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ON THE EXPONENTIATION OF LEADING INFRARED DIVERGENCES IN MASSLESS YANG-MILLS THEORIES

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Abstract : We derive, in the axial gauge, the effective U-matrix which governs the behaviour of leading infrared singularities in the self-energy functions of Yang-Mills particles. We then show in a very simple manner, that these divergences, which determine the leading singularities in massless Yang-Mills theories, exponentiate.

The important problem of the infrared behaviour of Yang-Mills theories has recently been investigated by many authors[1]. In particular, the leading infrared divergences of various processes were analysed and it was shown that these sum up into very simple expressions[2]. As discussed in reference (3), the leading infrared singularities in massless Yang-Mills theories occur, in non-covariant gauges, only in the self-energy parts of external lines. This fact, together with the gauge invariance of the S-matrix, lead then to a proof of the exponentiation of leading infrared divergences.

It is the purpose of this note to give a very simple argument for this behaviour. To this end, we derive the effective U-matrix which governs the behaviour of leading singularities in the self-energy of massless Yang-Mills particles. This derivation is based on the analysis of the Hamiltonian H^i which describes the interactions in our theory. In terms of H^i, the matrix U is given in the interaction picture by the well known formula :

$$U(t,t_0) = T \exp\left[-i\int_{t_0}^{t} H^i(t')\,dt'\right] \qquad (1)$$

where T denotes the usual time-ordering operator. As we will see, in our case, the T-operator does not affect the behaviour of the leading infrared singularities, and can therefore effectively be disregarded. It is precisely for this reason that the leading infrared singularities exponentiate in the self-energy functions in a simple way.

We will now consider the Hamiltonian which characterizes the Yang-Mills theory. To this end we start with the Lagrangean density[4] :

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \qquad (2a)$$

where μ, ν are Lorentz indices, a is an internal symmetry index and

$$F_{\mu\nu}^{a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \tag{2b}$$

In this equation, the fields A_μ^a denote the massless Yang-Mills vector bosons, g is the coupling constant and f^{abc} are the completely antisymmetric structure constants of the theory. (Our space-time metric is $\delta_{\mu\nu}$, 4-vectors being assigned imaginary fourth components. Summation over repeated indices is to be understood, unless otherwise stated). We will quantize the theory in the axial gauge(5), where there are no ghost particles, characterized by the gauge condition :

$$A_3^a = 0 \tag{3}$$

since it leads to a very simple interaction Hamiltonian(6).

From the definition of the canonical momenta Π_μ^a, which are given by $\delta\mathcal{L}/\delta(\partial_4 A_\mu^a)$ we see that Π_4^a vanishes. The field A_4^a is therefore not an independent variable, and must be eliminated with the help of the equations of motion. The only independent dynamical variables are A_i^a and Π_i^a (i=1,2), which satisfy the usual canonical equal-time commutation relations.

In the interaction picture, where the fields satisfy free-field equations, the free-field decomposition for A_i^a is given by :

$$A_i^a(x) = \sum_{k,\lambda=1,2} \frac{e_i(k,\lambda)}{\sqrt{2k_0 V}} \left(A_{k\lambda}^a e^{ikx} + A_{k\lambda}^{+a} e^{-ikx} \right) \tag{4}$$

where the momenta $k_\mu = (\vec{k}, ik_0)$ are on-shell : $k^2 = \vec{k}^2 - k_0^2 = 0$. $A_{k\lambda}^a$ and $A_{k\lambda}^{+a}$ denote, respectively, the annihilation and creation operators for the boson fields, and satisfy the commutation relation :

$$[A_{k\lambda}^a, A_{k'\lambda'}^{+a'}] = \delta^{aa'} \delta_{\lambda\lambda'} \delta_{kk'} \tag{5}$$

Then, it can be shown that we obtain the following expression for the interaction Hamiltonian density $\mathcal{H}^i$ in the interaction picture :

$$\begin{aligned}\mathcal{H}^i = {} & g f^{abc} [\tfrac{1}{2}(\partial_i A_j^a - \partial_j A_i^a) A_i^b A_j^c + A_4^b A_i^c (\partial_4 A_i^a - \partial_i A_4^a)] + \\ & + \tfrac{1}{2} g^2 f^{abc} f^{ab'c'} [\tfrac{1}{2} A_i^b A_j^c A_i^{b'} A_j^{c'} + \\ & + (\partial_4 A_i^b - \partial_i A_4^b) A_i^c \partial_3^{-2} (\partial_4 A_j^{b'} - \partial_j A_4^{b'}) A_j^{c'}\end{aligned} \tag{6a}$$

where the field A_4^a is given by :

$$A_4^a = -i \nabla^{-2} \partial_i \partial_0 A_i^a \tag{6b}$$

We can now pass to the discussion of the infrared singularities of the self-energy function S defined by (no summation over Λ, a) :

$$S = \lim_{t_0 \to -\infty;\, t \to \infty} \langle p, \Lambda, a \,|\, U(t, t_0) \,|\, p, \Lambda, a \rangle \tag{7}$$

