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EXTENDED MONOPOLES IN GAUGE FIELD THEORIES *

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ABSTRACT

We give a review of the 't Hooft monopole and briefly discuss the general topological considerations connected with monopoles. A method is presented for constructing explicit monopole solutions in any gauge theory. Some stability questions and time-dependent problems are also considered.

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INTRODUCTION

Recently there has been a steeply growing interest in monopole type solutions in classical gauge theories. Firstly they have appeared as important elements of the dual model giving a possibility for magnetic vortex lines with finite length. At the same time, monopoles arising from a gauge theory interact strongly and they are heavy, sharing many properties of "solitons", so we may think of hadrons as monopole solitons of a unified gauge theory of weak and electromagnetic interactions. In addition these monopoles have a topologically originated, conserved magnetic charge without the usual string singularity in the vector potential.

All these new ideas are based on a systematic theoretical progress in classical field theory that we are going to review briefly. In Section 1 we give an inductive discussion of the 't Hooft monopole together with the Julia-Zee dyon. In Section 2 we summarize the general topological considerations connected to monopoles. This is followed, in Section 3, by the presentation of a method for building explicit monopole models. This section includes some stability considerations as well as a brief discussion of the radial motion of the monopole solutions.

1. THE 'T HOOFT MONOPOLE

1.1 String-free gauge potential and magnetic charge

Let us consider the original 't Hooft argument [1,2] to understand how it is possible to construct string-free monopoles in general non-Abelian gauge theories. In ordinary electrodynamics with $U(1)$ symmetry, the vector potential describing the field of a magnetic monopole has a Dirac string ending on the pole:

$$A_\varphi = \frac{g}{r}\,\frac{1-\cos\vartheta}{\sin\vartheta}\ , \quad A_r = A_\vartheta = A_0 = 0\ . \tag{1.1}$$

Here g stands for the magnetic charge $(\hbar = c = 1)$ and φ, ϑ, r denote the components of the vector potential in a spherical coordinate system and ϑ, φ are the polar and asimuthal angles respectively. Consider the following gauge transformation on the wave function of a charge q moving in the monopole fields:

$$S = \exp(2iqg\varphi)\ .$$

S is well defined if the Dirac-charge quantization condition $gq = \frac{1}{2}n$ (n integer) is satisfied. This gauge transformation removes the string in A_φ (1.1) at $\vartheta = \pi$ but creates a new one at $\vartheta = 0$. This is inevitable since $U(1)$ is infinitely connected, thus it is impossible to construct a gauge transformation which changes continuously from S at $\vartheta = \pi$ to the identity at $\vartheta = 0$. Although the string has no observable effects [3] if the Dirac condition holds, it is impossible to get rid of it altogether.

We can always embed the $U(1)$ electromagnetic gauge theory in any non-Abelian gauge theory, when both the vector potential and the current have only one non-vanishing component in the internal space [4]. The gauge potential analogous to (1.1) can be written in matrix form as ("Abelian gauge")

$$-eA_\varphi = \frac{Q}{r}\,\frac{1-\cos\vartheta}{\sin\vartheta}\ .$$

Here Q is a constant matrix, a generator in the fundamental representation of the group, and e is the gauge coupling constant. The gauge transformation which removes the string along $\vartheta = \pi$ is given by

$$S = \exp(-2i\varphi Q)\ .$$

Now, if the gauge group G is simply connected, it is always possible to construct a gauge transformation which connects S at $\vartheta = \pi$ with the identity at $\vartheta = 0$.

Let us choose $G = SU(2)$, $Q = \frac{\tau_3}{2}$, i.e. let us take the following gauge potential:

$$A_\varphi = -\frac{1}{er}\,\mathrm{tg}\frac{\vartheta}{2}\,\frac{\tau_3}{2}\ , \quad A_\vartheta = A_r = A_0 = 0\ ,$$

and apply to it the gauge transformation [4,1]

$$\Omega(\vartheta,\varphi) = \exp\left(i\frac{\tau_3}{2}\varphi\right)\exp\left(i\frac{\tau_2}{2}\vartheta\right)\exp\left(-i\frac{\tau_3}{2}\varphi\right) . \qquad (1.2)$$

Recalling that $A'_\mu = \Omega A_\mu \Omega^{-1} - \frac{i}{e}(\partial_\mu\Omega)\Omega^{-1}$,

we find

$$A'_\varphi = \frac{1}{er}\left(-\frac{\tau_3}{2}\sin\vartheta + \frac{\tau_1}{2}\cos\vartheta\cos\varphi + \frac{\tau_2}{2}\cos\vartheta\sin\varphi\right) .$$

$$A'_\vartheta = -\frac{1}{er}\left(\frac{\tau_2}{2}\cos\varphi - \frac{\tau_1}{2}\sin\varphi\right) .$$

We see that the string type singularity has really disappeared from the gauge vectors. However, at the same time we have also lost the electromagnetic field, as the transformed gauge vectors have more than one component different from zero in the internal space. Clearly we need a gauge-invariant definition, which can be given with the aid of a gauge-covariant unit vector built up from the ingredients of the theory. Let us dispense with the explicit construction of this unit vector for the time being and concentrate instead on the electromagnetic field. Suppose we want to define the e.m.field in a gauge as

$$\mathcal{F}_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu .$$

If we have a $\Phi_0 = \Phi_0^i \frac{\tau_i}{2}$ with $\Phi_0^i = \delta^{i3}$, $\hat\Phi = \Omega\,\Phi_0\,\Omega^{-1}$, where $\Omega(x)$ is a general SU(2) transformation, then this $\mathcal{F}_{\mu\nu}$ can be written as

$$\mathcal{F}_{\mu\nu} = 2\,\mathrm{Tr}\left[\partial_\mu(\Phi_0 A^0_\nu) - \partial_\nu(\Phi_0 A^0_\mu)\right] , \quad A^0_\nu = A^3_\nu \frac{\tau_3}{2} .$$

This can be cast into a manifestly gauge-invariant form [1,4] :

$$\mathcal{F}_{\mu\nu} = 2\,\mathrm{Tr}\left[\hat\Phi F_{\mu\nu} - \frac{i}{e}\hat\Phi[D_\mu\hat\Phi, D_\nu\hat\Phi]\right] ,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu] , \qquad (1.3)$$

$$D_\mu\hat\Phi = \partial_\mu\hat\Phi - ie[A_\mu, \hat\Phi] .$$

This identification of the e.m.field yields for the previous point monopole in all gauges $\mathcal{F}_{\vartheta\varphi} = \frac{1}{er^2}$ with all other components vanishing.

