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ABSTRACT

A systematic quantization to the scalar-spinor instanton is given in a canonical formalism of Euclidean space. A basic idea is in the repair of the symmetries of the O(5) covariant system in the presence of the instanton. The quantization of the fermion is carried through in such a way that the fermion number should be conserved. Our quantization enables us to get well-defined propagators for both the scalar and the fermion, which are free from unphysical poles.

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I. INTRODUCTION

Recently classical solutions to the field equations have been used as a probe for the quantum significances of the system. The classical equations of motion are found by requiring the variation of the action S to be stationary,

$$\delta S(\phi_i) = 0 \quad . \tag{1.1}$$

In a Euclidean formulation some class of solutions ϕ_i^c might give the finite action. Then the corresponding classical field trajectory, through a path integral, would give a non-trivial contribution to the quantum formulation of the theory [1]. Furthermore, if the energy momentum tensor $\theta_{\mu\nu}$ happens to be vanishing for those field configurations, the solutions might be considered to be the vacuum of the system. New quantum fields ϕ_i' can be introduced by the shift

$$\phi_i = \phi_i^c + \phi_i' \quad , \tag{1.2}$$

and a quantum fluctuation around that non-trivial vacuum trajectory will be studied. The classical solution possessing these two properties, i.e. $S(\phi_i^c) < \infty$ and $\theta_{\mu\nu}(\phi_i{}^c) = 0$, *) may be called an instanton.

The instanton solution for a coupled system of scalar and fermion has been given [2]. In what follows we shall discuss the quantization of that instanton. The quantization is carried through in a canonical formalism in Euclidean space and, in consequence, well-defined propagators for the scalar and the fermion are obtained. The symmetries of the Lagrangian might seemingly be broken spontaneously by the shift (1.2). But such is not the case here. In the presence of the instanton, the symmetries are repaired in some sense. **) The main purpose of the present paper is towards a better understanding of such phenomena.

*) For the soliton, $\theta_{00}(x) > 0$ and in general is localized in space so that it cannot be viewed as the vacuum of the system.

**) Such repair of the symmetries might in general be called "tunnelling" when the amplitude is continued to Minkowski space. For the case of the BPST instanton, see Refs.1,3 and 4.

In Sec.II the instanton solution is regained in a manifestly O(5) covariant fashion; it is merely a constant solution. The equations of motion for the quantum fluctuations are derived. Quantization of the scalar field and the Goldstone-like modes are discussed in Sec.III. Cancellation of an unphysical pole in the propagator [5] is illustrated in detail. In Sec.IV quantization of the fermion is carried out in such a way that the fermion number should be conserved. It also leads to the well-defined propagator for the fermion. Finally, Sec.V is devoted to discussions on a zero eigenvalue for the fermion equation of motion. In the Appendix the Dirac formalism for a constrained system is given to complete the contents of Sec.III.

II. INSTANTON SOLUTION

Let us start from the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - 2\lambda^2\phi^4 + i\,\psi^\dagger\alpha\cdot\partial\psi + g\psi^\dagger\psi\phi \quad . \tag{2.1}$$

This Lagrangian is invariant under an O(5.1) conformal group. Since we are interested in the instanton solution, let us write (2.1) in an O(5) invariant form $\mathcal{L}_\Omega$ by using the relation

$$\int d^4x\,\mathcal{L} = \int d\Omega\,\mathcal{L}_\Omega \quad , \tag{2.2}$$

where $d\Omega$ is the solid angle of the sphere in five-dimensional Euclidean space. Now defining dimensionless fields by

$$\begin{cases} \varphi(\xi) \equiv \dfrac{(a^2+x^2)}{2a}\phi(x) & (2.3a) \\[2ex] \chi(\xi) \equiv \dfrac{(a^2+x^2)}{a^{3/2}}(a - i\alpha\cdot x)\psi(x) & (2.3b) \end{cases}$$

we get [6]

$$\mathcal{L}_\Omega = -\frac{1}{4}(L_{ab}\varphi)^2 + \varphi^2 - 2\lambda^2\varphi^4 - \frac{1}{8}\chi^\dagger(L_{ab}S_{ab}+2)\chi + \frac{1}{8}g\chi^\dagger\chi\varphi \quad . \tag{2.4}$$