where p denotes the on-shell momentum of a Yang-Mills particle with polarization Λ and internal symmetry quantum number a. We note that, substituting the interaction Hamiltonian $H^i = \int d^3x\, \mathcal{H}^i$ in equation (1), which gives the U-matrix, and using the free-field decomposition (4), there will appear energy denominators when the time integration is performed. As is well known(7), these denominators, for the matrix elements of the U-operator, will contain differences between the energies of the intermediate states and the initial state. The infrared singularities appear when these energies become degenerate and the energy denominators tend to zero. The leading infrared divergences, which are doubly logarithmic per loop, arise when soft massless quanta are emitted nearly parallel to the (hard) massless external particles, giving almost degenerated states. Now, for a given order in g, the 4-point vertices reduce the number of energy denominators, thereby reducing the maximum possible degree of divergences. Consequently, we can neglect these vertices(3) for our purposes. Hence, we need only to consider the trilinear vertices in (6), which can be written compactly as follows :

$$\mathcal{H}^i = \frac{1}{2} g f^{abc} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) A_\mu^b A_\nu^c \tag{8a}$$

with the fields A_μ^a given by :

$$A_\mu^a = \sum_{k, \lambda = 1,2} \frac{e_\mu(k,\lambda)}{\sqrt{2 k_0 V}} \left(A_{k\lambda}^a \, e^{ikx} + A_{k\lambda}^{\dagger a} \, e^{-ikx} \right) \tag{8b}$$

The polarization vector e_μ has its first two components as in (4), while $e_3 = 0$ in virtue of the gauge condition. Furthermore, from (6b) we see that $e_4(k,\lambda) = i k_j e_j(k,\lambda)/k_0$, so that we obtain the transversality property satisfied by the physical polarization vectors :

$$k_\mu e_\mu (k, \lambda) = 0 \tag{9}$$

We now observe that the various terms occuring in (8) can be grouped in two main groups. The terms in the first group contain only creation or annihilation operators of the three boson fields. The argument of the exponential function, which characterizes the time dependence of these terms, is proportional to the sum of the energies of the boson fields. Therefore, after performing the integration over time in equation (1), the energy denominators will not have the maximum singularity possible, so these terms are not relevant for our purposes. The terms of the other group contain both annihilation and creation operators of the boson fields, and have therefore an argument of the exponential function which is proportional to energy differences. The part of the interaction Hamiltonian containing these terms is, however, rather complicated. The expression which gives this part simplifies considerably, if we make the plausible assumption that, for the leading divergences, the relevant configuration arises when one gluon is soft with respect to another one in the triple vertex. Using momentum conservation, this gluon (momentum k) will be soft with respect to the other two, which will then carry practically the same momentum (K). In this case, we obtain :

$$H^{i}(t) = -g \sum_{\substack{K;k \\ \lambda=1,\lambda}} \frac{K_\mu}{K_0} Q^{a}_{K} \frac{\epsilon_\mu(k,\lambda)}{\sqrt{2k_0 V}} \left(A^{a}_{k\lambda}\, e^{i\frac{K k}{K_0}t} + A^{+a}_{k\lambda}\, e^{-i\frac{K k}{K_0}t} \right) \qquad (10a)$$

where Q^{a}_{K} is the internal symmetry charge operator given by (no summation over K) :

$$Q^{a}_{K} = -i f^{abc} A^{+b}_{K\lambda} A^{c}_{K\lambda} \qquad (10b)$$

This equation has a very simple physical interpretation : it essentially represents the coupling of the soft field A^{a}_{μ} to the (hard) conserved internal symmetry current $J^{a}_{\mu}(K) = K_\mu Q^{a}_{K}$.

In order to obtain the matrix U, we now substitute equation (10) into (1). We want to show that the time-ordering operator T does not affect the leading singularities. To this end we use the identity :

$$T \exp\left[-i\int_{t_0}^{t} H^{i}(t')dt'\right] = \exp\left[-i\int_{t_0}^{t} H^{i}(t')dt'\right] \times$$
$$\times \exp\left[-\frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \left[H^{i}(t'), H^{i}(t'')\right]\right] \ldots \qquad (11)$$

where ... denote terms involving higher order commutators. Now, if we expand the exponentials in (10) in a series of powers for small values of k, which are the relevant ones for the leading divergences, we see that the would-be leading terms are independent of time. Hence, these terms cancel in the commutators of equation (11), whence it follows that the exponentials containing commutators do not affect the leading divergences. (This is to be contrasted with the argument of the first exponential in (11), where the first term of the series gives a linear divergence in t, as t becomes large). The analysis above can be made more precise by calculating the exact expression for the commutators above, with the help of the commutations relations satisfied by the charge operator :

$$[Q^a_K, Q^{a'}_{K'}] = i\,\delta_{KK'}\, f^{aa'c}\, Q^c_{K'} \tag{12a}$$

$$[Q^a_K, A^{a'}_{K'\Lambda'}] = i\,\delta_{KK'}\, f^{aa'c}\, A^c_{K'\Lambda'} \tag{12b}$$

