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ABSTRACT

We relate here the geometric formulation of Hamilton's Principle presented in our previous paper [1] to the usual one in terms of a Lagrangian function. The exact conditions for their equivalence are obtained and a method is given for the construction of a Lagrangian function. The formalism is extended to spin particles and a local Lagrangian is constructed in this case also. However, this function cannot be extended to a global one.

АННОТАЦИЯ

Связываются различные способы определения принципа Гамилтона: геометрический, введенной в предыдущей статье [1] и обыкновенный, в виде функции Лагранжа. Выводятся точные условия их эквивалентности и показывается метод построения функции Лагранжа. Формальность распространяется и на частицы со спином. Однако, построенную в этих условиях, локальную функцию Лагранжа нельзя глобально определить.

KIVONAT

Kapcsolatot létesitettünk Hamilton elvének előző cikkünkben [1] bevezetett geometriai és szokásos Lagrange-függvényes megfogalmazása között. Pontos feltételeket kapunk azok ekvivalenciájára és módszert mutatunk a Lagrange-függvény konstrukciójára. A formalizmus kiterjeszthető a spines részecskékre is. Az ekkor konstruált lokális Lagrange-függvény azonban nem globalizálható.

1. INTRODUCTION

In our previous paper [1] we have present d a geometrical framework which made us possible to treat Hamilton's Principle in a coordinate-free manner.

A mechanical system was charecterized by its evolution space, E, and Lagrange-form, σ. The equations of motion are expressed as

$$\frac{d}{dt}(\tilde{g})(t) \in \text{Ker}\ \sigma_{\tilde{g}(t)} \qquad /1.1/$$

where $\tilde{g}$ denotes the lifted to E of the motion-curve g lying in configuration space, Q.

In classical case, when a Lagrangian function L is given, one can introduce a λ field of 1-forms called the system's <u>Cartan-form</u>, in terms of which the Euler-Lagrange-equations take the form:

$$\frac{d}{dt}(\tilde{g})(t) \in \text{Ker}\ d\lambda_{\tilde{g}(t)} \qquad /1.2./$$

Comparing 1.1 and 1.2 we said that a system given by (E,σ) has a local variational description if there exists at least locally /i.e. defined on a local chart/ a λ field of 1-forms for which $\sigma = d\lambda$

The necessary and sufficient condition for the existence of such a λ is $d\sigma = 0$ referred to by Souriau [2] as Maxwell's Principle.

Our most interesting result was to prove that although the equations of motion of a spin particle can be expressed in terms of a Lagrange-form σ [1] satisfying $d\sigma = 0$, there cannot exist a globally defined λ with $d\lambda = \sigma$.

However, the definition of the Cartan -form given in [1] was rather "ad hoc" and valid only in classical case. Also, it was not entirely clear what the above result means in terms of a Lagrangian function.

We propose here a slightly different framework which can easily be extended to include spin particles as well.

We shall study the world lines in QxR; the role of the evolution space will be played by a subset F of T(QxR). L will be substituted by a homogenized Lagrangian, α. Then we can state that there is a unique way to geometrize the problem, namely by using the field of 1-forms $\Lambda = \dot{d}\alpha$ /$\dot{d}$ denotes the operation of "vertical differentitation", see Klein [3],[4] or Godbillon [5] /. This Λ can be projected onto E; its image λ coincides with the Cartan-form introduced in [1]. Using Klein's results we can establish the necessary and sufficient conditions for the possibility to recover a Lagrangian function from a field of 1-forms λ defined on the evolution space. In local coordinates this sounds: λ must be of the form $a_\alpha \, dq^\alpha + b \, dt$ /$\alpha = 1,..n$/ and

$$v^\alpha \partial_{\dot{\beta}} a_\alpha + \partial_{\dot{\beta}} b = 0 \qquad / \beta = 1,..n/ \qquad /1.3./$$

must hold. In this case L has the local form:

$$L\,(q,v,t) = a_\alpha v^\alpha + b \qquad /1.4./$$

In this framework the generalized solutions of the variational problem will be constant dimensional submanifolds of F called evolution leaves. Their projections on space-time are C^∞ curves satisfying the Lagrange-equations. These leaves can be set in one-to-one correspondance with the evolution curves of E.

Hence our results obtained in [1] mean really that there cannot exist any globally defined Lagrangian function for the spin par 'cle; nevertheless, condition 1.3 is satisfied and we can construct a local Lagrangian function in this case as well.

2. GENERAL THEORY

We give here a short outline of a geometrical framework slightly differing of that presented in [1]. This is a kind of synthesis of the ideas of Souriau [2] and Klein [3],[4].