With the aid of $\mathcal{F}_{\mu\nu}$ (1.3) we can define the magnetic current $\partial^\mu\tilde{\mathcal{F}}_{\mu\nu} = k_\nu$. In the string-free gauge k_μ is built up entirely from $\hat\Phi$ and its derivatives [5] :

$$k_\mu = \frac{1}{2e}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{abc}\partial^\nu\left(\hat\Phi^a\partial^\rho\hat\Phi^b\partial^\sigma\hat\Phi^c\right) . \qquad (1.4)$$

By definition k_μ is conserved implying a conservation law for $g = \frac{1}{4\pi}\int k_0\, d^3x$. However, g does not generate a symmetry, as the conservation of k_μ is not a consequence of the dynamical equations [5] thus the Noether theorem does not apply. The conservation of g has a topological origin; to make it clear, let us express g in another form with the aid of (1.4) [5]:

$$4\pi e g = \lim_{R\to\infty}\int \frac{1}{2}\varepsilon_{abc}\hat\Phi^a\left(\frac{\partial\hat\Phi^b}{\partial\vartheta}\frac{\partial\hat\Phi^c}{\partial\varphi} - \frac{\partial\hat\Phi^b}{\partial\varphi}\frac{\partial\hat\Phi^c}{\partial\vartheta}\right)d\varphi\, d\vartheta ,$$

where we parametrized the x^i on the sphere $x^i x^i = R^2$ by two parameters $x^i = x^i(\vartheta,\varphi)$. The square of the integrand in this expression is the determinant G of the metric tensor of the unit sphere $\hat\Phi^i\hat\Phi^i = 1$, therefore the integrand is $\pm\sqrt{G}$. While the (ϑ, φ) parameters sweep through the whole surface of the sphere in coordinate space, $\hat{\phi}$ can cover the sphere in the isospace several times with the positive or negative sign of $\sqrt{G}$. The difference must be integer otherwise $\hat{\phi}$ would not be single valued: $4\pi e g = 4\pi n$, i.e. $g = n/e$. The n is called the Kronecker index of the mapping constituted by $\hat{\phi}$. The integer spectrum of g (in units of $1/e$) explains the conservation of the magnetic charge, since the – presumably continuous – time dependence of the ϕ (when we finally make them a part of a dynamical theory) can change g only continuously. This is compatible with the integer spectrum of g only if g is time independent.

1.2 The construction of finite energy monopoles in SU(2)

We have two problems left to solve; the first is the construction of $\hat{\phi}$ from the ingredients of a dynamical theory and the second is to construct finite energy solutions if possible. 't Hooft solved both problems at once [1]. Consider an SU(2) Yang — Mills theory coupled gauge invariantly to a scalar isotriplet which has a symmetry breaking self-interaction potential:

$$V(\phi) = \frac{\lambda}{4}\left(\phi_0^2 - 2\,\mathrm{Tr}\,\phi^2\right)^2 , \quad \phi = \phi^i(x)\frac{\tau_i}{2} .$$

The field equations for this theory in matrix notation are:

$$\partial^\mu F_{\mu\nu} - ie\left[A^\mu, F_{\mu\nu}\right] = -ie\left[\Pi_\nu, \phi\right] ,$$
$$\partial^\mu \Pi_\mu - ie\left[A^\mu, \Pi_\mu\right] = V'(\phi) , \qquad (1.5)$$

$$\Pi_\mu = D_\mu \phi .$$

A simple static solution is given by

$$A_\varphi = -\frac{1}{er}\,\mathrm{tg}\frac{\vartheta}{2}\,\frac{\tau_3}{2} , \quad \phi = \frac{\tau_3}{2}\phi_0 .$$

An obvious way of defining $\hat{\phi}$ is

$$\hat{\phi} = \frac{1}{\sqrt{2\,\mathrm{Tr}\,\phi^2}}\,\phi = \frac{\tau_3}{2} .$$

Transforming by the Ω of (1.2) to the non-singular gauge, we find the solutions

$$A_i = -\frac{1}{er^2}\varepsilon_{ijk}\frac{\tau_j}{2}x_k , \quad \phi = \frac{\tau_i x_i}{2r}\phi_0 .$$

It is easy to see that this ansatz describes an infinite energy point monopole.

We shall search for the finite energy solution by smoothing out the previous solution in the form

$$A_i = \varepsilon_{ijk}\frac{\tau_j}{2}x_k\,\frac{\xi(r)-1}{er^2} , \quad \phi = \frac{\tau_i x_i}{2}\,\frac{\eta(r)}{er^2} . \qquad (1.6)$$

The field equations take the following simple form for ξ and η:

$$x^2\xi'' = -\xi + \xi^3 + \xi\eta^2 ,$$
$$x^2\eta'' = 2\xi^2\eta - \frac{\beta}{2}\left(x^2 - \eta^2\right)\eta , \qquad (1.7)$$

where $x^2 = \phi_0^2 e^2 r^2$, $\frac{\beta}{2} = \frac{\lambda}{e^2}$, $\xi' = \frac{d\xi}{dx}$.

Of course this system of equations allows the two particular point solutions: a) $\xi = 0$, $\eta = x$, describing the previous point monopole, b) $\xi = \pm 1$, $\eta = 0$, describing the (symmetric) vacuum.

It is easy to calculate the energy functional in terms of ξ and η

$$E = -L_{tot} = \frac{4\pi M_W}{e^2} \int_0^\infty dx \left\{ 2(\xi')^2 + \frac{1}{x^2}\left[(1-\xi^2)^2 + 2\eta^2\xi^2 + (x\eta' - \eta)^2 \right] + \frac{\beta}{4}(\eta^2 - x^2)^2 \right\} . \tag{1.8}$$

This is a manifestly positive definite expression, from which Eqs. (1.7) can be obtained by applying the variational principle, and these facts suggest the existence of singularity-free solutions to (1.7) [6].

As we look for finite energy solutions, it is natural to demand that the integral (1.8) must not blow up either at $x \to 0$ or at $x \to \infty$, which in turn fixes the boundary conditions for the ξ, η functions

$$\begin{array}{lll} x \to 0 & & x \to \infty , \\ \xi \to 1 + a x^2 + \dots & & \eta \to x , \\ \eta \to b x^2 + \dots & a, b \text{ arbitrary} & \xi \to 0 . \end{array}$$

Methods to obtain numerical solutions to (1.7) with these boundary conditions are widely discussed in the literature [6], the characteristic behaviour of ξ and η is sketched in Fig.1.

Now we find that even in the finite energy case the only non-vanishing component of the e.m.field is still $\mathcal{F}_{\vartheta\varphi} = \frac{1}{e r^2}$, i.e. a point monopole. What, then, makes the energy finite? To

obtain a physical explanation let us split (gauge invariantly) the term $\mathrm{Tr}\, F_{ij} F^{ij}$ in the energy expression into two parts [4] :

$$(\mathcal{F}_{ij} - 4\pi \mathcal{M}_{ij})(\mathcal{F}^{ij} - 4\pi \mathcal{M}^{ij}) + \mathrm{Tr}\left\{ [\hat{\Phi}, F_{ij}][\hat{\Phi}, F^{ij}] \right\} . \tag{1.9}$$

Here we introduced the gauge invariant tensor $\mathcal{M}_{\mu\nu} = -\frac{i}{4\pi e} \mathrm{Tr}\{ \hat{\Phi} [D_\mu \hat{\Phi}, D_\nu \hat{\Phi}] \}$. As $\hat{\Phi}$ denotes the e.m. direction in the group space (1.3), only the charged vectors (that are orthogonal to $\hat{\Phi}$) contribute both to $\mathcal{M}_{\mu\nu}$ and to the second term in (1.9). The Maxwell equations take the form

$$\partial^\mu \mathcal{F}_{\mu\nu} = 4\pi \partial^\mu \mathcal{M}_{\mu\nu} + j_\nu .$$

This suggests defining a third gauge invariant tensor, as usual in electrodynamics $\mathcal{H}_{\mu\nu} = \mathcal{F}_{\mu\nu} - 4\pi \mathcal{M}_{\mu\nu}$. For our static solution we find that the only non-vanishing components of these tensors are

$$\mathcal{F}_{\vartheta\varphi} = B_r = \frac{1}{e r^2}, \quad \mathcal{M}_{\vartheta\varphi} = M_r = -\frac{\xi^2}{4\pi e r^2}, \quad H_{\vartheta\varphi} = H_r = -\frac{\xi^2 - 1}{e r^2} .$$