The equations of motion are

$$\begin{cases} (\frac{1}{2}L_{ab}^2 + 2)\varphi = 8\lambda^2\varphi^3 - \frac{1}{8}g\chi^\dagger\chi & (2.5a) \\[2ex] (L_{ab}S_{ab} + 2)\chi = g\varphi\chi \quad . & (2.5b) \end{cases}$$

We shall try to look for constant solutions of this coupled equation. Inserting an ansatz

$$\begin{cases} \varphi^c = \dfrac{c_3}{2} \;, \\[2ex] \chi^c = c_1 \;, \end{cases} \tag{2.6}$$

in Eq.(2.5) we have

$$\begin{cases} c_3 = 8\lambda^2\dfrac{c_3^3}{8} - \dfrac{1}{8}g\,c_1^\dagger c_1 \;, \\[2ex] 2c_1 = g c_1\dfrac{c_3}{2} \;. \end{cases} \tag{2.7}$$

By using Eq.(2.3), Eqs.(2.6) and (2.7) are written in the following form for $c_1 \neq 0$: *)

$$\phi^c(x) = \frac{c_3 a}{a^2+x^2} \quad , \quad \psi^c(x) = \frac{a^{3/2}}{(a^2+x^2)^2}(a + i\alpha\cdot x)\,c_1 \;, \tag{2.8}$$

with

$$c_1^\dagger c_1 = -\frac{32}{g^2}\left(1 - \frac{16\lambda^2}{g^2}\right) \;, \quad c_3 = \frac{4}{g} \;. \tag{2.9}$$

This is nothing but the instanton solution of Ref.2.

We shall study hereafter the quantum fluctuation about the instanton solution. Introducing quantum fields φ' and χ' by the shift

$$\varphi = c_3/2 + \varphi' \quad , \quad \chi = c_1 + \chi' \;, \tag{2.10}$$

with $\langle\varphi\rangle_0 = \frac{c_3}{2}$ and $\langle\chi^\dagger\chi\rangle_0 = c_1^\dagger c_1$, one can rewrite the Langrangian (2.4) in terms of φ' and χ'. Inserting Eq.(2.10) we get for the small coupling limits $\lambda \to 0$ and $g \to 0$,

*) For $c_1 = 0$, we get the Fubini instanton [5], $\phi(x) = \frac{1}{\lambda}\,\frac{a}{a^2+x^2}$.

To give a provisory meaning to Eq.(3.9), j has been put into $j+\varepsilon$; that is the origin of the pole in the propagator (3.8). We shall demonstrate in the following that the pole is cancelled by postulating a quantization appropriate to the instanton.

First we shall see the symmetry properties of the system. The original Lagrangian (2.1) is invariant under the O(5.1) (15-parameter group), while the vacuum (3.5) is invariant only under the O(5) [2] (10-parameter group). One would suspect that there might be the spontaneous breakdown of symmetry concerned with five degrees of freedom. The five generators responsible for it will be S_a , whose transformation properties are [5]:

$$[\varphi(\xi),\ S_a] = (\xi^c L_{ca} + i\xi_a)\,\varphi(\xi) \quad . \tag{3.10}$$

Taking the vacuum expectation value of this, one gets

$$\langle [\varphi(\xi),\ S_a] \rangle_0 = \frac{ic_3}{2}\,\xi_a \quad , \tag{3.11}$$

which is, of course, non-vanishing. The five modes, therefore, exist in the field φ . These five modes would be physical as the Goldstone modes if the symmetry were broken spontaneously. The Goldstone theorem is applied only for the case when the vacuum has a lower symmetry than the original Lagrangian. However, we must note that, as far as the field φ is concerned, both the Lagrangian (2.4) and the vacuum (3.5) have only O(5) invariance. There is no spontaneous breakdown mechanism in this system *) . The five modes are, therefore, unphysical and should be removed from the physical space in which the quantization will be carried out.