The basic point is to work with world lines in QxR rather then with motion in configuration space, Q, time being a fixed exterior parameter. This approach is natural in relativity and turns out to be useful in this context as well. Now we carry over our entire apparatus to this context.

2.1 Motions

To the motions in Q correspond the world lines parametrized by an arbitrary parameter τ. In local coordinates a world line is

$$h : \tau \to h(\tau) = (g(\tau),\ t(\tau)) \in Q\times R \qquad /2.1.1./$$

it is required that $\dot{t} > 0$ / the dot denotes the derivative of t by τ /.

2.2 The Hamiltonian Action

A change in parametrization shows that the Hamiltonian action has the form

$$\vartheta(g) = \int_{\tau_1}^{\tau_2} L\left[g(\tau),\ \frac{\dot{q}(\tau)}{\dot{t}(\tau)},\ t(\tau)\right] \dot{t}(\tau)\, d\tau \qquad /2.2.1./$$

If $x = (q,t)$ is a local chart for $Q\times R$, then the corresponding natural chart of $T(Q\times R)$ will be denoted by $(x,\dot{x}) = (q,t,\dot{q},\dot{t})$.

Define $F := \{(x,\dot{x}) \in T(Q\times R) \ :\ \dot{t} > 0\}$ and introduce the homogenized Lagrangian $\mathcal{L}$ as:

$$\mathcal{L} \ :\ F \ni (q,t,\dot{q},\dot{t}) \to L(q,\ \frac{\dot{q}}{\dot{t}},\ t\ )\ \dot{t} \qquad /2.2.2/$$

$\mathcal{L}$ is homogenous of order 1 in $\dot{x}$; using $\mathcal{L}$ the hamiltonian action is

$$\vartheta(g) = \int_{\tau_1}^{\tau_2} \mathcal{L}\ (h,\dot{h})d\tau \qquad /2.2.3./$$

2.3 States

In [1] we have identified the states of the system with the points of E. The points of F can be thought of as states as well; two points of F represent the same state and are called hence physically equivalent if their first coordinates coincide and second coordinates are proportional.

Define $\Pi\ :\ F \to E$ as $\Pi(q,t,\dot{q},\dot{t}) = (\ q,\frac{\dot{q}}{\dot{t}},t)$

then two points of F are physically equivalent if they are mapped to the same points of e by Π .

Let h be a curve in QxR parametrized by τ. Then define

$$\tilde{h}^{\tau}(\tau) := (h(\tau), \dot{h}(\tau)) \qquad /2.3.1./$$

to be the lifted if h to F according to τ. By changing the parameter we get an entire family of curves in F, which cover a 2 -dimensional submanifold lying in F. In particular, to the motion curves corresponds a system of two dimensional submanifolds with the property that through every point of F passes exactly one such submanifold called evolution leaf. Π sets up a 1-to-1 correspondance between these leaves and the evolution curves of E. Hence the system's evolution can be described alternatively by the evolution leaves of F.

2.4 The Cartan-form

The "geometrization " of the problem -which turned out to be useful in [1] - is based on the following lemma:

Lemma: Let us given an $\mathcal{L} : F \to R$ function which is homogenous of order 1 in $\dot{x}$. Then there exists a unique field of 1-forms Λ on F which satisfies

$$\int_{\tau_1}^{\tau_2} \mathcal{L}\,(h(\tau), \dot{h}(\tau))\, d\tau \qquad \int_{\tilde{h}^{\tau}} \Lambda \qquad /2.4.1./$$

In local coordinates this Λ is given as

$$\Lambda = \dot{d}\mathcal{L} = \partial_{\dot{\alpha}}\mathcal{L} \quad dx^{\alpha} \qquad /2.4.2./$$

the "vertical derivative" of $\mathcal{L}$.
/for the definition of $\dot{d}$ see Klein, [3],[4] or Godbillon [5] :

$\partial_{\dot{\alpha}}\mathcal{L}$ denotes $\frac{\partial \mathcal{L}}{\partial \dot{x}^{\alpha}}$ /.

One verifies at once that Λ is homogenous of order 0, semibasic, and $\dot{d}$ -closed, i.e. $\dot{d}\,\Lambda = 0$. Klein has shown in [3] that these conditions are also sufficient for Λ to be locally, i.e. on a local chart, of the form $\dot{d}\,\mathcal{L}$. In local coordinates if Λ is given as

$$\Lambda = A_{\alpha}\, dx^{\alpha} \text{ , then } \quad \mathcal{L} = A_{\alpha}\dot{x}^{\alpha} \qquad /2.4.3./$$

The above defined Λ can easily be related to the Cartan-form λ introduced in [1]

<u>Definition</u>: An α p-form on F is said to be projectable onto E iff there exists a unique β p-form on E for which $\alpha = \Pi^*(\beta)$

One shows that a semi-basic 1-form on F is projectable if it is homogenous of order 0 ; consequently our Λ defined in 2.4.2. is projectable.