So the energy density contains a term

$$\frac{1}{8\pi}\left(B_r H_r - 4\pi M_r H_r \right) ,$$

which is the usual form known from classical electrodynamics. The physical picture is that the monopole is put in a medium formed by the charged vectors, and the energy of the magnetic dipole density of the charged vector fields (M_r) gives an infinite contribution to the energy, which compensates the infinite energy of the point monopole. Let us mention that we are dealing with a spontaneously broken gauge theory where the charged vectors really do have an anomalouos magnetic moment which makes this identification possible. We can define the magnetic permeability for the medium formed by the charged vectors, this quantity behaves as

$$\mu(r) \to \frac{1}{2\alpha r^2} \quad r \to 0 \qquad \mu(r) \to 1 \quad r \to \infty .$$

1.3 Dyons

Dyons are hypothetical particles carrying simultaneously both electric and magnetic charges. It is natural to investigate how the 't Hooft monopole can be converted into a dyon. This was done in Ref.7 by extending the ansatz to include a static, non-vanishing zeroth component of the gauge field in the form $A_0 = \frac{\tau_i x_i}{2} \frac{\chi(r)}{e r^2}$ while keeping (1.6) for A_i and ϕ. This choice implies that $D_0 \phi \equiv 0$ and that the A_0 acts very much like another isotriplet, of course without the self-interaction potential. In this case one finds from the field equations:

$$
\begin{aligned}
x^2 \chi'' &= 2 \xi^2 \chi , \\
x^2 \eta'' &= 2 \xi^2 \eta - \frac{\beta}{2} (x^2 - \eta^2) \eta , \\
x^2 \xi'' &= - \xi + \xi^3 + \xi \eta^2 - \xi \chi^2 .
\end{aligned}
\qquad (1.10)
$$

The energy (or mass) of the dyon can be determined by $E = \int d^3x \, T_{00}$, where $T_{\mu\nu}$ is the symmetric stress energy tensor that couples to the gravitational field. The finiteness of the energy again fixes the boundary conditions for the functions in (1.10),

$$
\begin{array}{ll}
x \to 0 & x \to \infty , \\
\xi \to 1 + a x^2 + \dots & \eta \sim x , \\
\eta \to b x^2 + \dots & \xi \to 0 , \\
\chi \to c x^2 + \dots & \chi \to d .
\end{array}
$$

Our dyon has a radial electric field in the form $E_i = F_{0i} = -\frac{1}{e} \frac{x_i}{r} \frac{d}{dr} \left(\frac{\chi(r)}{r} \right)$, which implies a total charge $Q = -\frac{1}{e} \int_0^\infty r \frac{d^2\chi}{dr^2} dr$. At the present level of discussion there is no reason why this Q should be quantized [7] . On the other hand, $J_{\theta\varphi}$ remains the same and therefore $g = 1/e$. The physical picture of our dyon is that of a point monopole sorrounded by an extended cloud of electric charge.

In the case of dyons we encounter a new problem, namely - owing to the lack of topological considerations - the solution has no ab ovo stability against the dyon $(Q) \longrightarrow$ dyon $(Q-1) + W^+$ or dyon $(Q=1) \longrightarrow$ monople $+ W^+$ decays. However, numerical studies of the solutions to (1.10) revealed [6,7] that dyons are not much heavier than monopoles, which may imply the absence of the second decay. On the other hand, we proved [8] that dyons appropriately generalized to higher groups than SU(2) can be topologically stable. These dyons may play an important role as they - unlike monopoles - may bind quarks without magnetic charges.

2. TOPOLOGICAL CONSERVATION LAWS

2.1 General considerations

So far we have discussed the 't Hooft monopole based on a spontaneously broken SU(2) Yang-Mills theory. It is natural to ask how this can be generalized to higher groups, to the case of more scalar fields or scalar fields in arbitrary representation. It would also be very interesting to know how the previous topological considerations connected with the magnetic charge generalize for these cases. In answering these questions we shall follow the argument of Ref.9.

We consider the classical theory of the gauge fields of a compact, semisimple gauge group G coupled to a scalar multiplet in an n-dimensional unitary representation. From the gauge field strengths and covariant derivatives of the scalar fields we construct the gauge- invariant Lagrange density which defines our theory,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + D_\mu \phi^+ D^\mu \phi - U(\phi) . \tag{2.1}$$

Here $U(\phi)$ is the gauge-invariant self-interaction for the scalar fields and we assume $U(\phi) \geq 0$. The ground states of the theory are (up to a gauge transformation) the states, where all gauge fields vanish and ϕ is a constant and a zero of U. For such a ground state $\phi = \phi_0$, we define H (a subgroup of G) the unbroken group by saying that h is an element of H if and only if $h\phi_0 = \phi_0$. Further, we assume that all the zeros of U are of the form $g\phi_0$ for some g in G; which makes it possible to identify the set of the zeros of U with the coset space G/H [9,10].

We are now going to investigate under what conditions the space of finite energy solutions to (2.1) is disconnected in the genuine topological sense. As this is an invariant statement, we can work in the most suitable gauge, i.e. in the gauge specified by $A_0^a \equiv 0$ [9]. In this gauge the energy expression simplifies greatly:

$$E = \int d^3x \left\{ \frac{1}{2} (\partial_0 A_i^a)^2 + \partial_0 \phi^+ \partial_0 \phi + \right. \\ \left. + \frac{1}{4} (F_{ij}^a)^2 + D_i \phi^+ D_i \phi + U(\phi) \right\} . \tag{2.2}$$

Instead of focusing on non-singular, finite energy solutions, we shall focus on non-singular initial value data with finite energy. (In the $A_0^a \equiv 0$ gauge a complete and independent set of initial value data is given by the fields and their first time derivatives at any fixed time.) In doing so we do not lose any generality, as there is a theorem ensuring the one-to-one corresponedence between these two sets of objects. On the other hand, working at fixed time we can always choose a gauge such that $A_r^a = 0$ [9].

In this gauge we find from the formula for the energy (2.2)

$$E \geq \int_0^\infty r^2 dr \int_0^\pi \sin\vartheta \, d\vartheta \int_0^{2\pi} d\varphi \left(\partial_r \phi^+ \partial_r \phi + U(\phi) \right) .$$

Thus, if the energy is to be finite, $\lim_{r\to\infty} \phi(r,\vartheta,\varphi) = \phi^\infty(\vartheta,\varphi)$ must exist and must be an element of G/H for every ϑ, φ. The point is that $\phi^\infty(\vartheta,\varphi)$ can depend non-trivially on these angles, as the relevant expressions in the energy are the squares of the components of the covariant gradient and not those of the ordinary gradient, i.e. we must have

$$\lim_{r\to\infty} D_\vartheta \phi = \frac{1}{r}\frac{\partial \phi}{\partial \vartheta} - i e A^a_\vartheta T^a \phi = 0 \quad , \qquad (2.3)$$
$$\lim_{r\to\infty} D_\varphi \phi = \frac{1}{r \sin\vartheta}\frac{\partial \phi}{\partial \varphi} - i e A^a_\varphi T^a \phi = 0 \quad .$$

Now for any given, non-singular $\phi^\infty(\vartheta,\varphi)$ we can choose the gauge fields, so these expressions vanish for large r as fast as we please [9]. This implies that $A_\vartheta, A_\varphi \sim \frac{1}{r}$ for large r, which in turn involves $F^2_{\mu\nu} \sim \left(\frac{1}{r^2}\right)^2$, and this is perfectly consistent with the finiteness of the energy in the $r\to\infty$ region.