Let us remove the five unphysical modes ξ_a $(a = 1,\ldots,5)$ from the expansion (3.1). Such modes correspond to the first $j = 0$ (1 mode) and $j = 1$ (4 modes). It is confirmed easily by checking the following fact:

$$\frac{1}{2} L_{\mu\nu}L^{\mu\nu}\ \xi_5 = 0(0+2)\ \xi_5 \ , \tag{3.12a}$$

$$\frac{1}{2} L_{\mu\nu}L^{\mu\nu}\ \xi_\rho = 1(1+2)\ \xi_\rho \ , \tag{3.12b}$$

*) The parity is, of course, broken in O(5) space as is easily seen from the Lagrangian (2.4) and the fact that $\langle\varphi\rangle_0 = \frac{c_3}{2}$, analogous to the discussions of the usual Goldstone-like potential. However, we are concentrating on the continuous symmetry here.

where

$$\frac{1}{2} L_{\mu\nu}L^{\mu\nu} = -x^2\Box + (x\cdot\partial + 2)(x\cdot\partial)$$
$$\xi_5 = \frac{a^2 - x^2}{a^2 + x^2}$$
$$\xi_\rho = \frac{2a x_\rho}{a^2 + x^2} \quad . \tag{3.13}$$

As a consequence $j = 0$ and $j = 1$ modes should be removed from the physical space. This may be accomplished by the Dirac formalism for a constrained system. In the appendix we provide this calculation in detail. The desired commutation relation turns out to be

$$\left[a_{j\mu}^{(-)},\ a_{j'\mu'}^{(+)}\right] = \delta_{\mu\mu'}\left(\delta_{jj'} - \delta_{j0}\delta_{j'0} - \delta_{j1}\,\delta_{j'1}\right) \tag{3.14}$$

instead of (3.4). This form of the quantization makes manifest what is going on.

Now let us illustrate the cancellation of the unphysical pole in (3.8) by making use of our quantization. The propagator is defined by

$$\begin{aligned} G_5(z) &= \langle 0|\, R\,(\varphi'(\xi_1)\,\varphi'(\xi_2))\,|0\rangle \\ &= \theta(r_1 - r_2)\langle 0|\,\varphi'(\xi_1)\,\varphi'(\xi_2)\,|0\rangle + \theta(r_2 - r_1)\langle 0|\,\varphi'(\xi_2)\,\varphi'(\xi_1)\,|0\rangle \ . \end{aligned} \tag{3.15}$$

Eqs.(3.1) and (3.3) together with the commutation relation (3.14) lead to

$$\begin{aligned} G_5(z) = g(z) - \theta(r_1 - r_2)\Big\{ & f_0^{(-)}(r_1)\, f_0^{(+)}(r_2) \sum_\mu Y_{0\mu}^{*}(\alpha_1)\, Y_{0\mu}(\alpha_2) \\ & + f_1^{(-)}(r_1)\, f_1^{(+)}(r_2) \sum_\mu Y_{1\mu}^{*}(\alpha_1)\, Y_{1\mu}(\alpha_2) \Big\} - (1 \leftrightarrow 2) \ . \end{aligned} \tag{3.16}$$

From (3.9) one gets, by setting $j \to j + \varepsilon$,

$$\mathcal{L}_\Omega = -\frac{1}{4}(L_{ab}\varphi')^2 + (1-\frac{\mu}{2})\,\varphi'^2 - \frac{1}{8}\chi'^\dagger(S_{ab}L_{ab}+2)\chi' + \frac{1}{4}\chi'^\dagger\chi' \,, \qquad (2.11)$$

with $\mu = \dfrac{96\lambda^2}{g^2}$. Here we have assumed that the transitions between the classical and the quantum worlds are forbidden [2]. As a consequence the equations of motion for φ' and χ' are

$$\begin{cases} \left(\frac{1}{2}L_{ab}{}^2 + 2 - \mu\right)\varphi' = 0 \,, & (2.12a) \\ S_{ab}\,L_{ab}\chi' = 0 \,. & (2.12b) \end{cases}$$

Since μ is fixed to be 6 from the condition of the fermion-number conservation (see Eq.(4.16)), Eqs.(2.12) become

$$\begin{cases} \left(\frac{1}{2}L_{ab}{}^2 - 4\right)\varphi' = 0 \,, & (2.13a) \\ S_{ab}\,L_{ab}\chi' = 0 & (2.13b) \end{cases}$$

Eqs.(2.13) are the decoupled equations and are easier to solve.