In local coordinates we get

$$\Lambda = \left(\frac{\partial L}{\partial v^\alpha}\cdot\Pi\right) dq^\alpha + \left[\left(L - \partial_{\dot{t}}\dot{q} \frac{\partial L}{\partial v^\alpha}\right)\cdot\Pi\right]dt \qquad /2.4.4./$$

Thus its image under Π is exactly λ, the Cartan-form introduced in [1]. Conversely, if we are given a λ field of 1-forms on E, we can pull it back to F. If $\lambda = a_\alpha dq^\alpha + b\, dt$ is a semi-basic 1-form on E, then its pulled -back is

$$\Pi^x(\lambda) = (a_\alpha\cdot\Pi)\, dg^\alpha + (b\cdot\Pi)\, dt \qquad /2.4.5./$$

One verifies that $\dot{d}\,(\Pi^*(\lambda)) = 0$ iff

$$\underline{v^\alpha \partial_{\dot{\beta}}\, a_\alpha + \partial_{\dot{\beta}} b} = 0 \quad /\beta = 1,\dots n/ \qquad /2.4.7./$$

It this case the local Lagrangian function is given as

$$\underline{L(q,v,t) = a_\alpha v^\alpha + b} \qquad /2.4.8./$$

These last two formulae will be very important, for they not only express the necessary and sufficient conditions under which a local Lagrangian function can be recovered, but they give even its explicit form.

2.5 <u>Calculus of variations</u>

Our investigations in [1] were based on the heuristic idea that calculus of variations can be thougth of as differential calculus on an infinite-dimensional manifold. We will keep in mind this idea also now; the geometrical framework will be slightly different.

Let us choose two points in space-time, $x_1 = (q_1,t_1)$ and $x_2 = (q_2,t_2)$ and consider the C^∞ curves joining these points /denote their set again $\mathcal{P}$ /. The manifold-structure on $\mathcal{P}$ is defined by introducing a Banach-space structure on its tangent space. One of the basic postulates of classical mechanics is the absolute status of time; this property will be reflected by the requirement that time be not varied. Hence we define the tangent space of $\mathcal{P}$ at a curve h to be the set of all C^∞ vectorfields along h which can be extended onto a neighbourhood of h, vanish at the end-point's and have 0 time-component, i.e. have the local form $(Y_q, 0)$.

In order to calculate the stationary curves a very similar treatement as in [1] is applied. Finally we get: the directional derivative of ϑ by an element Y of the tangent space at a point g.

$$Y(\vartheta)(g) = \int_{\tilde{h}^{\tau}} d\Lambda(.,\tilde{Y}) \qquad /2.5.1./$$

where $\tilde{Y}$ denotes again the lifted of the vector field Y to F; in local coordinates

$$\tilde{Y} = (Y,\dot{Y}) \quad \text{with} \quad \dot{Y}^{\alpha} = \partial_{\beta} Y^{\beta} \dot{X}^{\alpha} \qquad /2.5.2./$$

/see Klein [3] /.

Now, the corresponding form of the Du Bois-Reymond lemma assures that

$$Y(\vartheta)(g) = 0 \quad \text{iff} \quad d\Lambda(\dot{\tilde{h}},\tilde{Y}) = 0 \quad \forall\, Y \in T_h \mathcal{D} \qquad /2.5.3./$$

Note that because time is not varied, this does not mean that

$\dot{\tilde{h}}^{\tau}(\tau) \in \operatorname{Ker} d\Lambda_{\tilde{h}^{\tau}(\tau)}$, nevertheless a similar line of thought as in [1] can be applied:

$$H_y := \{X \in T_y T(Q_x R) : d\Lambda(X,Y) = 0 \quad \forall Y \in T_y T(Q \times R) \text{ of the form } (Y_q, Y_{\dot{q}}, 0)\} \qquad /2.5.4./$$

then one verifies that g is a stationary curve if

$$\dot{\tilde{h}}^{\tau}(\tau) \in H_{h(\tau)} \qquad /2.5.5./$$

/ $h(\tau) = (g(\tau), t(\tau))$ /.