This way we have associated a function $\phi^\infty(\vartheta,\varphi)$ with every set of non-singular initial value data. $\phi^\infty(\vartheta,\varphi)$ constitutes a mapping of the sphere (S^2) into G/H. As the continuous time developement changes the associated mapping only continuously when there are two sets of initial value data with ϕ^∞_1 and ϕ^∞_2 such that they cannot be continuously deformed into each other, then the corresponding solutions cannot develop into each other in the course of time. Therefore, if there is a $\phi^\infty(\vartheta,\varphi)$ which cannot be continuously deformed into the trivial mapping of $S^2 \to G/H$, the space of finite energy solutions is disconnected [9].

In mathematics "continuously deformable" is called homotopic. Two continuous mappings $f_1(x), f_2(x)$ from X into Y are said to be homotopic if there exists a continuous $F(x,t)$ from $X \times [0,1]$ into Y such that $F(x,0) = f_1(x)$ and $F(x,1) = f_2(x)$. The set of all mappings homotopic to a given mapping defines its homotopy class. Clearly homotopy is an equivalence relation, and thus we can divide all mappings from X to Y into homotopy classes.

Summarizing what we have found, we may state that the connected parts of the space of the finite energy solutions are characterized by the homotopy classes of mappings $S^2 \to G/H$ [9]. Further, these homotopy classes can be labelled - provided G is simply connected - by a "topological number", which is an element of $\pi_1(H)$. (If G is not simply connected the situation is slightly more complicated. For detailed proofs and examples we refer the reader to Ref.9.) Here $\pi_1(H)$ is the first homotopy group, which is defined in the following way. Consider two mappings from the circle S^1 into H, such that $\phi_i(0) = \phi_i(2\pi) = \phi_0 \quad i = 1,2$, i.e. ϕ_1 and ϕ_2 are two closed paths in H, and each of them begins and ends at ϕ_0. Their product $\phi_1 \cdot \phi_2$ is a closed path obtained by first going around ϕ_1 and then going around ϕ_2. In equations

$$\phi_1 \cdot \phi_2 = \begin{cases} \phi_1(2\theta) & 0 \le \theta \le \pi \ , \\ \phi_2(2\theta - 2\pi) & \pi \le \theta \le 2\pi \ . \end{cases} \qquad (2.4)$$

We define two such mappings to be homotopic just as before except that we restrict the continuous deformation $F(\theta,t)$ to obey (2.4) for each t. This multiplication law is invariant under these homotopies, and thus defines a multiplication law among homotopy classes. This multiplication obeys all group axioms (identity=trivial homotopy, class, inverse.= going around the path in the opposite direction) and the group thus obtained is $\pi_1(H)$. The use of this abstract game is that generally it is easy to determine the elements of $\pi_1(H)$, which in turn - as we have seen before - characterize the disjunct components of the space of finite energy solutions. Two examples: $\pi_1(U(1)) = Z$, additive group of integers

$\pi_1(SO(3)) = Z_2$ — " — mod 2

<u>2.2 Connection with magnetic charge</u>

To see the connection of the previous discussion with magnetic charges we introduce the concept of path-dependent group elements [9, 11].

Let P be a path in space-time, going from x_0 to x_1 and let us consider a field ϕ such that its covariant derivative vanishes along P,

$$\frac{dx^\mu}{ds} D_\mu \phi = 0 \quad ,$$

where s is a parameter along the path. Now ϕ at the end of the path is given in terms of ϕ at the beginning of the path by [9] :

$$\phi(x_1) = g(P)\,\phi(x_0) \quad ,$$

where $g(P)$ is the path dependent $g(P) = T\exp\left(-ie\int_0^1 ds \frac{dx^\mu}{ds} A_\mu^a T^a\right)$ (T^a are the generators of G in the representation of ϕ and T indicates an ordering along the path). If we know $g(P)$ for every path, we can derive the gauge fields [9].

Now let us consider the situation described by (2.3) and the $\partial_r \phi = 0$ on a sphere of sufficiently large radius and define the following path-dependent gauge element [12]:

$$U(r,\Gamma) = T\exp\left(ie\int_{\Gamma(r)} A_\mu^a T^a dx^\mu\right) \quad ,$$

where Γ is a closed circle of fixed latitude on the sphere (Fig.2). Because of (2.3) and because Γ is closed, $\phi(x_0) = U(r,\Gamma)\,\phi(x_0)$ holds for all x_0 of Γ, i.e. $U(r,\Gamma)$ is an element of the unbroken group H. As Γ shrinks to a point on both the north and south pole of the sphere $U(r,N.P.) = U(r,S.P.) = 1$. Therefore when we move Γ from the north pole to the south pole on the sphere, $U(r,\Gamma)$ traces out a closed curve in both H (and in G). It is the homotopy classes of these closed paths which determine the topological number. On the other hand, this U is closely related to the magnetic charge [12,13].

To see this we introduce the following assumptions: 1) The local structure of H is $U(1)\otimes K$. 2) ϕ is in the adjoint representation, $\phi = \phi^i \frac{\lambda^i}{2}$ and we measure the mass in units such that for the asymptotic scalar field $2\,\mathrm{Tr}\,\phi_0^2 = 1$ (this <u>is not</u> a necessary assumption, for a more general case see [12,13]). 3) The electric charge, Q, the generator of the $U(1)$ factor of H is given by $Q = e\phi_0$.

We shall evaluate the $U(r,\Gamma)$ in a gauge where the asymptotic scalar field is a constant - at least to the north of Γ. This gauge can be achieved for all z to the north of Γ by [12]

$$\phi(z) = \Omega\,\phi_0(z)\,\Omega^{-1} = \phi'(N.P.) = \phi_0(N.P.) = \phi_0^i \frac{\lambda^i}{2} \quad ,$$

with $\Omega = T\exp\left(ie \int_z^{N.P.} dx^\mu A^a_\mu \frac{\lambda_a}{2}\right)$. Here the integration runs along the meridian line from z to the north pole. In this gauge from the vanishing of the covariant derivatives (2.3) we find that $A'_\mu = A^{a'}_\mu \frac{\lambda_a}{2} = \mathcal{A}'_\mu \hat{\Phi}_0 + B^i_\mu K_i$, where K_i are the generators of K factor of H , which satisfy [12] :

$[K_i, \hat{\Phi}_0] = 0 , \quad \mathrm{Tr}\, K_i \hat{\Phi}_0 = 0 .$ Due to the vanishing $K_i\ \Phi_0$ commutator, $U'(r,\Gamma)$ factorizes:

$$U'(r,\Gamma) = \exp\left(ie\hat{\Phi}_0 \oint_\Gamma \mathcal{A}'_\mu dx^\mu\right) T\exp\left(ie \oint_\Gamma B^i_\mu K_i dx^\mu\right) \tag{2.5}$$

Applying the Stokes theorem to the first factor yields $\oint_\Gamma \mathcal{A}_\mu dx^\mu = \int_{\hat{S}} d\sigma^*_{\mu\nu} \mathcal{F}^{\mu\nu}$, where $\hat{S}$ is the surface of the sphere to the north of Γ and $\mathcal{F}^{\mu\nu}$ is the electromagnetic field defined in (1.3) with $\frac{\tau_i}{2}$ replaced by $\frac{\lambda_i}{2}$. Now, when we move Γ to the south pole this integral tends to $4\pi g$, while the second factor of (2.5) remains an element of K :

$$\exp(iQ4\pi g) \otimes k = 1 .$$

This expression implies for k that it must commute with all the elements of K , and so lie in what is called the centre of K [12] . If K is SU(N-1) then k must be of the form $\exp(2\pi i n/N-1)\, I_{N-1}$ for $n = 0, \dots N-2$, implying eigenvalues $eg = n/2(N-1)$ for the charge (Q) .