III. QUANTIZATION OF THE SCALAR FIELD

The quantization of the scalar field φ' that satisfies Eq.(2.13a) has been discussed in Ref.5. Here we only briefly mention the essence of it with regard to our subject. The field $\varphi'(\xi)$ is considered in polar co-ordinates of four-dimensional Euclidean space, $\varphi'(\xi) = \varphi'(r,\alpha)$ and is expanded in spherical harmonics,

$$\varphi'(r,\alpha) = \sum_{j=0}^{\infty}\sum_{\mu}\left(a_{j\mu}^{(-)}f_j^{(-)}(r)\,Y_{j\mu}^*(\alpha) + a_{j\mu}^{(+)}f_j^{(+)}(r)\,Y_{j\mu}(\alpha)\right), \qquad (3.1)$$

where $Y_{j\mu}(\alpha)$ satisfies

$$\begin{cases} \frac{1}{2}L_{\rho\sigma}L^{\rho\sigma}\,Y_{j\mu}(\alpha) = j(j+2)\,Y_{j\mu}(\alpha) & (3.2) \\ \sum_{j=0}\sum_{\mu}Y_{j\mu}^*(\alpha)\,Y_{j\mu}(\alpha') = \delta^3(\alpha-\alpha') \,. & (3.3) \end{cases}$$

The quantization is performed by means of the commutation relation

$$\left[a_{j\mu}{}^{(-)}\,,\,a_{j'\mu'}{}^{(+)}\right] = \delta_{jj'}\,\delta_{\mu\mu'} \,, \qquad (3.4)$$

with the definition of an O(5) invariant vacuum $|0\rangle$ [2]

$$a_{j\mu}{}^{(-)}|0\rangle = 0 \,. \qquad (3.5)$$

This procedure of the quantization has been applied to obtain a propagator of the scalar field satisfying

$$\left\{\frac{1}{2}L_{ab}{}^2 - n(n+3)\right\}\varphi' = 0 \,. \qquad (3.6)$$

The result is

$$\begin{aligned} g(\xi) &= \langle 0|\,R(\varphi'(\xi_1)\,\varphi'(\xi_2))\,|0\rangle \\ &= \frac{-1}{8\pi^2}\left(\frac{n-\varepsilon+1}{\varepsilon}\right)\frac{1}{2\pi i}\oint\left(\frac{1-w^2}{z}\right)^{-(2+n)}(w-z)^{n-\varepsilon}\,dw \,, \end{aligned} \qquad (3.7)$$

with $z = \xi_1\cdot\xi_2$. Consequently, for Eq.(2.13a) (i.e. n = 1) it becomes

$$g(z) = \frac{1}{8\pi^2}\left\{\frac{3z}{\varepsilon} - 3 - 3z\log(1-z) + \frac{1}{1-z}\right\}. \qquad (3.8)$$

The unphysical pole $3z/8\pi^2\varepsilon$ has appeared due to the fact that f_j in Eq.(3.1) becomes singular for $j = 0$ and $j = 1$. More precisely, f_j is given by

$$f_j^{(+)}(r) = \sqrt{\frac{j(j+2)(j-1)(j+3)}{8(j+1)}}\cdot\frac{ar}{a^2+r^2}\left\{\frac{(r/a)^{(j-1)}}{(j-1)j} + 2\,\frac{(r/a)^{(j+1)}}{j(j+2)} + \frac{(r/a)^{j+3}}{(j+2)(j+3)}\right\} \qquad (3.9)$$

with

$$f_j^{(-)}(r) = f_j^{(+)}(1/r) \,.$$

$$f_0^{(+)}(r) = \sqrt{\frac{-3}{4\varepsilon}}\, \frac{r^2 - a^2}{r^2 + a^2} \,, \quad f_0^{(-)}(r) = \sqrt{\frac{-3}{4\varepsilon}}\, \frac{a^2 - r^2}{a^2 + r^2} \,. \tag{3.17}$$

So

$$f_0^{(-)}(r_1) f_0^{(+)}(r_2) = \frac{1}{\varepsilon}\frac{3}{4}\, \frac{a^2 - r_1^2}{a^2 + r_1^2}\, \frac{a^2 - r_2^2}{a^2 + r_2^2} \,. \tag{3.18}$$

Similarly it follows that

$$f_1^{(-)}(r_1) f_1^{(+)}(r_2) = \frac{1}{\varepsilon}\frac{3}{4}\, \frac{a r_1}{a^2 + r_1^2}\, \frac{a r_2}{a^2 + r_2^2} \,. \tag{3.19}$$