Suppose H_y has constant dimension for every $y \in F$; one verifies then that if $X, Z \in H_y$ then their Lie Bracket, $[X,Z]$ belongs also to H_y; hence we can apply the corresponding theorem of fiber spaces /see Souriau [2]/ and get:

Proposition: Through every point of F passes exactly one leaf, whose tangent space coincides there with the H- subspace ordered to this point : we call these leaves the leaves belonging to Λ /or $d\Lambda$/.

One verifies, that if we start with L, then codim H_y equals to rang of the matrix $|\partial_{\dot{\alpha}\dot{\beta}}L|$; the problem will be called to be a regular one if this rang is n ; the leaves are then twodimensional submanifolds of F. We shall consider them as the generalized solutions of the variational problem. By a straightforward calculation one verifies, that

Theorem: Suppose the problem is regular; then the leaves belonging to $d\dot{d}\mathcal{L}$ have the following property: their projections onto QxR are C^∞-curves satisfying the Lagrange-equations.

2.6 Relation to symplectic mechanics

We see that the two-form $d\dot{d}\mathcal{L}$ plays a fundamental role similar to that of σ in [1]. This is the consequence of the following.

Proposition: $\Sigma := d\dot{d}\mathcal{L}$ can be projected onto E and
$d\,\Sigma = 0$, $\dot{d}\Sigma = 0$.
if $\Lambda = \Pi^*(\lambda)$, then $\Sigma = \Pi^*(d\lambda)$

The fundamental result of Klein in [4] states, that if Σ can be projected on E, and $d\Sigma = 0$, $\dot{d}\Sigma = 0$ both are valid, then there exists a local $\mathcal{L}$ with $\Sigma = d\dot{d}\mathcal{L}$. The most general $\mathcal{L}$ is given as $\mathcal{L} \to k\mathcal{L} + \partial_\alpha f \ \dot{x}^\alpha$ where f is a function on QxR alone. /here we recognize the well-known gauge transformation/. In terms of L this means:

$$L \to kL + \frac{\partial f}{\partial q^\alpha} v^\alpha + \frac{\partial f}{\partial t}$$

Conversely, if we are given a pair (E,σ), we may pull σ back to F and define so a Σ ; $d\sigma = 0$ iff $d\Sigma = 0$. Our local variational principle proposed in [1] states the existence of a locally defined λ with $d\lambda = \sigma$; as we have just seen, this does not mean automatically the existence of a Lagrangian function. In order to decide whether this stronger requirement has to be made or not, we have to study the spin particle, for in classical cases the problem is solved.

3. VARIATIONAL FORMALISM FOR SPIN PARICLES

After this general framework we are ready to turn to the construction of a variational formalism for the spin particles. Remember, that in this case $E = T(Q) \times R \times S^2$, and the Lagrange-form is given as

$$\sigma = \{ (mdv - E\,dt)\wedge(dq - vdt) + \langle B, dq \times dq\rangle \} + + d(\mu\langle s,B\rangle dt) - \lambda\langle s, ds \times ds\rangle) \qquad /3.1./$$

/ $\langle .,. \rangle$ the scalar product in 3-space/
Here the first curly bracket is the Lagrange-form of a particle moving in electromagnetic field; in what follows we shall study only the second one representing the spin-interaction.

As spin can be thought of as an inner degree of freedeom, it seems to be natural to treat it together with configuration space; we will simply substitute Q by $Q \times S^2$. A local Lagrangian function would be then an L defined on a local chart of $T(Q \times S^2) \times R$. To L would correspond a homogenized $\mathcal{L}$ according to the formula

$$\alpha(q,s,t,\dot q,\dot s,\dot t) := L(q,s,\frac{\dot q}{\dot t}, \frac{\dot s}{\dot t}, t)\ \dot t \qquad /3.2./$$

which yields the Cartan-form

$$\Lambda = \frac{\partial \mathcal{L}}{\partial \dot q^\alpha}\, dq^\alpha + \frac{\partial \mathcal{L}}{\partial \dot t}\, dt + \frac{\partial \mathcal{L}}{\partial \dot s^\alpha}\, ds^\alpha \qquad /3.3./$$

the exterior derivative of this must coincide with the pulled back to F of the system's Lagrange-form. The evolution space would be $T(Q \times S^2) \times R$. Our original E was merely $T(Q) \times R \times S^2$, but this does not matter as σ can be trivially extended to $T(Q \times S^2) \times R$ as it has no $d\dot s$ component.

In order to find a variational formalism for the spin particle we have only to find an additional term the spin part of σ, σ_{spin}.