3. EXPLICIT MODEL BUILDING

3.1 The generalized 't Hooft monopole

The topological conservation laws briefly summarized in the previous section give us an enormous amount of information

about the structure of the space of finite energy solutions, yet they have some limitations. 1) They ensure the existence of finite energy (non-dissipative [9]) solutions, not time-independent ones. 2) They are sufficient conditions for the existence, not necessary ones. 3) They do not give us explicit models. If we want to have explicit models, say for higher monopoles or monopoles in SU(N) gauge theory we must follow another way [14]. In what follows we give a direct method to achieve this; we shall also obtain some interesting byproducts as a proof of the uniqueness the 't Hooft monopole under the assumption of spherical symmetry [15] and the possibility of describing the radial motion of the monopole solutions.

Let us assume that the Φ scalar field in (2.1) is in the adjoint representation of the G group, then the field equations are of the form of (1.5) with $\tau_i/2$ replaced by X^i , the group generators, satisfying $[X^i, X^j] = iC^{ij}_k X^k$. First, we deal with the static solutions to these equations [14].

It is possible to show that by a suitable singular gauge transformation [4] 't Hooft's monopole solution - when a static A_0 component is added - can be written in the following form:

$$A_r = 0 , \qquad A_\theta = -\frac{\tau_2}{2}\frac{\xi(r)}{er} ,$$

$$A_\varphi = \frac{\tau_1}{2}\frac{\xi(r)}{er} + \frac{\tau_3}{2}\frac{1}{er}\cot g\,\vartheta ,$$

$$A_0 = \frac{\chi(r)}{er}\frac{\tau_3}{2} , \qquad \Phi = \frac{\tau_3}{2}\frac{\eta(r)}{er} .$$

The advantage of this gauge is the independence from the azimuthal angle φ. Therefore, generalizing these expressions, we shall seek solutions in the general case in the following form:

$$A_r = 0 \, , \quad A_\theta = \frac{a(r)}{er} \, , \quad A_0 = \frac{h(r)}{er} \, ,$$
$$A_\varphi = \frac{b(r)}{er} + \frac{H(r)}{er} \cot g\,\vartheta \, , \quad \Phi = \frac{g(r)}{er} \, . \tag{3.1}$$

Here a, b, H, h, g are functions taking their values in the space spanned by X_i. We fix the gauge by the form of our ansatz (3.1) which is an alternative possibility instead of the usual way of imposing a subsidiary condition.
When we compute the components of $F_{\mu\nu}$ and $D_\mu \Phi$ in terms of this ansatz we find that they contain some angular dependence in the form of certain $\cot g\,\vartheta$ terms. As the next step we impose the requirement of spherical symmetry of our ansatz in the form of demanding that the gauge covariant quantities $(F_{\mu\nu}, D_\mu \Phi)$ should depend on the radial distance only. This makes all coefficients of $\cot g\,\vartheta$ vanish [14]:

$$\frac{dH}{dr} = 0 \, , \quad [h, H] = 0 \, , \quad [H, g] = 0 \, , \quad b = i[a, H] \, . \tag{3.2}$$

We get the same conditions - at least for the vector part - if we impose the Bianchi identities $D^\mu \tilde{F}_{\mu\nu} = 0$ everywhere outside the origin.

Now using (3.2) in the field equations, from the radial component we get two constraints:

$$\frac{d}{dr}\left(a - [H,[H,a]]\right) = 0 \, , \tag{3.3}$$

$$\left[h, \frac{dh}{dr}\right] = \left[g, \frac{dg}{dr}\right] + \left[a, \frac{da}{dr}\right] - \left[[a,H], \left[\frac{da}{dr}, H\right]\right] . \tag{3.4}$$

It is the first of these which is crucial in disproving the existence of higher charge magnetic monopoles in SU(2) under the assumption of spherical symmetry and finite energy. In principle, to describe a spherically symmetric monopole with magnetic charge n/e one would modify the Ansatz (3.1) by multiplying the A_φ term by n [5]. To convince ourselves of the correctness of this ansatz we apply $\Omega = \exp(-i\vartheta \frac{\tau_2}{2}) \otimes \exp(-in\varphi \frac{\tau_3}{2})$ to the modified ansatz, yielding:

$$\Phi = \left(\sin\vartheta \cos n\varphi \frac{\tau_1}{2} + \sin\vartheta \sin n\varphi \frac{\tau_2}{2} + \cos\vartheta \frac{\tau_3}{2}\right) \frac{\eta(r)}{er} \, ,$$
$$A_\vartheta = \frac{\xi(r)-1}{er}\left(\sin n\varphi \frac{\tau_1}{2} - \cos n\varphi \frac{\tau_2}{2}\right) ,$$
$$A_\varphi = \frac{\xi(r)-1}{er}\, n \left(\cos\vartheta \left\{\cos n\varphi \frac{\tau_1}{2} + \sin n\varphi \frac{\tau_2}{2}\right\} - \sin\vartheta \frac{\tau_3}{2}\right) .$$

The normalized $\hat{\Phi}$ constructed from this scalar field really gives a mapping with Kronecker index n, and it is possible to show [15] that it corresponds to the most general situation.

Now, for the modified ansatz, (3.3) yields

$\frac{d}{dr}\left(\xi(r)\,[n^2-1]\frac{\tau_3}{2}\right)=0$. This is trivially satisfied for $n=\pm 1$, while for $n \neq \pm 1$ it implies $\xi(r)=\xi_0$ (constant). From a constant ξ it follows that all $F_{\mu\nu}$ components vanish except $F_{\vartheta\varphi}=\frac{n}{er^2}(\xi_0^2-1)\frac{\tau_3}{2}$.

The energy expression contains $\int \mathrm{Tr}\, F_{\vartheta\varphi}^2\, d^3x$, which is finite if $\xi \longrightarrow \pm 1$ for $r \longrightarrow 0$, i.e. $\xi_0 = \pm 1$. However, this implies that even $F_{\vartheta\varphi}$ vanishes, i.e. the above ansatz describes a pure gauge and hence vanishing magnetic charge. Q.E.D.

The general solution of (3.3) is

$$[H,[H,a(r)]]=a(r)+C\ ,$$

where C is an arbitrary constant matrix. We have three points for choosing $C=0$:

1) As we shall see, the $a(r)\to 0$ for $r \longrightarrow \infty$ behaviour is advantageous, implying $C=0$.