Combining these with the addition formula

$$\sum_{\mu} Y_{j\mu}^{*}(\alpha_1) Y_{j\mu}(\alpha_2) = \frac{j+1}{2\pi^2}\, C_j^{1}(\alpha_1 \cdot \alpha_2) \,, \tag{3.20}$$

one gets

$$G_s(z) = g(z) - \{\theta(r_1 - r_2) + \theta(r_2 - r_1)\} \frac{1}{\varepsilon}\frac{3}{4}\frac{1}{2\pi^2} \left\{ \frac{a^2 - r_1^2}{a^2 + r_1^2}\, \frac{a^2 - r_2^2}{a^2 + r_2^2} + \frac{4a^2 r\alpha_1 \cdot r\alpha_2}{(a^2 + r_2^2)(a^2 + r_1^2)} \right\}$$

$$= g(z) - \frac{1}{8\pi^2} \cdot \frac{3z}{\varepsilon} \,. \tag{3.21}$$

Finally, with (3.8) one can see that the pole is exactly cancelled to give

$$G_s(z) = \frac{1}{8\pi^2} \left\{ -3 - 3z \log(1-z) + \frac{1}{1-z} \right\} \,. \tag{3.22}$$

This is the propagator of the field φ' which satisfies Eq.(2.13a). We may write it in the form:

$$\left\{ \frac{1}{2} L_{ab}{}^2 - 4 \right\} G_s(z) = \delta(\xi_1 - \xi_2) \,. \tag{3.23}$$

The well-defined propagator (3.22) is the main outcome of the quantization (3.14).

IV. QUANTIZATION OF FERMION FIELD

Since the fermion field has been shifted by the form (2.10), one could suspect that the fermion-number conservation would have been broken. We shall analyse this problem here.

In addition to the conformal invariance, the Lagrangian (2.1) has the gauge invariance of the first kind. The charge of such a transformation is r-independent and is given by

$$Q_F = \int \chi^\dagger \chi \, d\alpha \,. \tag{4.1}$$

Thus we get

$$[\chi, Q_F] = q_F \chi \tag{4.2}$$

with $q_F = 1$. For the pair state we have

$$[\chi^\dagger \chi, Q_F] = q_F' \chi^\dagger \chi \tag{4.3}$$

with $q_F' = 0$. The O(5) invariant nature of the vacuum for the fermion could be considered in the form [2)]

$$\langle [\chi^\dagger \chi \,,\, J_{ab}] \rangle_0 = 0 \,, \tag{4.4}$$

where J_{ab} is the generator of the O(5) rotation. Analogous to it, we get from (4.3)

$$\langle [\chi^\dagger \chi \,,\, Q_F] \rangle_0 = q_F' \langle \chi^\dagger \chi \rangle_0 = q_F' \cdot c_1^\dagger c_1 \,. \tag{4.5}$$

This suggests that the constant mode should be removed from χ to preserve the fermion-number conservation.

To quantize the field χ', let us expand it in the form:

$$\chi'(r,\alpha) = \sum_{j\mu} \sum_{s=1}^{2} \left\{ u_s(j,\mu)\, a_{j\mu}^{(s)} f_j^{(-)}(r) Y_{j\mu}^{*}(\alpha) + v_s(j,\mu)\, b_{j\mu}^{(s)} f_j^{(-)}(r) Y_{j\mu}(\alpha) \right\} , \tag{4.6}$$

where we have put into evidence the helicity index s. The constant mode corresponds to the $j = 0$ mode. Then $j = 0$ should be removed from the expansion (4.6). The similar analysis described in Sec.III leads us to the following quantization:

$$[a_{j\mu}^{(s)}, a_{j'\mu'}^{(s')*}]_+ = [b_{j\mu}^{(s)}, b_{j'\mu'}^{(s')*}]_+ = \delta_{ss'}\delta_{\mu\mu'}(\delta_{jj'} - \delta_{j0}\delta_{j'0}) \quad . \tag{4.7}$$