The pulled-back of the spin -part to F is simply

$$\Pi^*(\sigma_{spin}) = d(\mu\langle s,b\rangle dt) - \lambda\langle s, ds \times ds\rangle \qquad /3.4./$$

/ λ s the proper angular - μs the proper magnetic momentum; see [1] or [2]; λ, μ real constants/
Thus we have to solve the equation

$$d\dot{\mathcal{L}}_{spin} = \Pi^*(\sigma_{spin}) \qquad /3.5./$$

Our most intersting result in [1] was that there cannot exist a globally defined Θ field of 1-forms with $d\Theta = \sigma_{spin}$; 3.5 shows then that this means really that no global Lagrangian, function can exist. In order to construct a local Lagrangian we must find first a local potential for the spin-part of σ. The magnetic field-spin - interaction part is itself a total differential, hence a /global/ potential for it:

$$\lambda^{(1)}_{spin} = \mu \langle s,B \rangle \, dt \qquad /3.6./$$

Let us study now the second term. Let's introduce the atlas of S^2 consisting of charts F_+ and F_- : $R^2 \to S^2$:

$$\left[F_{\pm} (s) \right]^{\alpha} = \frac{2s^{\alpha}}{1 + |s|^2} \qquad /\alpha = 1,2 \; ; \; s = (s^1, s^2) \; /$$

$$\left[F_{\pm} \; s^3 \right] = \pm \frac{1 - |s|^2}{1 + |s|^2} \qquad /3.7./$$

One verifies at once that on these charts the form

$$\lambda^{(2)}_{spin} = \pm 2 \lambda \frac{s^1 \, ds^2 - s^2 \, ds^1}{1 + (s^1)^2 + (s^2)^2} \qquad /3.8./$$

is a potential for $\sigma^{(2)}_{spin}$. The results of 2.4 can be then applied : 2.4.7 is verified, hence by 2.4.8 the corresponding local Lagrangian functions:

$$L^{(1)}_{spin} (s,\dot{s}) = \mu \langle s,B \rangle \qquad /3.9./$$

$$L^{(2)}_{spin} (s,\dot{s}) = \pm \frac{2 \lambda}{1+|s|^2} \left[-s^2 \dot{s}^1 + s^1 \dot{s}^2 \right] \qquad /3.10./$$

Summing up, the local Lagrangian function for spin particle is given in the above coordinates as

$$L(q,s,v,\dot{s},t) =$$

$$\left\{ \frac{1}{2} \left[mv^2 + \langle A,q \rangle \right] - V(q) \right\} + \left\{ \mu \langle s,B \rangle \pm \frac{2 \lambda}{1+|s|^2} \left[-s^2 \dot{s}^1 + s^1 \dot{s}^2 \right] \right\} \qquad /3.11./$$

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with gauge transformation

$$L \to kL + \frac{\partial f}{\partial q^{\alpha}} v^{\alpha} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial s^{\alpha}} \dot{s}^{\alpha}$$

/here f is any real function on $Q \times S^2 \times R$ /.

From 3.10 we see at once that none of these local Lagrangians can be extended onto the entire evolution space. This can be demonstrated by observing, that

$$s_1 = 0,\ s_2 \to \infty,\ \dot{s}_1 = s_2,\ \dot{s}_2 = 0$$
$$s_1 = 0,\ s_2 \to \infty,\ \dot{s}_1 = 0,\ \dot{s}_2 = 0$$

both tend to the same point of $T\,S^2$ /namely to the 0 -tangent vector at the lower resp. upper pole/, but L tends in the first case to ∓ 1, in the second case to 0. Hence L cannot have a limit.

In this case the rang of $|\partial_{\dot{\alpha}\dot{\beta}} L|$ is $2n+2$, H_y is four-dimensional, the leaves belonging to L are thus four-dimensional submanifolds of $T(Q\times S^2 \times R)$. Their projections onto $Q\times S^2\times R$ are C^∞ curves satisfying the Thomas-equations /[1],[2]/.

4. CONCLUSION

We have shown that even the spin particle has a local Lagrangian function. Thus we are lead to the following modification of our local variational principle:

Definition: A mechanical system given by (E,σ) is said to have a local variational description if there exists a λ field of 1-forms with the following properties:

- i./ $d\lambda = \sigma$
- ii./ λ is semi-basic
- iii./ the pulled-back of λ to F is $\dot{d}$-closed: $\dot{d}(\Pi^*(\lambda)) = 0$.

Under these conditions there exists a Lagrangian function L on each local chart of E and the system's Lagrange-form is the projection of $d\dot{d}\mathcal{L}$ onto E.

The necessary and sufficient condition for (E,σ) to have a local variational description is

- i./ $d\sigma = 0$ /Maxwell's Principle/
- ii./ $\dot{d}(\Pi^\times\sigma) = 0$

The local variational principle states that these conditions are verified for any mechanical system.

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