2) For $C \neq 0$ the compatibility of the ϑ, φ components of the gauge field equations yields further unwanted (r dependent) constraints, while for $C=0$ they are identical

3) In a certain sense (see Appendix A of Ref.14) the single valuedness of our ansatz demands $C=0$. Therefore we substitute (3.3) by

$$[H,[H,a(r)]]=a(r)\ . \tag{3.5}$$

Using this in the field equations we finally obtain:

$$r^2\frac{d^2a}{dr^2}-[[a,H],[a,[a,H]]]+a+[h,[h,a]]=[[a,g],g]\ ,$$

$$r^2\frac{d^2h}{dr^2}+[a,[h,a]]-[[a,H],[h,[a,H]]]=[[h,g],g]\ ,$$

$$r^2\frac{d^2g}{dr^2}-[a,[a,g]]+[[a,H],[[a,H],g]]+[h,[h,g]]=$$
$$=X^a\frac{\partial V}{\partial\varphi^a}\,e\,r^2\ . \tag{3.6}$$

3.2 Finite energy solutions

Since Eqs. (3.6) are entirely general, it is important to know what subsets of X_i are admitted for a, H, g, h by these equations without specifying the Lie algebra $L(G)$. Now we show that, together with the additional requirement that the solution should have finite energy, Eqs. (3.6) are really restrictive, especially for H, the coefficient matrix of the string. This additional requirement is purely physical, as at present we are mainly interested in finding extended objects with finite energy (solitons) in spontaneously broken gauge theories.

Let us turn to the $r \longrightarrow \infty$ region first. The finiteness of $E_{Y.M.}$ implies $\lim_{r\to\infty}\left\{\begin{matrix} h \\ a\end{matrix}\right\}\to 0$ or $\left\{\begin{matrix} h_\infty \\ a_\infty\end{matrix}\right\}$, while from $E_{Higgs}<+\infty$ it follows that $g \to g_\infty r$ and $[a^\infty, g^\infty], [h^\infty, g^\infty]=0$. These last equations express that the asymptotic scalar field is in its ground state

$(V(\phi)=0\ ,\ D_r\phi=0)$. When we assume $[h,g]=0$ for all r , this, together with $[h,H]=0$ and $[H,g]=0$ following from the spherical symmetry, can be satisfied at once choosing $H,\ h,\ g$ from an abelian subalgebra. This choice reduces (3.4) to

$$\left[a,\frac{da}{dr}\right]=\left[[a,H],\left[\frac{da}{dr},H\right]\right].$$

In the $r\to 0$ we find from the convergence of $\int d^3x\, \mathrm{Tr}\, F^2_{\vartheta\varphi}$ that $\lim_{r\to 0} a(r)=A$ must exist and must satisfy $[A,[A,H]]=H$. ($[[A,H],H]-H=\tilde{C}$, $\mathrm{Tr}\,\tilde{C}^2=0$ is automatically excluded for semi-simple Lie algebras.) Now from (3.5) and (3.2) we find

$$\lim_{r\to 0} a(r)=A=[H,[H,A]]\ ,\qquad \lim_{r\to 0} b(r)=B=i\,[A,H]\ . \tag{3.7}$$

Combining these together we see that $A,\ B,\ H$ must form an SU(2) or SO(3) subalgebra, i.e. H is not arbitrary! The finiteness of the scalar contribution yields that $g,\ h$ are of $\mathcal{O}(r^2)$ for $r\to 0$.

An immediate consequence of the presence of this SO(3), SU(2) subalgebra is that we can always describe a solution with it, the trivial embedding of the previous dyon solution into our theory based on G [14] . Secondly, we can always transform away the string from A_φ via a gauge transformation [14]:

$$\Omega=\exp(-iH\varphi)\exp(iA\vartheta)\ .$$

This transformation also gives an angular dependence to the transformed asymptotic scalar field. If we use the $\left\{\begin{matrix} a(\infty)=0 \\ h(\infty)=0\end{matrix}\right\}$ boundary conditions, the transformed form of the asymptotic gauge field tensor is $F_{ij}=\varepsilon_{ijk}\frac{x_j T_k}{e r^2}$, where $T_k\ (k=1,2,3)$ are the A,B,H matrices in an appropriate order. As a third consequence, we mention that now the Bianchi identities are satisfied everywhere, even at the origin.

3.3 SU(N) monopole solutions

Let us concretize Eqs. (3.6) for the $G=SU(N)$ case, i.e. take X^i to be the $N\times N$ λ_i matrices; their representation and the necessary commutation relations are given in [14] . In the light of the criteria found in the previous section the most natural choices for H are the following matrices:

$$H^{(m)}=\mathrm{diag}\left(\frac{m-1}{2},\frac{m-3}{2},\dots -\frac{m+1}{2},0\dots 0\right)=\sum_{k=2}^{m}\frac{k}{2}\sqrt{\frac{k-1}{2k}}\,\lambda_{k^2-1}\ ,$$

with $m=2,\dots N$, describing $N-1$ different, irreducible representations of I_3 . Constructing the corresponding $A^{(m)},B^{(m)}$ matrices, we are finally led to the following ansatz [14] satisfying all constraints:

$$a(r)=\sum_{k=2}^{m}\frac{\sqrt{(k-1)(m-k+1)}}{2}\lambda_{k^2-3}f_k(r),\quad h(r)=\sum_{k=2}^{N}\sqrt{\frac{k-1}{2k}}\lambda_{k^2-1}h_k(r).$$

$$g(r)=\sum_{k=2}^{N}\sqrt{\frac{k-1}{2k}}\lambda_{k^2-1}g_k(r).$$

In terms of these functions the system (3.6) reduces to

$$r^2 f_k'' - f_k\Big[(k-1)(m-k+1)f_k^2-\frac{k(m-k)}{2}f_{k+1}^2-\frac{(k-2)(m-k+2)}{2}f_{k-1}^2\Big]$$

$$+f_k+f_k\Big(h_k-\frac{k-2}{k-1}h_{k-1}\Big)^2=f_k\Big(g_k-\frac{k-2}{k-1}g_{k-1}\Big)^2$$

$$r^2 h_\ell''-\ell(m-\ell)f_{\ell+1}^2\Big(\frac{\ell-1}{\ell}h_\ell-h_{\ell+1}\Big)$$

$$+\ell(m-\ell+1)f_\ell^2\Big(\frac{\ell-2}{\ell-1}h_{\ell-1}-h_\ell\Big)=0$$

$$r^2 g_\ell''-\ell(m-\ell)f_{\ell+1}^2\Big(\frac{\ell-1}{\ell}g_\ell-g_{\ell+1}\Big)+\ell(m-\ell+1)f_\ell^2\Big(\frac{\ell-2}{\ell-1}g_{\ell-1}-g_\ell\Big)=V_\ell$$

$$V_\ell=\frac{1}{2}\mathrm{Tr}\sqrt{\frac{2\ell}{\ell-1}}\lambda_{\ell^2-1}X^a\frac{\partial V}{\partial\phi^a}er^3. \tag{3.8}$$

Here $k=2,\dots m$, $\ell=2,\dots N$ and in the second and third equations all $f_\ell\equiv 0$ for $\ell>m$. The finite energy requirement demands that all $f_k(r)\to\pm1$ at $r\to 0$. We also assume that the symmetry breaking minimum of $V(\phi)$ lies on a λ_{N^2-1}-like orbit in SU (N) :

$$V(\phi)=\frac{c_1^2}{2}\left(d^2-\mathrm{Tr}\,\phi^2\right)^2+\frac{c_2^2}{2}\mathrm{Tr}\left(\phi^2-\frac{N-2}{\sqrt{2N(N-1)}}d\phi-\frac{1}{2N}d^2 I\right)^2.$$

(Here c_1, c_2, d are constants and I stands for the $N\times N$ unit matrix.) This form implies that $\phi^\infty=\lim_{r\to\infty}\frac{g(r)}{er}$ can be normalized, $2\,\mathrm{Tr}\,\hat\phi^\infty\hat\phi^\infty=1$, which in turn means that

$$\hat\phi^\infty_{(p)}=\frac{1}{2}\sqrt{\frac{N}{N-1}}\left\{-\sqrt{\frac{p-1}{p}}\lambda_{p^2-1}+\sum_{\ell=p+1}^{N}\frac{1}{\sqrt{\ell(\ell-1)}}\lambda_{\ell^2-1}\right\},\quad p=1,\dots N$$

are the possibilities.