Let us elucidate that the quantization (4.7) surely restores the fermion-number conservation. The shift (2.10) and the same assumption that leads to Eq.(2.11) imply

$$\chi^\dagger\chi = c_1^\dagger c_1 + \chi'^\dagger\chi' \quad . \tag{4.8}$$

Consequently the charge (4.1) becomes

$$Q_F = c_0 + \int \chi'^\dagger\chi' \, d\alpha \tag{4.9}$$

with

$$c_0 = 2\pi^2 c_1^\dagger c_1 \quad . \tag{4.10}$$

Inserting (4.6) together with a little algebra, one can verify that *)

$$\int \chi'^\dagger\chi' \, d\alpha = \sum_{j\mu} \sum_{s=1}^{2} [a_{j\mu}^{(s)*}a_{j\mu}^{(s)} - b_{j\mu}^{(s)*}b_{j\mu}^{(s)}] - 1 \quad . \tag{4.11}$$

This is the usual form of the charge apart from (-1) on the right-hand side. Such an additional term (-1) comes about from the extra term in the quantization (4.7), (i.e. $j = j' = 0$). Now the definition of the O(5) invariant vacuum,

$$a_{j\mu}^{(s)}|0\rangle = b_{j\mu}^{(s)}|0\rangle = 0 \tag{4.12}$$

leads to the result

$$Q_F|0\rangle = (c_0 - 1)|0\rangle \quad . \tag{4.13}$$

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*) The product $\chi'^\dagger\chi'$ should be interpreted to be the normal ordered product to ensure $\langle\chi'^\dagger\chi'\rangle_0 = 0$. As a consequence bb^* can be replaced by $-b^*b$ unambiguously.

When c_0 is chosen to be

$$c_0 = 1 \ , \tag{4.14}$$

one finally gets

$$Q_F|0\rangle = 0 \quad . \tag{4.15}$$

Hence the vacuum does not carry the fermion number. If the usual quantization is used instead of (4.7), the vacuum would be endued with the fermion number c_0 (apart from the trivial case $c_0 = 0$) as is clear from the foregoing results. It will inevitably violate the fermion-number conservation in physical processes. Note that from Eqs.(4.14), (4.10) and (2.9) we get

$$\mu \equiv \frac{96\lambda^2}{g^2} = 6 + \frac{3g^2}{32\pi^2} \longrightarrow 6 \tag{4.16}$$

for the small coupling limit. *) Thus Eq.(2.13a) is achieved.

We now turn our attention to the propagator. In principle it could be obtained by means of the quantization (4.7), but a simpler derivation is given below. From the equation of motion (2.13b), the propagator should satisfy

$$S_{ab} L_{ab} G_F(z) = \delta(\xi_1 - \xi_2) \quad . \tag{4.17}$$

By using the identity

$$(L_{ab}S_{ab})(L_{ab}S_{ab} + 3) = (L_{ab}S_{ab})^2 + 3L_{ab}S_{ab} = \frac{1}{2} L_{ab}{}^2 \tag{4.18}$$

we get

$$G_F(z) = (S_{ab}L_{ab} + 3)\, G(z) \tag{4.19}$$

with

$$\frac{1}{2} L_{ab}{}^2 G(z) = \delta(\xi_1 - \xi_2) \quad . \tag{4.20}$$

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*) Even for $\mu = 6$ we get $\langle\chi^\dagger\chi\rangle_0 = c_1^\dagger c_1 = \frac{1}{2\pi^2}$ due to $g \to 0$, so that the situation is different from that of de Alfaro and Furlan (Ref.6) where $\langle\chi^\dagger\chi\rangle_0 = 0$.

Hence $G(z)$ is nothing else than the propagator for the scalar (3.7) of $n = 0$. It reads explicitly

$$G(z) = \frac{1}{8\pi^2} \left\{ \frac{1}{\epsilon} - \log (1-z) + \frac{1}{1-z} \right\} . \qquad (4.21)$$

Although we do not proceed with the calculations here, we can verify that, as in Sec.III, the removal of the $j = 0$ mode is accompanied exactly by the cancellation of the pole $1/8\pi^2\epsilon$. Thus one can finally reach the result

$$G(z) = \frac{1}{8\pi^2} \left\{ -\log (1 - z) + \frac{1}{1-z} \right\} . \qquad (4.22)$$

To summarize, the major outcomes of our quantization procedure are the well-defined propagators (3.22) and (4.19) together with (4.22). These propagators will provide paramount tools to a systematic study of the quantum structure of the system.