which follows from equation (5). The resulting expression is rather complicated, the relevant point to notice being the appearence of sine functions involving the above arguments. In consequence, as can be shown by a detailed analysis, these terms will give rise only to non-leading divergences, and can therefore be disregarded for our purposes. Hence, with the help of equations (10) and (11), we obtain the following relation for the part of the U-matrix which governs the behaviour of the leading singularities of the self-energy functions :

$$U^L(t) = \exp g \sum_{K,k,\lambda} \frac{K.e(k,\lambda)}{K.k} Q^a_K \frac{1}{\sqrt{2k_0 V}} \left(A^a_{k\lambda}\, e^{i\frac{K.k}{K_0}t} - A^{+a}_{k\lambda}\, e^{-i\frac{K.k}{K_0}t}\right) \tag{13}$$

(In deriving this expression, we have introduced, as usual, a factor $\exp \varepsilon t$, with $\varepsilon \to 0$, in order to assure the convergence of (1) when $t_0 \to -\infty$. The presence of factors $\pm i\varepsilon$ in the denominator $K.k$ of this equation is to be understood).

Using this expression, we will now proceed to show that the leading infrared divergences (see equation (7)) exponentiate. To this end, we use the fact that the gluon state with momentum p and polarization Λ is an "eigenvector" of the operator U^L. We find :

$$U^L(t)\,|p,\Lambda,a\rangle = [\exp W(p,t)]_{ba}\,|p,\Lambda,b\rangle \tag{14}$$

The operator $W(p,t)$ is linear in the soft (with respect to the gluon of momentum p) gluon creation and annihilation operators, being given by :

$$W(p,t) = F^c\,[\omega_{k\lambda}(p,t)\,A^c_{k\lambda} - \omega^*_{k\lambda}(p,t)\,A^{+c}_{k\lambda}] \tag{15a}$$

where F^c denotes the hermitian matrix, whose elements are $(F^c)_{mn} = -i f^{cmn}$ and the c-number function $\omega_{k\lambda}(p,t)$ is :

$$\omega_{k\lambda}(p,t) = \frac{g}{\sqrt{2k_0 V}}\;\frac{p\,e(k,\lambda)}{p.k}\; e^{i\frac{p.k}{p_0}t} \tag{15b}$$

With the help of these relations, we see that the calculation of the leading infrared divergences of the self-energy function S reduces to the calculation of the vacuum-expectation values of the operator $[\exp W(p,t)]_{aa}$ (no summation over a) :

$$S^L = \lim_{t\to\infty} \langle 0|\,[\exp W(p,t)]_{aa}\,|0\rangle \tag{16}$$

In order to calculate S^L, we use a well known identity, which expresses a function of the type exp(linear form in A and $A^\dagger$) in terms of the commutators and normal-ordered products of these operators. In the leading approximation, using the commutation relation (5), we obtain :

$$\exp W(p,t) = \exp\left[-\frac{1}{2} C_{YM}\, \omega^*_{k\lambda}(p,t)\, \omega_{k\lambda}(p,t)\right] \times$$
$$\times \exp\left[-F^c \omega^*_{k\lambda}(p,t) A^{+c}_{k\lambda}\right] \times \exp\left[F^c \omega_{k\lambda}(p,t)\, A^c_{k\lambda}\right] \quad (17)$$

where C_{YM} denotes the Casimir operator given by $C_{YM} I = F^c (F^c)^\dagger$. Substituting this equation into (16), we then readily obtain that the leading infrared singularities in the self-energy function exponentiate :

$$S^L = \exp\left[-\frac{1}{2} C_{YM}\, \omega^*_{k\lambda}(p,t)\;\; \omega_{k\lambda}(p,t)\right] \quad (18)$$

In order to define the sum in the argument of the exponential, we will use the dimensional regularization scheme[8], where the integrations are performed in a n-dimensional space-time. Making the replacement $\sum_k \rightarrow \frac{V}{(2\pi)^{n-1}} \int d^{n-1}k$ and using the transversality property of the physical polarization vectors (see equation (9)), we obtain for the above exponential the result :

$$S^L = \exp\left[-\frac{g^2}{4\pi^2} C_{YM} \frac{1}{(n-4)^2}\right] \quad (19)$$

which exhibits the double pole structure associated with the leading infrared divergences.

Finally, we will briefly discuss the case of the Coulomb gauge. In this gauge, the interaction Hamiltonian contains an infinite number of terms[9], which are equivalent to the ghost particles. Since these are not relevant for the leading divergences, we are presumably left in this approximation with a simple interaction Hamiltonian similar to the one described in equation (8). Furthermore, the commutation relation (5) holds here as well, and, since we have in the interaction picture $A^c_0 = 0$, we see that the gauge condition $\vec{k}\cdot\vec{\varepsilon}(k) = 0$ is equivalent to the transversality property satisfied by the physical polarization vectors (see equation (9)). Since all we have used in the above analysis were these relations, it follows that the exponentiation of the leading infrared divergences in the self-energy function of Yang-Mills particles occurs also in the Coulomb gauge, in accordance with the conclusions of reference (3).

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