The asymptotic electromagnetic field, defined in analogy with (1.3) is given by $\mathcal{F}_{\mu\nu}=2\,\mathrm{Tr}\,\hat\phi^\infty_{(p)}F_{\mu\nu}$. From Eqs.(3.8) it follows that all $D_\mu\phi$ and $F_{\mu\nu}$ vanish exponentially, except $F_{\theta\varphi}=-H^{(m)}/er^2$. Thus asymptotically we have a radial magnetic field with magnetic charge $g=(-2\,\mathrm{Tr}\,\hat\phi_{(p)}H^{(m)})/e$. Varying m and p in their corresponding intervals we obtain $(N-1)$ different magnetic charges: $g_0\le kg_0\le(N-1)g_0$, $g_0=e^{-1}\sqrt{N/2(N-1)}$.

To discuss the charge quantization, we couple a fermion field in the fundamental representation (quarks) in a minimal way to the gauge fields. The electric charges of the quarks are then given by the eigenvalues of $e\hat\phi$ [12,13], i.e. they are $e_0=\frac{e}{\sqrt{2N(N-1)}}$ and $e'=-e\sqrt{\frac{N-1}{2N}}$ for the non-singlet and singlet (with respect to the unbroken SU(N-1)) quarks respectively. We have seen above that the possible magnetic charges are given by $g=\frac{k}{e}\sqrt{\frac{N}{2(N-1)}}$, $k=1,\dots(N-1)$ so

$$ge_0=\frac{k}{2(N-1)},\qquad ge'=-\frac{k}{2}.$$

It is possible to interpret the anomalous first relation [12] if one identifies the unbroken SU(N-1) with colour. Then one observes that all monopoles with $k = 1, \dots N-2$ carry both magnetic and colour flux, thus they can interact with the coloured quarks not only electromagnetically but through the SU(N-1) colour fields too. On the other hand, they couple to the colour singlet quark only electromagnetically, and so for these states the charge quantization is satisfied in Dirac's sense.

It is clear from the form of $\mathcal{F}_{\mu\nu}$ that if we want our solutions to carry electric charge also (i.e. to be dyons) we must change the $h_i \to 0$ assumption for $h_i \to C_i$. This is a choice allowed by Eqs.(3.8) and by the finite energy requirement. As for the time being we are unable to describe a dyon-antidyon system orbiting around each other, we find no reason why the electric charge of the dyons should obey a quantization rule.

3.4 Stability considerations

Very little is known about the stability of these solutions under small perturbations. Of course the conservation of the magnetic charge ensures that the solutions cannot radiate away all their energy. However, all solutions so far have been obtained under the assumption of spherical symmetry, and therefore we have no information about the effects of spherically non-symmetric perturbation. The first step in this direction was made in Ref. 16 for the case of SU(2); here we apply the

same method for our case. As a result we can state that certain dyon-type solutions in the $V(\phi) \equiv 0$ case are stable because they are absolute minima of the energy. Here, following Ref.16, we define stability to include the possibility of neutral equlibrium.

Three conserved quantities are important for the proof: the energy, the electric charge of the solution $Q = \lim_{r\to\infty} \frac{1}{4\pi} \int d\sigma_i \, 2\,\mathrm{Tr}(\phi F_{0i})$ and the magnetic charge $g = \lim_{r\to\infty} \frac{1}{4\pi} \int d\sigma_i \, 2\,\mathrm{Tr}(\phi \frac{1}{2} \varepsilon_{ijk} F^{jk})$ (we measure the length in such units that $d^2 = \phi_0^2 = 1$).

The proof rests in a theorem of Lagrangian mechanics: Given a system (q^a) with Lagrangian of the form:

$$L = \frac{1}{2} \dot{q}^{a^2} + B^a(q)\,\dot{q}^a - V(q),$$

then the minima of the energy are points of stable equilibrium [16]. If the theory has other conserved quantities, then the same is true for the minima of E if these quantities are held fixed. The Lagrangian of our theory has this form if we choose a gauge where $A_0 = \bar{A}_0$, where $\bar{A}_0$ is the supposed known value of the solution to (3.8) [17].

We shall seek solutions which are minima of the energy, keeping g and Q fixed. From the Bianchi identities follows $D_i \frac{1}{2} \varepsilon_{ijk} F_{jk} = 0$, implying $\int d^3x \, 2\,\mathrm{Tr}(D_i\phi \, \frac{1}{2} \varepsilon_{ijk} F^{jk}) = 4\pi g$. On the other hand, partially integrating the zeroth component of the gauge field equations, we obtain $\int d^3x \, 2\,\mathrm{Tr}(D_i\phi \, F_{0i}) = 4\pi Q$. Now we can give a lower bound for the energy [16]:

$$E \geq \int d^3x \, 2 \left\{ \mathrm{Tr} \left(F_{0i} - \sin\alpha \, D_i \phi \right)^2 + \mathrm{Tr} \left(\tfrac{1}{2} \varepsilon_{ijk} F_{jk} - \cos\alpha \, D_i \phi \right)^2 \right\} +$$

$$+ 4\pi Q \sin\alpha + 4\pi g \cos\alpha \geq 4\pi Q \sin\alpha + 4\pi g \sin\alpha$$

(α arbitrary).

The bound is saturated and therefore the corresponding solution is stable if $V(\phi) \equiv 0$, $D_0 \phi = 0$ and $F_{0i} = \sin\alpha \, D_i \phi$, $\frac{1}{2} \varepsilon_{ijk} F_{jk} = \cos\alpha \, D_i \phi$.

These requirements impose the following equations for our ansatz (3.1) :

$$h = -\sin\alpha \, g \qquad\qquad -\frac{1}{er}\frac{da}{dr} + \frac{\cos\alpha}{er^2} [[a,H],g] = 0 ,$$

$$\frac{1}{r^2}\left([a,[a,H]]-H\right) - \cos\alpha \frac{d}{dr}\left(\frac{g}{r}\right) = 0 \qquad -\frac{1}{er}\left[\frac{da}{dr}, H\right] + \frac{\cos\alpha}{er^2}[a,g] = 0 .$$

The two equations in the second column are compatible if the (3.3)-(3.4) constraints hold. For the f_m, g_n functions we find from these equations a first-order system of equations that replaces (3.8) :

$$-f_n' + \frac{\cos\alpha}{r} f_n \left(g_n - \frac{n-2}{n-1} g_{n-1} \right) = 0 \qquad n = 2, \dots m ,$$

$$-\cos\alpha \frac{d}{dr}\left(\frac{g_s}{r}\right) + \frac{s}{r^2}\left\{ (m+1-s) f_s^2 - \frac{1}{2} - (m-s) f_{s+1}^2 \right\} = 0$$
$$s = 2, \dots m ,$$

$$-\cos\alpha \frac{d}{dr}\left(\frac{g_s}{r}\right) = 0 , \qquad m < s \leq N .$$

If $m = N = 2$ this system of equations can be integrated analytically, yielding the Prased-Sommerfield solution [18] .