The cancellation of the unphysical poles in the propagators is reminiscent of the situation encountered in the soliton physics. The "naive propagator" has an infra-red divergence that has originated in certain zero frequency modes. Several methods have been proposed to overcome this difficulty.[7)]

V. DISCUSSION

As a peculiar problem of the instanton that includes the fermion, we shall finally comment on the vanishing functional fermion determinant [3),8)]. When the vacuum-to-vacuum amplitude is considered in path integral, a functional fermion determinant, det M_χ , comes out after the functional integration of the fermion field. From Eq.(2.11),

$$M_\chi \chi' = S_{ab} L_{ab} \chi' . \qquad (5.1)$$

The determinant will be computed by an eigenvalue equation

$$M_\chi \chi' = \lambda\chi' . \qquad (5.2)$$

The zero eigenvalue $\lambda = 0$ corresponds to the $n = 0$ mode in the O(5) spherical harmonics as is easily seen through the identity (4.18) and Eq.(3.6). However, the $n = 0$ mode (i.e. constant mode) has already been

removed as the result of the discussions in Sec.IV. So the zero eigenvalue $\lambda = 0$ is eliminated automatically in our formalism. The functional fermion determinant is therefore non-vanishing, so that the amplitude does not vanish. *)

Our simple example suggests a clue to the zero eigenvalue problem of the fermion in the presence of the BPST pseudoparticle potential [3),8)]. It has been hitherto oscure. The zero eigenvalues are in general closely related to the symmetries of the system. One can discuss it unambiguously only after having solved the equations of motion exactly. Thus the coupled equation of the gauge and the spinor fields should be solved in an exact way to give a precise conclusion to it. We would hopefully conjecture, however, that the statement mentioned above for the scalar-spinor theory will also be applicable to the gauge-spinor theory.

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*) It is easy to see that quite the same argument holds for the zero eigenvalue of the scalar.

APPENDIX

The aim of this appendix is to derive Eq.(3.14). The removal of $j = 0$ and $j = 1$ modes from the expansion (3.1) is achieved by imposing the subsidiary conditions

$$\int \varphi'(r,\alpha) Y_{0\mu\nu}(\alpha)\, d\alpha = 0 \ , \quad \int \pi'(r,\alpha) Y^{*}_{0\mu\nu}(\alpha)\, d\alpha = 0 \tag{A.1}$$

and

$$\int \varphi'(r,\alpha) Y_{1\mu\nu}(\alpha)\, d\alpha = 0 \ , \quad \int \pi'(r,\alpha) Y^{*}_{1\mu\nu}(\alpha)\, d\alpha = 0 \ . \tag{A.2}$$

Here $\pi'(r,\alpha)$ is the canonical conjugate to $\varphi'(r,\alpha)$,

$$\pi'(r,\alpha) = r \frac{d}{dr} \varphi'(r,\alpha) \tag{A.3}$$

and the canonical bracket takes the following form *) according to Ref.5

$$\{ \varphi'(r,\alpha_1), \pi'(r,\alpha_2) \} = \left(\frac{a^2+r^2}{2ar} \right) \delta^3(\alpha_1 - \alpha_2) \ . \tag{A.4}$$

Eqs.(A.1) and (A.2) give ten conditions in all and we write them in the following order **) :

$$\begin{aligned}
\psi_1 &= \int \varphi'(r,\alpha) Y_{000}(\alpha)\, d\alpha = 0 \ , & \psi_2 &= \int \pi'(r,\alpha) Y^{*}_{000}(\alpha)\, d\alpha = 0 \\
\psi_3 &= \int \varphi'(r,\alpha) Y_{100}(\alpha)\, d\alpha = 0 \ , & \psi_4 &= \int \pi'(r,\alpha) Y^{*}_{100}(\alpha)\, d\alpha = 0 \\
\psi_5 &= \int \varphi'(r,\alpha) Y_{11-1}(\alpha)\, d\alpha = 0 \ , & \psi_6 &= \int \pi'(r,\alpha) Y^{*}_{11-1}(\alpha)\, d\alpha = 0 \\
\psi_7 &= \int \varphi'(r,\alpha) Y_{110}(\alpha)\, d\alpha = 0 \ , & \psi_8 &= \int \pi'(r,\alpha) Y^{*}_{110}(\alpha)\, d\alpha = 0 \\
\psi_9 &= \int \varphi'(r,\alpha) Y_{111}(\alpha)\, d\alpha = 0 \ , & \psi_{10} &= \int \pi'(r,\alpha) Y^{*}_{111}(\alpha)\, d\alpha = 0 \ .
\end{aligned} \tag{A.5}$$

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*) This should be considered as the Poisson bracket.

