. Läuter, A. Stampa, H. Tuczek, Proc. XIth Int. Conf. Phenomena in Ionized Gases, Prague, 1973. 320 pp

[4] E.N. Krivoirutski, V.G. Makhankov, V.N. Tsytovich, Nucl. Fusion 9 97 /1969/

[5] A.K. Nekrasov, Nuclear Fusion 16, 1 /1976/

[6] V.N. Tsytovich, Theory of turbulent plasma, Consultants Bureau, New York 1977.

[7] B.B. Kadomtsev, Plasma turbulence, Academic Press, Oxford 1965.

[8] Formula 416.17 H.B. Dwight, Tables of integrals, Macmillan New York 1961.

[9] B.D. Fried, S.D. Conte, The plasma dispersion function, Academic Press, Oxford 1961.

[10] V.N. Fadeeva, N.M. Terentev, Tables of values of the probability integral for complex arguments, Gostehizdat, Moskva 1954.

Appendix

As an example we calculate a term of the A -s. By the notation (26) and (27) the Poisson equation with the source (22) is the following

$$\varepsilon(x)\psi^r_{1x} = \int dx' A(x,x')\left(<\psi^t_{1x'}\psi^t_{0\,x-x'}> + <\psi^t_{0x'}\psi^t_{1\,x-x'}>\right) +$$

$$+\int dx'\,dx''\left\{B(x,x',x'')\left(\psi^r_{1x''}<\psi^t_{0x'}\psi^t_{0\,x-x'-x''}> + \right.\right. \qquad (A.1)$$

$$\left.\left. + \psi^r_{1\,x-x'-x''}<\psi^t_{0x'}\psi^t_{0x''}> + \psi^r_{1x'}<\psi^t_{0x''}\psi^t_{0\,x-x'-x''}>\right)\right\}$$

Introducing $\psi^t_{1x'}$ form (24) we get

$$\psi^t_{1x'} = \frac{1}{\varepsilon(x')}\,\times$$

$$\times\int dx''\left(\psi^r_{1x''}\psi^t_{0\,x-x'} + \psi^r_{1\,x'-x''}\psi^t_{0x''}\right)A(x',x'') \qquad (A.2)$$

and a similar expression for ψ^t_{1x} . For the regular part of ψ^r_{1x} we can write

$$\varepsilon(x)\psi^r_{1x} = \int dx' A(x,x')\int dx''\left\{\frac{1}{\varepsilon(x')}\left[\psi^r_{1x''}<\psi^t_{0\,x'-x''}\psi^t_{0\,x-x'}\,\times\right.\right.$$

$$\left.\times A(x',x'')> + \psi^r_{1\,x'-x''}<\psi^t_{0x''}\psi^t_{0\,x-x'}A(x',x'')>\right] +$$

$$+\frac{1}{\varepsilon(x-x')}\left[\psi^r_{1x''}<\psi^t_{0x'}\psi^t_{0\,x-x'-x''}A(x-x';x'')> + \right. \qquad (A.3)$$

$$\left.\left. + \psi^r_{1\,x-x'-x''}<\psi^t_{0x'}\psi^t_{0x''}A(x-x',x'')>\right]\right\} +$$

$$+ \text{ terms with } B\text{-s}$$

The averaging upon the ensemble /see for example ref. [6]/ gives

$$<\psi^t_{0x'}\psi^t_{0x''}> = |\psi^t_{0x'}|^2\,\delta(x'+x'') \qquad (A.4)$$

Let us calculate the term $< \varphi^{+}_{0\,x'-x''}\, \varphi^{+}_{0\,x-x'}\, A(x',x'') >$
Then by (A.4), the δ -function reduces the expression for the case $x = x''$,

$$\int dx'\, dx'' \left\{ \frac{A(x,x')\, A(x',x'')}{\varepsilon(x')}\, \varphi^{-}_{1x''} < \varphi^{+}_{0\,x'-x''}\, \varphi^{+}_{0\,x-x'} > \right\} =$$

$$= \int dx' \left\{ \frac{A(x,x')\, A(x',x)}{\varepsilon(x')} \left| \varphi^{+}_{0\,x-x'} \right|^2 \varphi^{-}_{1x} \right\} ;$$

Now put $x' = x - u'$, so

$$- \int du' \left\{ \frac{A(x,x-u')\, A(x-u',x)}{\varepsilon(x-u')} \left| \varphi^{+}_{0u'} \right|^2 \varphi^{-}_{1x} \right\} =$$

$$= - \int dx' \left\{ \varphi^{-}_{1x} \left| \varphi^{+}_{0x'} \right|^2 \frac{A(x,x-x')\, A(x-x',x)}{\varepsilon(x-x')} \right\} \qquad (A.5)$$

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Szegő Károly
Szakmai lektor: Kóta József
Nyelvi lektor: Harwey Shenker
Példányszám: 260 Törzsszám: 1978-311
Készült a KFKI sokszorosító üzemében
Budapest, 1978. március hó