The second aspect of stability is the question of, in principle, dynamically possible processes like decay: monopole $(kg) \longrightarrow k$ monopoles (g) . If $E(kg) \geq k\,E(g)$, then the higher charge monopole is dynamically unstable, it can decay by fission into the topologically equivalent case of several separated unit monopoles. Precisely this situation was found [19] in the SU(2) case for $k \geq 2$ We have a feeling that this phenomenon generalizes to the SU(N) cases, since for the infinite energy-point solutions $\left(g_\ell(r) \equiv g_\ell(\infty) ; f_m = 0 , m = 2, \dots, N \right)$ we find the <u>energy densities</u>:

$$\mathcal{E}([m-1]g_0) = \frac{4\pi}{e^2}\frac{1}{4r}\left\{ 2(m-1)^2 + \sum_{k=2}^{m-1} [2k-(m+1)]^2 \right\} >$$
$$> (m-1)\,\mathcal{E}(g_0) = (m-1)\frac{4\pi}{e^2}\frac{1}{4r^2}\cdot 2 .$$

<u>3.5 Radial motion-time dependent monopoles</u>

So far we have studied the static solutions only. With an eye on the quantization of these models it is necessary to investigate the non-static case also. Because this problem is too complicated to discuss in full complexity we shall restrict ourselves to studying only the radial motion of the previous static <u>monopoles</u>, i.e. we now choose our ansatz by modifying (3.1) in such a way that all a, b, H, g depend on both x_0 and r and we assume $h \equiv 0$ [17] .

As this ansatz has the same angular dependence as the static case, the corresponding time-dependent solution will describe the "breathing" of the static monopole, preserving the radial symmetry.

Of course, changing the ansatz (3.1) modifies some $F_{\mu\nu}$ and Π_μ

components, from the requirement of spherical symmetry we obtain for this case [14] :

$$\frac{\partial H}{\partial \tau} = 0 \,, \quad \frac{\partial H}{\partial x_0} = 0 \,, \quad b = i\,[a, H] \,, \quad [H, g] = 0 \,.$$

From the field equations we find that (3.4) and (3.5) remain unchanged, the zeroth component of the gauge field equations yields a further constraint:

$$\left[a, \frac{\partial a}{\partial x_0}\right] - \left[[a, H], \left[\frac{\partial a}{\partial x_0}, H\right]\right] = \left[\frac{\partial g}{\partial x_0}, g\right] ,$$

and finally

$$\tau^2\left(\frac{\partial^2 a}{\partial \tau^2} - \frac{\partial^2 a}{\partial x_0^2}\right) = \big[[a, H], [a, [a, H]]\big] - a - \big[[a, g], g\big]$$

$$\tau^2\left(\frac{\partial^2 g}{\partial \tau^2} - \frac{\partial^2 g}{\partial x_0^2}\right) - \big[a, [a, g]\big] + \big[[a, H], [[a, H], g]\big] = X^a \frac{\partial V}{\partial \phi^a} e\tau^3 \,.$$

We note that this system is the same as (3.6) except for the replacement of the operator $d^2/d\tau^2$ by $\partial^2/\partial\tau^2 - \partial^2/\partial x_0^2$

The finite energy requirement preserves all its previous consequences and in addition also imposes $\lim\limits_{\tau \to 0,\infty} \frac{\partial a}{\partial x_0} = 0$, $\lim\limits_{\tau \to 0,\infty} \frac{\partial g}{\partial x_0} = 0$. This last condition ensures that the time dependence affects only the intermediate τ region keeping the boundary conditions at $\tau = 0$ $\tau = \infty$ and fixed.

Specializing for the 't Hooft monopole, the equations of radial motion can be obtained from (1.7) by the above-mentioned substitution of d^2/dx^2 for $\partial^2/\partial x^2 - \partial^2/\partial t^2$ [4] .

As even in the time dependent case $\hat{\phi} = \frac{\tau_i x_i}{2\tau}$ (provided $\eta(r,t) \neq v$) the previously obtained expressions for the gauge invariant magnetic field and the dipole density remain valid:

$$\mathcal{F}_{\vartheta\varphi} = B_r = \frac{1}{e r^2} \,, \quad \mathcal{M}_{\vartheta\varphi} = M_r = -\frac{\xi^2(r,t)}{4\pi e r^2}$$

while the electric field $\mathcal{F}_{0i}$ remains zero, implying the absence of radiation. Therefore the time dependence affects only the dipole density and not the magnetic field of the monopole.

The first step towards quantization requires a detailed knowledge of small oscillations around the static solutions.
To this end we introduce

$$\xi = \xi_0 + \xi_1(x)\, e^{i\omega t} \qquad \eta = \sqrt{2}\, u = \sqrt{2}\,(u_0 + u_1(x)\, e^{i\omega t}) \,,$$

where ξ_0 , $\sqrt{2}u_0 = \eta_0$ denote the previous static solutions and linearize the equations in $\xi_1(x)$ and $u_1(x)$. The resulting equations can be written in the form of a Schrödinger equation for the two-component vector $\sigma(x) = \begin{pmatrix} \xi_1 \\ u_1 \end{pmatrix}$ [4] :

$$x^2(\sigma'' + \omega^2\sigma) = \underline{A}\,\sigma + 4\,\xi_0 u_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma$$

$$\underline{A} = \begin{pmatrix} 3\xi_0^2 - 1 + 2u_0^2 & 0 \\ 0 & 2\xi_0^2 - \frac{\beta}{2}(x^2 - 6u_0^2) \end{pmatrix} .$$

The ω^2 eigenvalues can be determined only numerically as we know ξ_0 and u_0 analytically only in the unphysical $\beta = 0$ limit.

The usual argument relying on the translational invariance of the theory for the existence of a $\omega^2 = 0$ eigenvalue corresponding to the eigenfunctions constructed from the gradients of the static solutions cannot be applied here since Eq.(1.7) is spherically symmetric and the translated monopole is not. The determination of the ω^2 spectra as well as the investigation of the spherically non-symmetric oscillations go beyond the scope of the present paper.

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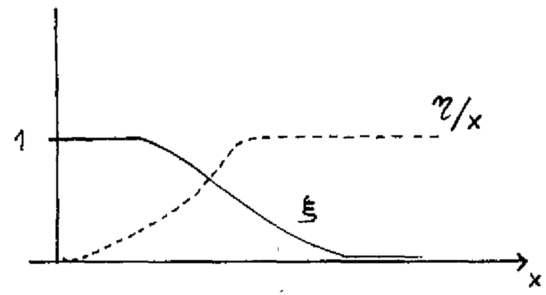

Fig.1

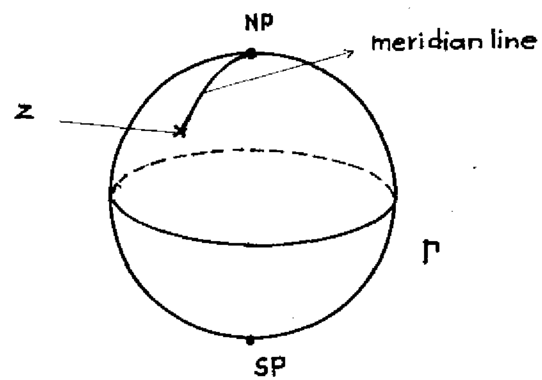

Fig.2