**) The author wants to thank F. Legovini for discussions about the construction of the Dirac matrix.

Thus the conditions (A.5) do not commute. They are the second-class constraints in Dirac's formalism of constrained system.[9] Actually the 10 × 10 matrix of the brackets of ψ_a,

$$C_{ab} = \{ \psi_a , \psi_b \} \ , \tag{A.6}$$

does not vanish weakly and the non-vanishing elements are

$$C_{ab} = \begin{pmatrix}
0 & 1 &&&&&&&& \\
-1 & 0 &&&&&&&& \\
&& 0 & 1 &&&&&& \\
&& -1 & 0 &&&&&& \\
&&&& 0 & 1 &&&& \\
&&&& -1 & 0 &&&& \\
&&&&&& 0 & 1 && \\
&&&&&& -1 & 0 && \\
&&&&&&&& 0 & 1 \\
&&&&&&&& -1 & 0
\end{pmatrix} \times \left(\frac{a^2+r^2}{2ar} \right)^2 \ . \tag{A.7}$$

The inverse is easy to find,

$$C^{-1}_{ab} = \begin{pmatrix}
0 & -1 &&&&&&&& \\
1 & 0 &&&&&&&& \\
&& 0 & -1 &&&&&& \\
&& 1 & 0 &&&&&& \\
&&&& 0 & -1 &&&& \\
&&&& 1 & 0 &&&& \\
&&&&&& 0 & -1 && \\
&&&&&& 1 & 0 && \\
&&&&&&&& 0 & -1 \\
&&&&&&&& 1 & 0
\end{pmatrix} \times \left(\frac{2ar}{a^2+r^2} \right)^2 \ . \tag{A.8}$$

The new canonical bracket is defined as [9]

$$\{ \varphi'(r,\alpha_1), \pi'(r,\alpha_2) \}^{*} = \left(\frac{a^2+r^2}{2ar} \right)^2 \delta^3(\alpha_1 - \alpha_2) - \{ \varphi'(r,\alpha_1), \psi_a \} C^{-1}_{ab} \{ \psi_b, \pi'(r,\alpha_2) \} \ . \tag{A.9}$$

Inserting (A.5) and (A.8), one gets

$$\begin{aligned}
\{ \varphi'(r,\alpha_1), \psi_a \} C^{-1}_{ab} \{ \psi_b, \pi'(r,\alpha_2) \} &= \left(\frac{a^2+r^2}{2ar} \right)^2 \left\{ \sum_{\mu} Y^{*}_{0\mu}(\alpha_1) Y_{0\mu}(\alpha_2) + \sum_{\mu} Y^{*}_{1\mu}(\alpha_1) Y_{1\mu}(\alpha_2) \right\} \\
&= \left(\frac{a^2+r^2}{2ar} \right)^2 \left\{ \frac{1}{2\pi^2} (1 + 4\alpha_1 \cdot \alpha_2) \right\} \ .
\end{aligned} \tag{A.10}$$

Then (A.9) becomes

$$\{\varphi'(r,\alpha_1), \pi'(r,\alpha_2)\} = \left(\frac{a^2+r^2}{2ar}\right)^2 \left\{ \delta^3(\alpha_1-\alpha_2) - \frac{1}{2\pi^2}(1+4\alpha_1\cdot\alpha_2) \right\} . \tag{A.11}$$

One can easily verify that (A.11) is equivalent to

$$\{a_{j\mu}^{(-)}, a_{j'\mu'}^{(+)}\} = \delta_{\mu\mu'}(\delta_{jj'} - \delta_{j0}\delta_{j'0} - \delta_{j1}\delta_{j'1}) \tag{A.12}$$

by using Eq.(3.3) and so on. Taking into account the correspondence principle, one may finally arrive at the result (3.14).

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